Differential Geometry

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Contents

1 Smooth manifolds

2 Tangent Spaces

3 Partitions of Unity

4 The Derivative

5 The Tangent Bundle

6 Submanifolds

7 The Whitney Theorems

8 Vector Fields

9 Flows

10 Lie Groups

11 The Lie Algebra of a Lie Group

12 Smooth Actions of Lie Groups

13 Homogeneous Spaces

14 Distributions and Integrability

15 Foliations and the Frobenius Theorem

16 Bundles

17 The Fibre Bundle Construction Theorem

18 Associated Bundles

19 Tensor and Exterior Algebras

20 Sections of Vector Bundles

21 Tensor Fields

22 The Lie Derivative Revisited

23 The Exterior Differential

24 Orientations and Manifolds With Boundary

25 Smooth Singular Cubes

26 Stokes' Theorem

27 The Poincaré Lemma and the de Rham Theorem

28 Connections

29 Parallel Transport

30 The Equivalence of Connections and Parallel Transport

31 Covariant Derivatives

32 Holonomy

33 Curvature

34 The Holonomy Algebra

35 Reinterpreting Curvature

36 Exterior Covariant Differentials

37 Riemannian Vector Bundles

38 Characteristic Classes

39 Connections on Principal Bundles

40 The Connection Form

41 The Curvature Form

42 The Ambrose–Singer Holonomy Theorem

43 Geodesics and Sprays

44 The Exponential Map of a Spray

45 Torsion-free Connections

46 The Levi-Civita Connection

47 Symmetries of the Curvature Tensor

48 Sectional, Ricci, and Scalar Curvature

Problem Sheet A

Problem Sheet B

Problem Sheet C

Problem Sheet D

Problem Sheet E

Problem Sheet F

Problem Sheet G

Problem Sheet H

Problem Sheet I

Problem Sheet J

Problem Sheet K

Problem Sheet L

Problem Sheet M

Problem Sheet N

Problem Sheet O

Problem Sheet P

LECTURE 1

Smooth manifolds

Let us begin with a short history lesson on how you learned to identify (continuously) differentiable functions.

(i) **(High school)** A function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous if its graph doesn't have any jumps. The derivative $f'(t)$ at a point t is the slope of the graph of $f(t) = s$ at the point t.

(ii) **(First class in Analysis)** The (ε, δ) definition of continuity. A function $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at the point r if the limit

$$\lim_{s \to 0} \frac{f(t+s) - f(t)}{s}$$

exists. This limit is denoted by $f'(t)$. The function f is continuously differentiable if $t \mapsto f'(t)$ is itself a continuous function.

(iii) **(Second class in Analysis)** Now you learned how to handle functions with more than one variable. Suppose $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function. Then f is differentiable at $p \in \mathbb{R}^m$ if there exists a *linear* map $\ell\colon \mathbb{R}^m \to \mathbb{R}^n$ (that is, an $n \times m$ matrix) such that

$$\lim_{\|\xi\| \to 0} \frac{\|f(p+\xi) - f(p) - \ell\xi\|}{\|\xi\|} = 0. \tag{1.1}$$

We denote ℓ by $Df(p)$. It is the matrix of partial derivatives of $f = (f^1, \dots, f^n)$ at the point $p = (u^1, \dots, u^m)$:

$$Df(p) = \begin{pmatrix} \frac{\partial f^1}{\partial u^1}(p) & \cdots & \frac{\partial f^1}{\partial u^m}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^n}{\partial u^1}(p) & \cdots & \frac{\partial f^n}{\partial u^m}(p) \end{pmatrix}$$

Of course, this reduces to the same definition as before if $m = n = 1$, since a 1×1 matrix is just a number, and in this case $Df(p)$ is simply multiplication by the number $f'(p)$. As before, the function f is continuously differentiable if $p \mapsto Df(p)$ is a continuous function (this is now a function $\mathbb{R}^m \to \{n \times m \text{ matrices}\} \cong \mathbb{R}^{mn}$).

(iv) **(First class in topology)** Suppose now X and Y are topological spaces and $f\colon X \to Y$ is a function. You learned that f is continuous if $f^{-1}(U)$ is an open set in X for every open set U in Y. If X and Y are metric spaces then this reduces to the old (ε, δ) definition of continuity. But how does one define differentiability in this setting? Equation (1.1) *does not make sense* any more, since in an arbitrary topological space one cannot simply "add" points, and there is no such thing as a "linear" map!

Here endeth the history lesson.

TL;DR:

- It's easy to differentiate functions on Euclidean spaces.
- Most topological spaces are not Euclideam spaces.
- Bummer.

Indeed, this is a real shame. Measuring the rate at which things change – that is, differentiating them – is absolutely crucial to all applications of mathematics (and is arguably the single most important concept in theoretical physics). However most "real life" systems are *not* defined on open sets in Euclidean spaces (the whole point of your topology course was to introduce classes of spaces appropriate for such models).

This is where differential geometry comes in. Our first aim is to define a special type of topological space, called a **smooth manifold**, on which it *is* possible to make sense of differentiating a continuous function. The definition of a smooth manifold will:

- Include open sets in Euclidean spaces as a special case.
- Be sufficiently general so that the topological spaces that occur in "real life" systems (in theoretical physics, economics, computer science, robotics, genetics, cooking etc) are smooth manifolds.

So let's get started.

In fact, we will define smooth manifolds in two stages. We will first define a **topological manifold**, which is a topological space that locally resembles Euclidean space. We will then endow a topological manifold with an additional piece of data called a **smooth structure**. The smooth structure is what will allow us to actually go ahead and differentiate things. A topological manifold equipped with a smooth structure is then called a **smooth manifold**.

We first recall a few concepts from point-set topology.

Definition 1.1. A topological space X is said to be **metrisable** if there exists a metric on X which induces the given topology.

Thus a metrisable topological space is a topological space which is homeomorphic to a metric space. Non-metrisable topological spaces crop up quite frequently in functional analysis and algebraic topology. In geometry, however, such spaces are abominations, and we will exclude them right from the start.

Exercise: Find examples thereof.

Definition 1.2. A metrisable topological space X is said to be **separable** if there exists a countable dense subset.

It is easy to come up with examples of non-separable metrisable spaces – for example, an uncountable disjoint union of metrisable spaces.

Definition 1.3. A topological space X is said to be **locally Euclidean of dimension** m if every point has a neighbourhood which is homeomorphic to $\mathbb{R}^m$.

We use the convention that a neighbourhood of a point is an *open* set containing that point.

Thus a topological space is locally Euclidean of dimension m if locally it "looks" like the Euclidean space $\mathbb{R}^m$. We are now ready for the first key definition of the course.

Definition 1.4. A **topological manifold of dimension** m is a separable metrisable topological space which is locally Euclidean of dimension m.

As already alluded to, the most important part of the definition of a topological manifold is the locally Euclidean part. Metrisability and separability are included solely to rule out pathologies. In general the phrase "topological manifold" means a topological manifold of some unspecified dimension m.

Convention. We will typically use the letters M, N, and L to denote manifolds. Unless specified otherwise, the dimension of a manifold should be assumed to be the corresponding lowercase letter. Thus M has dimension m, and N has dimension n, and so on.

At the end of this lecture, there is an additional "bonus" section that contains additional background information on the point-set topological properties of manifolds. All of this material is *non-examinable*.

This is a general practice that we will follow throughout the course: most lectures will conclude with additional bonus material, and it will always be non-examinable. There are various reasons for relegating content to the bonus section:

- it is only tangentially related to the course,
- it is rather technical or difficult,
- it is just a sketch,
- it requires more background knowledge (eg. algebraic topology, functional analysis, etc) than the rest of the course assumes.

In any case, you are welcome to ignore the bonus material.

Remark 1.5. Suppose $m \neq n$ are two non-negative integers. Is it possible for a topological space to be locally Euclidean of dimension m *and* locally Euclidean of dimension n? Equivalently, is $\mathbb{R}^m$ homeomorphic to $\mathbb{R}^n$ for $m \neq n$? The answer to this is "no", but this is surprisingly difficult to prove. This result is called the *Invariance of Domain Theorem*, and was first proved by Brouwer in 1912. The easiest proof uses tools from algebraic topology.

Examples 1.6. Here are some examples.

(i) $\mathbb{R}^m$ is trivially a topological manifold of dimension m. More generally, any m-dimensional vector space is a topological manifold of dimension m.

We use the convention that all vector spaces are real and finite dimensional, unless specified otherwise.

(ii) The *open* ball

$$B^m := \{p \in \mathbb{R}^m \mid \|p\| < 1\}$$

is a topological manifold of dimension m. More generally, every non-empty open subset of a topological manifold of dimension m is also a topological manifold of dimension m.

(iii) A **non-example**: The *closed* unit ball

$$D^m := \{p \in \mathbb{R}^m \mid \|p\| \leq 1\}$$

is *not* a topological manifold of dimension m. In fact, D^m is an example of a more general concept of a **manifold with boundary** that we will come back to later in Lecture 21.

It is an illustrative exercise to try and work out why.

We will see more interesting examples later in this lecture.

Let us now get back to the point of view discussed at the beginning of the lecture: we are trying to develop a class of topological spaces for which it is possible to differentiate functions on. One might naively believe that the locally Euclidean condition built into the definition of a topological manifold is enough. Indeed, to check whether a function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at a point $p \in \mathbb{R}^m$, we need only examine f in a small neighbourhood of p – this is clear from (1.1). Thus if we are given a continuous map between two topological manifolds, we can locally view it as a continuous map between two Euclidean spaces, and thus we could conceivably say our original map is differentiable if this local map is. But herein lies a problem: a topological manifold is only locally *homeomorphic* to Euclidean space, and a different choice of homeomorphism might affect whether the local map is differentiable or not.

The solution to this is to introduce more structure. Before doing so, let us recall the chain rule for continuously differentiable functions between Euclidean spaces. We will give two different versions: one for the total differential $Df(p)$ (the matrix) and one for the partial derivatives $\frac{\partial f^i}{\partial u^j}$.

Proposition 1.7 (The Chain Rule). *Let $\mathcal{O} \subset \mathbb{R}^m$, $\Omega \subset \mathbb{R}^n$ be open sets. Let $f\colon \mathcal{O} \to \mathbb{R}^n$ and $g\colon \Omega \to \mathbb{R}^l$ be continuously differentiable functions satisfying $f(\mathcal{O}) \subset \Omega$.*

(i) *The function $g \circ f$ is also continuously differentiable, and its derivative at the point p is given by*

$$D(g \circ f)(p) = Dg(f(p)) \circ Df(p).$$

(ii) *Write $(u^1, \dots, u^m)$ for the coordinates on $\mathbb{R}^m$ and $(v^1, \dots, v^n)$ for the coordinates on $\mathbb{R}^n$, and write $f = (f^1, \dots, f^n)$ and $g =$*

$(g^1, \dots, g^l)$. *Then the partial derivatives of* $g \circ f$ *are given by*

$$\frac{\partial (g^i \circ f)}{\partial u^j}(p) = \sum_{k=1}^{n} \frac{\partial g^i}{\partial v^k}(f(p)) \frac{\partial f^k}{\partial u^j}(p), \qquad \text{for all } 1 \le i \le l, 1 \le j \le m.$$

We now define higher order derivatives.

Definition 1.8. Let $\mathcal{O} \subset \mathbb{R}^m$ and $\Omega \subset \mathbb{R}^m$ be open sets and suppose $f\colon \mathcal{O} \to \Omega$ is a differentiable map. We say that f is of class C^k if each partial derivative $\frac{\partial f^i}{\partial u^j}$ is an $(k-1)$-times continuously differentiable function. We say that f is **smooth** or of class C^∞ if f is of class C^k for every $k \ge 1$. If f is both smooth and bijective and the inverse function is also smooth then we say that f is a **diffeomorphism**.

It follows from part (ii) of Proposition 1.7 that the composition of smooth functions defined on open sets in Euclidean spaces is again a smooth function.

Remark 1.9. If f is a diffeomorphism then necessarily $m = n$. This follows immediately from part (i) of Proposition 1.7, which tells us that if f is a diffeomorphism then $Df(p)$ is an invertible matrix. (Its inverse is given by $D(f^{-1})(f(p))$.) An $n \times m$ matrix can only be invertible if $m = n$. Thus in particular $\mathbb{R}^m$ cannot be *diffeomorphic* to $\mathbb{R}^n$ for $m \ne n$ (compare to Remark 1.5).

With these preliminaries in hand, let us get started on the definition of a smooth manifold.

Definition 1.10. Let M be a topological manifold of dimension m. A **smooth atlas** on M is a collection

$$\mathcal{X} = \{x_a\colon U_a \to \mathcal{O}_a \mid a \in A\}$$

where $\{U_a \mid a \in A\}$ is an open cover of M, each $\mathcal{O}_a$ is an open set in $\mathbb{R}^m$, and each $x_a\colon U_a \to \mathcal{O}_a$ is a homeomorphism such that the following *compatibility condition* is satisfied: Suppose $a, b \in A$ are such that $U_a \cap U_b \ne \emptyset$. Then the composition

$$x_b \circ x_a^{-1}\colon x_a(U_a \cap U_b) \to x_b(U_a \cap U_b)$$

should be a diffeomorphism. This makes sense, since both $x_a(U_a \cap U_b)$ and $x_b(U_a \cap U_b)$ are open subsets of $\mathbb{R}^m$. We call the maps x_a the **charts** of the atlas $\mathcal{X}$, and the compositions $x_b \circ x_a^{-1}$ the **transition functions** of the atlas.

Convention. We typically denote points in manifolds by the letters p and q, and charts on manifolds by the letters x and y. The phrase "let (U, x) be a chart about p" is short for: let $x\colon U \to \mathcal{O}$ be a chart on M with $p \in U$.

We say that two smooth atlases $\mathcal{X}$ and $\mathcal{Y}$ are **equivalent** if their union is also a smooth atlas, that is, if given any chart x of $\mathcal{X}$ and any chart y of $\mathcal{Y}$ such that the domains of x and y intersect, the composition $y \circ x^{-1}$ is also a diffeomorphism. It is immediate that this notion defines an equivalence relation on the set of smooth atlases on a given topological manifold.

Definition 1.11. A **smooth structure** on a topological manifold is an equivalence class of smooth atlases.

Remark 1.12. Given an equivalence class of smooth atlases, there is a unique *maximal* smooth atlas in that class (simply take the union of all the atlases in the given equivalence class). Thus there is a one-to-one correspondence between smooth structures and maximal smooth atlases. Since dealing with equivalence relations can be tedious, it is usually more convenient to regard a smooth structure as a maximal smooth atlas, and we will do so without further comment.

We now finally arrive at the main definition of this first lecture.

Definition 1.13. A **smooth manifold of dimension** m is a pair $(M, \mathfrak{X})$ where M is a topological manifold of dimension m and $\mathfrak{X}$ is a smooth structure on M.

Since a smooth atlas is contained in a unique maximal smooth atlas, it is sufficient when defining a smooth manifold to specify a smooth atlas on the underlying topological manifold. Whenever possible we will omit the $\mathfrak{X}$ from the notation and just write M. For smooth manifolds the fact that the dimension is well-defined is much easier than for topological manifolds (we only need Remark 1.9, which does not require any algebraic topology).

Example 1.14. The standard smooth structure on $\mathbb{R}^m$ is the one containing the smooth atlas consisting of exactly one chart: the identity map $\mathrm{id}\colon \mathbb{R}^m \to \mathbb{R}^m$. The reason for the word "standard" will become clear by the end of the lecture. More generally, if V is any m-dimensional real vector space, then the standard smooth structure on V is the one induced by the smooth atlas consisting of a single chart $\ell\colon V \to \mathbb{R}^m$, where ℓ is some linear isomorphism.

Exercise: Why is this independent of the choice of ℓ?

Just as with topological manifolds, an open subset of a smooth manifold is also a smooth manifold:

Lemma 1.15. *Let M be a smooth manifold of dimension m and let $W \subset M$ be a non-empty open set. Then W naturally inherits the structure of a smooth manifold of dimension m.*

Proof. We have already remarked in part (i) of Example 1.6 that W is a topological manifold of dimension m. Let $\mathfrak{X} = \{x_a\colon U_a \to \mathcal{O}_a \mid a \in A\}$ be a smooth atlas on M. Then

$$\{x_a|_{W\cap U_a}\colon W \cap U_a \to x_a(W \cap U_a) \subset \mathcal{O}_a \mid a \in A\}$$

is a smooth atlas for W. ∎

Thus any open subset of a vector space is a smooth manifold. Let us now consider a slightly less trivial example. Recall we denote by S^m the unit sphere:

$$S^m := \left\{p \in \mathbb{R}^{m+1} \mid \|p\| = 1\right\}.$$

Proposition 1.16. *The sphere S^m is a compact smooth manifold of dimension m.*

See Definition 1.28 for the definition of compact.

Proof. We give S^m the subspace topology from $\mathbb{R}^{m+1}$. Then S^m is certainly a separable and metrisable. We will directly exhibit a smooth atlas on S^m (thus proving at the same time that S^m is a topological manifold). Let $p_N = (0,\dots,0,1)$ denote the "north pole" and let $p_S := (0,\dots,,0,-1)$ denote the "south pole". Let $U_N = S^m \setminus \{p_N\}$ and $U_S := S^m \setminus \{p_S\}$. Then $\{U_N, U_S\}$ is an open cover of $S^{\cdot}$. Define charts

$$x_N \colon U_N \to \mathbb{R}^m, \qquad x_N(u^1,\dots,u^{m+1}) := \frac{1}{1-u^{m+1}}(u^1,\dots,u^m)$$

and

$$x_S \colon U_S \to \mathbb{R}^m, \qquad x_S(u^1,\dots,u^{m+1}) := \frac{1}{1+u^{m+1}}(u^1,\dots,u^m).$$

The maps x_N and x_S are stereographic projection from the north and south pole respectively. Both the transition maps

$$\begin{aligned} x_N \circ x_S^{-1} &\colon \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}, \\ x_S \circ x_N^{-1} &\colon \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\} \end{aligned}$$

are given by

$$(u^1,\dots,u^m) \mapsto \frac{1}{\sum_{i=1}^m (u^i)^2}(u^1,\dots,u^m)$$

which is obviously a diffeomorphism. Thus we have defined a smooth atlas on S^m. We refer to this smooth structure as the standard smooth structure on S^m. ∎

All we really needed to do in the previous proof was check differentiability of the transition function $x_N \circ x_S^{-1}$. This is because (as a subset of $\mathbb{R}^{m+1}$), S^m already carried a nice topology. Sometimes however we will want to build a smooth manifold "from scratch". For this, the next result is very useful.

Proposition 1.17 (Constructing smooth manifolds). *Let M be a set. Suppose we are given a collection $\{U_a \mid a \in A\}$ of subsets of M together with bijections $x_a \colon U_a \to \mathcal{O}_a$, where $\mathcal{O}_a$ is an open subset of $\mathbb{R}^m$. Assume in addition that:*

(i) *For any $a, b \in A$, $x_a(U_a \cap U_b)$ is open in $\mathbb{R}^m$.*

(ii) *If $U_a \cap U_b \neq \emptyset$, the map $x_b \circ x_a^{-1} \colon x_a(U_a \cap U_b) \to x_b(U_a \cap U_b)$ is a diffeomorphism.*

(iii) *Countably many of the U_a cover M.*

(iv) *If $p \neq q$ are points in M then either there exists a such that p and q both belong to U_a, or there exists a, b with $U_a \cap U_b = \emptyset$ such that $p \in U_a$ and $q \in U_b$.*

Then M has a unique smooth manifold structure for which the collection $\{x_a \colon U_a \to \mathcal{O}_a \mid a \in A\}$ is a smooth atlas.

The proof is essentially trivial: we simply took the definition of a smooth manifold and inserted it into the hypotheses.

Proof. Define a topology on M by declaring all the x_a to be homeomorphisms. That this is well-defined topology follows from the fact that the x_a are bijections, together with (i) and (ii). The locally Euclidean property is then immediate. Properties (iii) and (iv) guarantee this topology is metrisable and separable, thus turning M into a topological manifold. Finally the fact that $\{x_a \colon U_a \to \mathcal{O}_a \mid a \in A\}$ is a smooth atlas on M is clear from (ii). ∎

REMARK 1.18. Historically, a manifold M (smooth or topological) was called *open* if M was non-compact and *closed* if M was compact. This however is bad terminology for two reasons:

(i) Thought of as an abstract topological space, every manifold is both open and closed! (This is true of any topological space.)

(ii) If however our given manifold M is a subspace of a larger space N, then it does make sense to ask whether M is open or closed in the subspace topology of N. For example, the unit ball B^m is open in $\mathbb{R}^m$ and the unit sphere S^m is closed in $\mathbb{R}^{m+1}$. Historically, all manifolds were thought of as subspaces – actually *submanifolds* – of some Euclidean space $\mathbb{R}^m$, and in fact any manifold can be embedded inside Euclidean space. However even then the terminology "open" and "closed" does not make sense! For instance, if we identify $\mathbb{R}^2$ with the set of points in $\mathbb{R}^3$ whose last coordinate is zero then $\mathbb{R}^2$ is closed as a subspace of $\mathbb{R}^3$, but $\mathbb{R}^2$ is not compact as a manifold.

We will define submanifolds precisely in Lecture 5.

In the smooth case, this is known as the *Whitney Embedding Theorem*, which we will prove in Lecture 7.

Thus throughout this course, we will only use the words "open" and "closed" in their topological context (i.e. to speak of open sets and closed sets). If we wish to indicate a given manifold is compact, we will use the rather more logical terminology "compact manifold".

The only caveat to this is that when we define (both smooth and topological) manifolds with boundary later on (Lecture 21), we will need to differentiate between the terms "compact manifold with boundary" and "compact manifold without boundary". Indeed, as we have already mentioned, the closed unit ball D^m is an example of a compact smooth manifold with boundary.

On Problem Sheet A there are many more examples (and non-examples) of smooth manifolds for you to play with. Going back to the general theory, we have now achieved the goal we set out at the beginning of the lecture: to come up with an appropriate class of topological spaces for which it makes sense to say whether a map is differentiable or not.

DEFINITION 1.19. Let $\varphi \colon M \to N$ be a continuous map between two smooth manifolds. We say that φ is of class C^k if for every point $p \in M$, if (U, x) is any chart on M with $p \in U$ and (V, y) is any chart on N with $\varphi(U) \subset V$, the composition

$$y \circ \varphi \circ x^{-1} \colon x(U) \to y(V)$$

is of class C^k. If φ is of class C^k for all k then we say φ is **smooth** (or of class C^∞). If φ is smooth and bijective and the inverse function $N \to M$ is also smooth then φ is said to be a **diffeomorphism**.

It follows from the definition of smooth atlases that it does not matter which charts we use to check differentiability (i.e. we could replace "any chart" with "every chart" above).

See Problem A.1.

EXAMPLES 1.20.

(i) If $(M, \mathcal{X})$ is a smooth manifold and $x \colon U \to \mathcal{O}$ belongs to $\mathcal{X}$, then if we think of U and O as smooth manifolds in their own right (using Lemma 1.15 and Example 1.14) then x is a diffeomorphism.

(ii) Similarly if $W \subset M$ is any open set (endowed with the smooth structure from Lemma 1.15) then the inclusion map $\imath \colon W \hookrightarrow M$ is a smooth map.

The next result also follows immediately from the chain rule in Euclidean spaces (Proposition 1.7).

PROPOSITION 1.21. *Let M, N and L be smooth manifolds, and suppose $\varphi \colon M \to N$ and $\psi \colon N \to L$ are smooth maps. Then $\psi \circ \varphi \colon M \to L$ is smooth.*

Proof. Let $p \in M$. Let (U, x) be a chart on M containing p, let (V, y) be a chart on N containing $\varphi(p)$, and let (W, z) be a chart on L containing $\psi(\varphi(p))$. We want to show that the composition $z \circ (\psi \circ \varphi) \circ x^{-1}$ is smooth where defined. But

$$z \circ (\psi \circ \varphi) \circ x^{-1} = \left(z \circ \psi \circ y^{-1}\right) \circ \left(y \circ \varphi \circ x^{-1}\right),$$

and by assumption each of the two bracketed terms on the right-hand side is a smooth map. Since the composition of smooth maps (defined on open sets in Euclidean space) is smooth, the left-hand side is also smooth. ∎

REMARK 1.22. Consider the following curiosity. We have defined what it means for a continuous map between two smooth manifolds to be differentiable (Definition 1.19), but we have not defined what the derivative $D\varphi(p)$ is yet! This is somehow backwards – in normal calculus one first defines the derivative $Df(p)$ and then says the map is differentiable if the derivative $Df(p)$ always exists. In fact, the definition of the derivative of a map between two smooth manifolds is a little tricky, and this is what we will do in the next three lectures.

A smooth structure is defined as an equivalence class of smooth atlases. We can take this one step further and look at equivalence classes of smooth structures.

DEFINITION 1.23. We say that two smooth structures $\mathcal{X}_1$ and $\mathcal{X}_2$ on a given topological manifold M belong to the same **diffeomorphism class** if there exists a diffeomorphism $(M, \mathcal{X}_1) \to (M, \mathcal{X}_2)$. This is clearly another equivalence relation. We write $\mathcal{S}(M)$ for the set of diffeomorphic classes of smooth structures on M.

Example 1.24. As an example to show that smooth structures and diffeomorphism classes really are different concepts, take $M = \mathbb{R}$. Let $\mathcal{X}$ denote the maximal smooth atlas containing the chart $t \mapsto t^3$. On Problem Sheet A you will check that this is *not* the same smooth structure as the standard one described in Example 1.14. However, there is an obvious diffeomorphism between the two smooth structures (namely, $t \mapsto t^3$). Thus they belong to the same diffeomorphism class.

Remark 1.25. Does every topological manifold admit a smooth structure (i.e. can every topological manifold be turned into a smooth manifold)? Can a topological manifold admit more than one diffeomorphism class? These questions are typically very hard to solve (and there are many open problems). Here are some interesting facts, all of which are way too hard to prove in this course.

(i) If M is a topological manifold of dimension 0,1,2 or 3 then $\mathcal{S}(M)$ consists of exactly one element.

(ii) In higher dimensions, there may be more than one diffeomorphism class. For example, $\mathcal{S}(S^7)$ has exactly 28 elements (15 if one ignores orientations), and there are more than sixteen million different elements in $\mathcal{S}(S^{31})$! On the other hand. $\mathcal{S}(S^{61})$ consists of exactly one element, but for any odd number $m \geq 63$, one has $\#\mathcal{S}(S^m) \geq 2$.

(iii) For any $m \neq 4$, $\mathbb{R}^m$ admits a unique diffeomorphism class. However $\mathcal{S}(\mathbb{R}^4)$ has infinitely many elements. In general the most "wild" phenomena occur in dimension 4.

(iv) There exist topological manifolds that do not admit *any* smooth structures at all: $\mathcal{S}(M) = \emptyset$.

Bonus Material for Lecture 1

Defining topological manifolds as separable metrisable spaces that are locally Euclidean has the advantage of being concise, but in practice it can be hard to check. In this bonus section we recall some additional material from point-set topology, and explore alternative ways to define topological manifolds.

Definition 1.26. Let X be a topological space. We say that X is **Hausdorff** if for every pair $p \neq q$ of points in X, there are open subsets $U, V \subset X$ such that $p \in U$, $q \in V$ and $U \cap V = \emptyset$.

Any metrisable space is Hausdorff.

Definition 1.27. A topological space X is said to be **connected** if it is not the disjoint union of nonempty open sets. A topological space X is said to be **path connected** if for any two points $p, q \in X$ there exists a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

A path connected space is connected, but the converse need not hold.

In general any topological space can be decomposed into its **connected components** (resp. **path components**), where the connected component (resp. path component) containing a given point x is the union of all the connected (resp. path connected) sets containing x.

Recall that an **open cover** of a topological space X is a collection $\{U_a \mid a \in A\}$ of open subsets of X, where A is some index set, such that $X = \bigcup_{a \in A} U_a$. If the index set A is a finite set, we say that the open cover is a **finite cover**. A **subcover** of an open cover $\{U_a \mid a \in A\}$ consists of a subset $A' \subset A$ such that the collection $\{U_a \mid a \in A'\}$ is still an open cover.

DEFINITION 1.28. Let X be a topological space. We say that X is **compact** if every open cover has a finite subcover.

Compact spaces are typically the most "useful" class of topological spaces, in the sense that many powerful theorems only hold for compact spaces. Unfortunately, since manifolds include Euclidean spaces as a special case, they are certainly not always compact.

A subset $K \subset \mathbb{R}^m$ is compact if and only if it is closed and bounded – this is the Heine-Borel theorem.

We therefore introduce a weaker condition, which requires two more preliminary definitions about covers. Suppose $\{U_a \mid a \in A\}$ is an open cover. A **refinement** is another open cover $\{V_b \mid b \in B\}$ with the property that for every $b \in B$ there exists $a \in A$ such that $V_b \subset U_a$. Next, an open cover $\{U_a \mid a \in A\}$ of X is said to be **locally finite** if for every $x \in X$ there exists a neighbourhood W of x such that the set $\{a \in A \mid U_a \cap W \neq \emptyset\}$ is a finite set.

DEFINITION 1.29. A topological space X is said to be **paracompact** if every open cover has a locally finite refinement.

Thus compact spaces are obviously paracompact, but the latter is more general. For instance, $\mathbb{R}^m$ is paracompact, but as we have just observed, not compact. In fact, the following result holds.

THEOREM 1.30. *Every metrisable space is paracompact.*

If X is a topological space then a **basis** for the topology on X is a set $\mathcal{B}$ of open sets of X with the property that every open set in X is a union of sets in $\mathcal{B}$.

DEFINITION 1.31. A topological space is said to be **second countable** if it admits a countable basis.

A metrisable space is second countable if and only if it is separable (consider balls of rational radii).

The following proposition gives two alternative characterisations of topological manifolds, which often are easier to verify.

PROPOSITION 1.32. *Let M be a locally Euclidean topological space. The following are equivalent:*

(i) *M is a topological manifold.*

(ii) *M is Hausdorff, paracompact, and has at most countably many connected components.*

(iii) *M is Hausdorff and second countable.*

Here are some more point-set topology definitions.

Definition 1.33. A topological space is said to be **Lindelöf** if every open cover has a *countable* subcover.

Clearly compact ⇒ Lindelöf.

Any locally compact paracompact space with at most countably many components is Lindelöf.

Definition 1.34. A topological space X is **normal** if given any two closed disjoint subsets K_1, K_2 of X there are open sets U_1, U_2 of X such that $K_i \subset U_i$ for $i = 1, 2$ and $U_1 \cap U_2 = \emptyset$.

If $\{x\}$ is closed for all x in X then normal ⇒ Hausdorff.

Every paracompact Hausdorff space is normal.

Definition 1.35. A topological space X is said to be **locally compact** if for every point $p \in X$ there exists a compact set K and a neighbourhood U of x such that $U \subset K$.

Clearly compact ⇒ locally compact.

If the topological space is Hausdorff, this is equivalent to asking that every point has a neighbourhood with compact closure.

Definition 1.36. A topological space X is **locally path connected** if for every point $p \in X$ and every neighbourhood U of p, there exists a path connected neighbourhood V of p with $V \subset U$.

For a locally path connected space, the path components and the connected components coincide.

Topological manifolds enjoy all of these properties.

Proposition 1.37. *Topological manifolds are normal Lindelöf spaces which are both locally compact and locally path connected. Moreover a topological manifold is connected if and only if it is path connected.*

We conclude this lecture with another somewhat esoteric remark about infinite-dimensional manifolds. This is for *interest only* – we will not use infinite-dimensional manifolds in this course.

Definition 1.38. Fix a Banach space E. We say that a topological space X is **locally modelled on** E if every point in X has a neighbourhood which is homeomorphic to an open set in E.

Definition 1.39. A **topological Banach manifold** is a separable metrisable topological space which is locally modelled on some Banach space E.

Since any Euclidean space is a Banach space, any topological manifold is also a topological Banach manifold.

A smooth Banach manifold is defined similarly – here we use the fact that differentiating functions on Banach spaces works in exactly the same way as differentiating functions on Euclidean spaces.

You should compare this to how you initially learned linear algebra. To begin with all vector spaces were finite-dimensional and linear operators were just matrices. Then two years later they told you that actually things could be infinite-dimensional. All the theorems you knew and loved from linear algebra continued to hold (provided a few more assumptions were made), only the proofs were much harder and it was no longer called "linear algebra", it was called "functional analysis". The same is true in differential geometry – infinite-dimensional differential geometry is sometimes referred to as "global analysis".

EXAMPLE 1.40. As a concrete example of an infinite-dimensional manifold, let M and N be two finite-dimensional smooth manifolds, and let $0 \leq k < \infty$. Then the space $C^k(M, N)$ of maps from M to N of class C^k is an infinite-dimensional Banach manifold.

A constant thorn in the side of global analysists is the fact that the space $C^\infty(M, N)$ of smooth maps from M to N is *not* a Banach manifold.

This is because $C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ is not a Banach space.

LECTURE 2

Tangent Spaces

The goal of the next few lectures is to associate to an m-dimensional smooth manifold an m-dimensional vector space, denoted by T_pM, to each point $p \in M$. We call T_pM the **tangent space** to M at p. Although it *won't* be immediate from the definition why, the tangent space is what you would naturally "guess" it would be. See Figure 2.1 for the case of S^2 (which should be thought of as sitting inside $\mathbb{R}^3$).

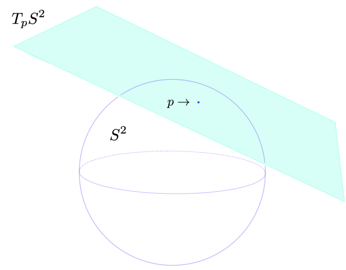


Figure 2.1: The tangent space to S^2 at a point p

We will use this construction to define the *derivative* of a smooth map $\varphi\colon M \to N$: this will be a *linear* map $D\varphi(p)\colon T_pM \to T_{\varphi(p)}N$ for each $p \in M$. In Lecture 5 we will "glue" the vectors spaces together to form one larger space called the **tangent bundle** of M. This will be smooth manifold of twice the dimension of M. A smooth map $\varphi\colon M \to N$ will then induce a smooth map $D\varphi\colon TM \to TN$. In Lecture 6 we will look at submanifolds – it will not be until then that we can rigorously prove that the tangent space we define in this lecture really is the actual "tangent space" as in Figure 2.1 (cf. Example 6.16).

Definition 2.1. A **smooth function** on a manifold is a smooth map $f\colon M \to \mathbb{R}$ in the sense of Definition 1.19, where $\mathbb{R}$ is given the standard smooth structure from Example 1.14. Thus f is a smooth function if for any chart $x\colon U \to \mathcal{O}$ on M, the composition $f \circ x^{-1}\colon \mathcal{O} \to \mathbb{R}$ is a smooth function (in the normal sense).

Convention. We typically use the symbols φ, ψ to denote smooth maps from one manifold to another, and f, g for smooth maps from a manifold to a Euclidean space.

We denote by $C^\infty(M)$ the space of smooth functions. If $W \subset M$ is an open set, we define $C^\infty(W)$ to be the space of smooth functions that are only defined on W (where W is thought of as a smooth manifold in the sense of Lemma 1.15). The space $C^\infty(M)$ is an **algebra** (and thus in particular a ring and a vector space), under the operations

That is, a vector space where you can also multiply two elements together.

$$(f+g)(p) := f(p) + g(p), \qquad (fg)(p) := f(p)g(p),$$

and $(cf)(p) := cf(p)$ for $c \in \mathbb{R}$.

We will always write $\circ$ to denote composition, meanwhile juxtaposition indicates the pointwise product. See Definition 19.18 if you are unfamiliar with algebras.

Before going any further, let us go back to $\mathbb{R}^m$ and introduce some more notation. To begin with, this will feel somewhat redundant, but we will see next lecture that it makes the various formulae easier to understand. Slightly abusively, we denote by $u^i\colon \mathbb{R}^m \to \mathbb{R}$ the function

$$(u^1, \dots, u^m) \mapsto u^i. \tag{2.1}$$

Let e_i denote the ith standard basis vector in $\mathbb{R}^m$, so that

$$u^i(e_j) = \delta^i_j, \tag{2.2}$$

where δ_i^j is the *Kronecker delta* defined by

$$\delta_i^j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Now suppose $f \colon \mathcal{O} \subset \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map defined on an open subset $\mathcal{O}$ of $\mathbb{R}^m$. If $p \in \mathcal{O}$ and $\xi \in \mathbb{R}^m$ then the vector $Df(p)\xi$ can be thought of as the partial derivative of f in the direction ξ:

$$Df(p)\xi = \lim_{t \to 0} \frac{f(p + t\xi) - f(p)}{t}.$$

Definition 2.2. We abbreviate $Df(p)e_j$ by $D_j f(p)$:

$$D_j f(p) = Df(p)e_j = \lim_{t \to 0} \frac{f(p + te_j) - f(p)}{t}.$$

Let us summarise the various different ways we can write the derivative:

Let $f \colon \mathcal{O} \subset \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map, and let $p \in \mathcal{O}$. Then:

- $Df(p)$ is a $n \times m$ matrix.
- $D_j f(p)$ is an element of $\mathbb{R}^n$. It is the jth column of the matrix $Df(p)$.
- $D(u^i \circ f)(p)$ is a linear map from $\mathbb{R}^m$ to $\mathbb{R}$. One can think of it as the ith row of the matrix $Df(p)$.
- $D_j(u^i \circ f)(p)$ is a number. It is the (i,j)th entry of the matrix $Df(p)$.

In more familiar notation

$$D_j(u^i \circ f)(p) = \frac{\partial f^i}{\partial u^j}(p). \tag{2.3}$$

In general we will prefer the slightly more cumbersome expression on the left-hand side of (2.3). This is because next lecture the symbol $\frac{\partial}{\partial x^i}$ will take on a special meaning, cf. Example 3.6.

Remark 2.3. In our new notation, part (ii) of the chain rule in Euclidean spaces (Proposition 1.7) reads:

$$D_j(u^i \circ g \circ f)(p) = \sum_{k=1}^{n} D_k(u^i \circ g)(f(p)) D_j(u^k \circ f)(p).$$

Going back to manifolds, we can use the (u^i) to give examples of smooth functions.

Example 2.4. If $x \colon U \to \mathcal{O}$ is a chart on M, for each $i = 1, \dots, m$ the function $u^i \circ x$ is a smooth function on U.

This type of smooth function is especially important, so it gets its own special name.

Definition 2.5. If $p \in M$ and x is a chart defined on a neighbourhood of p then we write

$$x^i := u^i \circ x.$$

We call the functions (x^i) the **coordinates** of the chart x, and we say that the (x^i) are **local coordinates about** p.

We use the convention that the local coordinates of a chart are always written with the same letter: thus if y is another chart then $y^i := u^i \circ y$. Since the local coordinates uniquely determine the chart, this convention also works backwards. Thus the phrase "let (z^i) be local coordinates about p" is shorthand for: let z be a chart on M containing p, and set $z^i := u^i \circ z$.

Remark 2.6. Consider $\mathbb{R}^m$ as a smooth manifold with the single chart $\mathrm{id}\colon \mathbb{R}^m \to \mathbb{R}^m$ (cf. Example 1.14). Then the local coordinates of id are simply the (u^i).

> Let us say a few words on the philosophy behind the notation, which may help you in lectures to come.
>
> Differential Geometry is essentially a way of formalising calculus so that it makes sense on smooth manifolds. The formalism is designed to make things "look" as similar as possible to calculus on Euclidean space. This means that if you are ever stuck when trying to compute something (for instance, a derivative), you can just "pretend" that everything is actually defined on Euclidean space, and then simply follow the normal rules of multivariable calculus. Magically, it just works!

Definition 2.7. Let M be a smooth manifold and let $p \in M$. Let U and V be two neighbourhoods of p, and suppose $f \in C^\infty(U)$ and $g \in C^\infty(V)$. We say that f and g have the same **germ** at p if there exists a smaller neighbourhood $W \subset U \cap V$ of p such that

$$f|_W \equiv g|_W.$$

One can think of this as follows: define an equivalence relation on the set of smooth functions defined on a neighbourhood of p by saying that $(U, f) \sim (V, g)$ if there exists a neighbourhood $W \subset U \cap V$ such that $f|_W \equiv g|_W$. A germ is then an equivalence class under this relation. We denote the germ by $\underline{f}$ and we let $\mathcal{F}_p M$ denote the set of germs at p.

In fact, $\mathcal{F}_p M$ is another algebra. We can add germs together: if $\underline{f}$ and $\underline{g}$ are two germs with representatives (U, f) and (V, g) respectively, then $\underline{f} + \underline{g}$ is the germ represented by $(U \cap V, f + g)$. Similarly $\underline{f}\,\underline{g}$ is the germ represented by $(U \cap V, fg)$, and for a real number c, $c\underline{f}$ is the germ represented by (U, cf). We denote by $\underline{c}$ the germ of any function which is constant and equal to c in a neighbourhood of p. The map $\mathbb{R} \to \mathcal{F}_p M$ given by $c \mapsto \underline{c}$ is then an inclusion of algebras.

A germ at p has a well-defined value at p (although nowhere else), and this gives us map

In fact, $\mathcal{F}_pM$ is a local ring; see Lemma 2.15.

$$\mathrm{eval}_p\colon \mathcal{F}_pM \to \mathbb{R}, \qquad \mathrm{eval}_p(\underline{f}) := f(p), \tag{2.4}$$

where (U, f) is any representative of $\underline{f}$. The next example motivates the approach we will take to tangent vectors.

EXAMPLE 2.8. Let $f\colon \mathcal{O} \to \mathbb{R}$ be smooth, where $\mathcal{O} \subset \mathbb{R}^m$ is open. Let $p \in \mathcal{O}$ and $\xi \in \mathbb{R}^m$. The usual interpretation of the derivative is that the matrix $Df(p)$ eats the vector ξ to produce a real number $Df(p)\xi$. However we could flip this on its head and think of (p, ξ) as being fixed, and instead let f vary. To this end, let us denote by

$$\xi_p\colon C^\infty(\mathcal{O}) \to \mathbb{R}, \qquad \xi_p(f) := Df(p)\xi.$$

It follows from equation (1.1) that differentiability is a local property, in the sense that the value of $\xi_p(f)$ depends only on the germ of f at p. Thus we can think of ξ_p as defining a *linear* map

The formal definition of a "local property" will come in Lecture 16.

$$\xi_p\colon \mathcal{F}_p\mathcal{O} \to \mathbb{R}, \qquad \xi_p(\underline{f}) := Df(p)\xi$$

(here we are thinking of $\mathcal{O}$ as a smooth manifold). In fact, the map $\xi_p\colon \mathcal{F}_p\mathcal{O} \to \mathbb{R}$ is not just any linear map, it is also a *derivation* in the sense that

$$\xi_p(\underline{f}\,\underline{g}) = \mathrm{eval}_p(\underline{f})\xi_p(\underline{g}) + \mathrm{eval}_p(\underline{g})\xi_p(\underline{f}).$$

Indeed, this is just a fancy way of expressing the *Leibniz rule*:

$$D(fg)(p)\xi = f(p)Dg(p)\xi + g(p)Df(p)\xi.$$

Following Example 2.8, we define a tangent vector as a derivation on the space of germs.

DEFINITION 2.9. Let M be a smooth manifold and let $p \in M$. A **tangent vector at** p is a linear map

$$\xi\colon \mathcal{F}_pM \to \mathbb{R}$$

which satisfies the *derivation property*:

$$\xi(\underline{f}\,\underline{g}) = \mathrm{eval}_p(\underline{f})\xi(\underline{g}) + \mathrm{eval}_p(\underline{g})\xi(\underline{f}).$$

CONVENTION. We typically denote tangent vectors with the symbols ξ, ζ.

Since a tangent vector is a linear map from the vector space $\mathcal{F}_pM$ to $\mathbb{R}$, the set of tangent vectors is itself a vector space, and we denote it by T_pM.

The next observation follows directly from the definition:

LEMMA 2.10. *Let M be a smooth manifold and let $W \subset M$ be a non-empty open set. Regard W as a smooth manifold in its own right, as in Lemma 1.15. Then for any $p \in W$ there is a canonical identification $T_pM = T_pW$.*

Proof. Since W is open it there is a canonical isomorphism $\mathcal{F}_pM \cong \mathcal{F}_pW$. ■

Here is an easy lemma about derviations.

LEMMA 2.11. *Suppose* $\xi\colon \mathcal{F}_pM \to \mathbb{R}$ *is a tangent vector at* p *and* $\underline{c} \in \mathcal{F}_pM$ *is a constant germ. Then* $\xi(\underline{c}) = 0$.

Proof. Since $\underline{c} = c\,\underline{1}$ we have $\xi(\underline{c}) = c\,\xi(\underline{1})$ by linearity. But by the derivation property:

$$\xi(\underline{1}) = \xi(\underline{1}\,\underline{1}) = 2\ \mathrm{eval}_p(\underline{1})\xi(\underline{1}) = 2\,\xi(\underline{1})$$

and thus $\xi(\underline{1}) = 0$. Thus also $\xi(\underline{c}) = 0$. ■

In the special case where $\mathcal{O} \subset \mathbb{R}^m$ is an open set, Example 2.8 showed that every vector $\xi \in \mathbb{R}^m$ defines an element of $\xi_p \in T_p\mathcal{O}$ (in the sense of Definition 2.9). In fact, these are *all* the elements of $T_p\mathcal{O}$, although this requires a bit of work to see. More generally, one has:

THEOREM 2.12. *Let* M *be a smooth manifold of dimension* m *and let* $p \in M$*. Then the vector space* T_pM *has dimension* m*.*

Theorem 2.12 is not immediate. Indeed, from Definition 2.9 it is not remotely clear why T_pM should even be finite-dimensional! We will prove Theorem 2.12 in the next lecture by explicitly finding a basis of T_pM.

Bonus Material for Lecture 2

In this bonus section we explore two further properties of the algebras $C^\infty(M)$ and $\mathcal{F}_pM$.

LEMMA 2.13. *Let* M *be a manifold of dimension* $m > 0$*, and let* $W \subset M$ *be a non-empty open set. Then as a real vector space,* $C^\infty(W)$ *is always infinite-dimensional.*

Proof. Let $f \in C^\infty(W)$ be any smooth function which is not constant on some connected component of W. Then $f(W)$ is an infinite subset of $\mathbb{R}$ (since it contains an interval).

If you are worried why such a function exists, use Lemma 3.2 from the next lecture.

Consider now the vector space $\mathbb{R}[t]$ of all polynomials. This is an infinite-dimensional vector space – a basis is the set of monomials $\{t^k \mid k \geq 0\}$. Any polynomial $P(t)$ is completely determined by its values on an infinite set, and thus if $P \in \mathbb{R}[t]$ then P is completely determined by its values on $f(W)$. Therefore

$$\{P \circ f \mid P \in \mathbb{R}[t]\} \subset C^\infty(W)$$

is an infinite-dimensional subspace of $C^\infty(W)$. ■

Now for some abstract ring theory. This material is not remotely relevant to the course, so ignore it if the terms are not familiar.

Definition 2.14. A ring is said to be a **local** ring if it contains a unique maximal left ideal.

Lemma 2.15. *The ring $\mathcal{F}_pM$ is a local ring.*

Proof. The map $\mathrm{eval}_p \colon \mathcal{F}_pM \to \mathbb{R}$ from (2.4) is clearly a ring homomorphism. Thus the kernel of eval_p is an ideal in the ring $\mathcal{F}_pM$. Since the map eval_p is surjective (as $\mathrm{eval}_p(\underline{c}) = c$), this is actually a maximal ideal. In fact, it is the unique maximal ideal, since if $\mathrm{eval}(\underline{f}) \neq 0$ then $\underline{f}$ is invertible in $\mathcal{F}_pM$. Indeed, if (U, f) is a representative of $\underline{f}$ then there exists $V \subset U$ such that f is never zero on V. Thus there is a well-defined function $g \coloneqq 1/f \colon V \to \mathbb{R}$, and $\underline{g}$ is then an inverse to $\underline{f}$. This completes the proof. ∎

LECTURE 3

Partitions of Unity

We begin this lecture by reformulating the definition of a tangent vector in a slightly more convenient way. Since germs are defined via equivalence classes, they are often tedious to work with, and we would like to dispense with them.

DEFINITION 3.1. Let M be a smooth manifold, let $p \in M$, and let W be any neighbourhood of p (for instance W could be all of M). A **derivation of $C^\infty(W)$ at p** is a linear map $\zeta \colon C^\infty(W) \to \mathbb{R}$ which satisfies the *derivation property*

$$\zeta(fg) = f(p)\zeta(g) + g(p)\zeta(f).$$

If $\xi \in T_pM$ then ξ naturally defines a derivation ζ of $C^\infty(W)$ for any open W containing p by setting

$$\zeta(f) := \xi(\underline{f}). \tag{3.1}$$

In fact, the converse is also true, as we will prove in Proposition 3.3 below. First we need a preliminary lemma. To state it, recall that for a smooth function $f \colon M \to \mathbb{R}$, we denote by $\mathrm{supp}(f)$ the **support** of f, defined by:

$$\mathrm{supp}(f) := \overline{\{p \in M \mid f(p) \neq 0\}}.$$

LEMMA 3.2 (Bump functions). *Let M be a smooth manifold and let $K \subset U$ be subsets, where K is closed and U is open. Then there exists a smooth function $\chi \colon M \to \mathbb{R}$ such that:*

(i) $0 \le \chi(p) \le 1$ *for all* $p \in M$,

(ii) $\mathrm{supp}(\chi) \subset U$,

(iii) $\chi(p) = 1$ *for all* $p \in K$.

A function χ satisfying the three conditions of Lemma 3.2 is called a **bump function**. The proof of Lemma 3.2 will be carried out at the end of this lecture, when we discuss **partitions of unity**. It is *not* obvious—as we will see this is the main reason we imposed the additional point-set topological conditions (metrisable and separable) on top of the locally Euclidean property.

PROPOSITION 3.3. *Let M be a smooth manifold, let $p \in M$, and let W be any neighbourhood of p. Then there is a linear isomorphism between T_pM and the space of derivations of $C^\infty(W)$ at p.*

Proof. Let W be a neighbourhood of p. We prove the result in three steps.

1. Let $\zeta \colon C^\infty(W) \to \mathbb{R}$ be a derivation at p. Suppose $f \in C^\infty(W)$ is identically zero on a neighbourhood $V \subset W$ of p. We claim that

$\zeta(f) = 0$. For this, choose a bump function $\chi\colon M \to \mathbb{R}$ such that $\chi(p) = 1$ and $\operatorname{supp}(\chi) \subset V$. Let $g = \chi f$, thought of as a function $W \to \mathbb{R}$. Then g is identically zero, and hence $\zeta(g) = 0$ by linearity. But by the derivation property

That is, apply Lemma 3.2 with $K = \{x\}$ and "U" equal to V.

$$\begin{aligned} \zeta(g) &= \zeta(\chi f) \\ &= \chi(p)\zeta(f) + f(p)\zeta(\chi) \\ &= \zeta(f) \end{aligned}$$

since $\chi(p) = 1$ and $f(p) = 0$. Thus $\zeta(f) = 0$.

2. Suppose now $\underline{f} \in \mathcal{F}_p M$. We claim that we can always find a representative for $\underline{f}$ with domain W, i.e. a smooth function $g\colon W \to \mathbb{R}$ such that $\underline{g} = \underline{f}$. For this, let (V, f) be any representative of $\underline{f}$. By shrinking V if necessary, we may assume that $V \subset W$. Now choose a smaller neighbourhood U of p with $\overline{U} \subset V \subset W$. Our goal now is to *extend* f to a smooth function g defined on W such that $g|_U = f$. For this, we apply Lemma 3.2 again, this time with $K = \overline{U}$ and "U" equal to V. Now consider the smooth function

The existence of such a set U follows from the fact M is metrisable.

$$g\colon W \to \mathbb{R}, \qquad g(p) := \begin{cases} \chi(p)f(p), & x \in V, \\ 0, & x \in W \setminus V. \end{cases}$$

Since $g|_U = f$, we certainly have $\underline{g} = \underline{f}$.

3. We now complete the proof. Let $\zeta\colon C^\infty(W) \to \mathbb{R}$ be a derivation at p. We define a linear map $\xi\colon \mathcal{F}_p M \to \mathbb{R}$ by setting

$$\xi(\underline{f}) := \zeta(f), \qquad \text{where } (W, f) \text{ is any representative of } \underline{f}.$$

That such a representative (W, f) exists was the content of Step 2, and the fact that ξ is well-defined follows from Step 1. Indeed, if (W, h) was another representative of $\underline{f}$ then by assumption there exists a smaller neighbourhood V of p such that $f|_V \equiv h|_V$. Then by linearity $\zeta(f) - \zeta(h) = \zeta(f - h)$ and $\zeta(f - h) = 0$ by Step 1. Finally, it is clear that ξ is a derivation. This association $\zeta \mapsto \xi$ obviously inverts (3.1), and thus this completes the proof. ■

Thanks to Proposition 3.3, we will from now always regard a tangent vector ξ as a derivation of $C^\infty(W)$ at p for any open W containing p. We emphasise that Proposition 3.3 implies that it doesn't matter which W we choose. Typically we take W either to be the domain of a chart, or the whole manifold M. The next statement is a reformulation of Lemma 2.11 (or alternatively Step 1 of Proposition 3.3).

Corollary 3.4. *Let M be a smooth manifold, let $p \in M$, and let $f \in C^\infty(W)$ for some open W containing p. If f is constant in a neighbourhood of p then $\xi(f) = 0$ for all $\xi \in T_p M$.*

Remark 3.5. Given Proposition 3.3, you may wonder why we didn't immediately define $T_p M$ as the vector space of derivations of $C^\infty(M)$ at p. There are (at least) four reasons:

(i) Using germs better encapsulates the fact that differentiation is a local property.

(ii) An advantage of the germ approach was that Lemma 2.10 is tautological; if we had defined T_pM directly as derivations of $C^\infty(M)$ at p then Lemma 2.10 would have required proof — namely, one would have had to directly show that derivations of $C^\infty(M)$ at p are isomorphic to derivations of $C^\infty(W)$ at p. This is essentially the statement of Proposition 3.3, and thus we wouldn't have saved any time by avoiding germs.

(iii) In certain other geometric categories, the analogue of Lemma 3.2 is false. For instance, there is an analogous theory of **analytic manifolds**, which are defined in exactly the same way as smooth manifolds, except the word "smooth" should be replaced with "real-analytic" everywhere (thus an analytic manifold has a real-analytic atlas, and maps between real-analytic manifolds are required to be real-analytic, etc). We will not study analytic manifolds in this course, although they are very important in certain fields. In the real-analytic category, Lemma 3.2 is false: there do *not* exist real-analytic bump functions. Thus for analytic manifolds, Proposition 3.3 is false, and one is forced to work with germs to define the tangent space.

Exercise: Why?

(iv) Later in the course (Lecture 17) we will discuss **sheaves**, and germs are a motivating example for the construction of the **stalk** of a sheaf.

Let us give now give a concrete example of a tangent vector.

EXAMPLE 3.6. Let M be a smooth manifold of dimension n, and let (U, x) be a chart on M with local coordinates (x^i). Let p be any point in U. Define a derivation of $C^\infty(U)$ at p by:

$$\frac{\partial}{\partial x^i}\Big|_p : C^\infty(U) \to \mathbb{R}, \qquad \frac{\partial}{\partial x^i}\Big|_p (f) := D_i(f \circ x^{-1})(x(p)),$$

where the right-hand side uses the convention from Definition 2.2. We will shortly prove that the collection $\frac{\partial}{\partial x^i}\big|_p$ for $i = 1, \dots, m$ form a basis of T_pM, thus establishing Theorem 2.12.

Let us now get started on the proof of Theorem 2.12. We will need the following easy lemma from multivariable calculus. Recall an open set $\mathcal{O} \subset \mathbb{R}^m$ such that $0 \in \mathcal{O}$ is said to be **star-shaped** if given any $p \in \mathcal{O}$, the line segment from 0 to p is also contained in $\mathcal{O}$.

LEMMA 3.7. *Let $\mathcal{O} \subset \mathbb{R}^m$ be a star-shaped open set. Suppose $h \colon \mathcal{O} \to \mathbb{R}$ is a smooth function. Then there exist m smooth functions $g_i \colon \mathcal{O} \to \mathbb{R}$ for $i = 1, \dots, m$ such that $g_i(0) = D_i h(0)$ and such that*

$$h = h(0) + \sum_{i=1}^m u^i\, g_i,$$

where u^i is as in (2.1).

Proof. Fix $q = (a^1, \dots, a^m) \in \mathcal{O}$ and consider the line segment $\gamma(t) = tq$. Set $\delta \coloneqq h \circ \gamma \colon [0,1] \to \mathbb{R}$. Then by the chain rule

$$\delta'(t) = \sum_{i=1}^{m} a^i \, D_i h(tq).$$

Thus

$$h(q) - h(0) = \delta(1) - \delta(0) = \int_0^1 \delta'(t)\, dt = \sum_{i=1}^{m} a^i \int_0^1 D_i h(tq)\, dt.$$

Since $a^i = u^i(q)$ by definition, the claim follows with

$$g_i(q) \coloneqq \int_0^1 D_i h(tq)\, dt. \qquad \blacksquare$$

Theorem 2.12 from the last lecture follows immediately from the next statement.

PROPOSITION 3.8. *Let M be a smooth manifold of dimension m. Let $x \colon U \to \mathcal{O}$ be a chart on M, and fix $p \in U$. Then any tangent vector $\xi \in T_pM$ can be uniquely written as a linear combination*

$$\xi = \sum_{i=1}^{m} a^i \left.\frac{\partial}{\partial x^i}\right|_p .$$

In fact, $a^i = \xi(x^i)$. Thus $\left\{ \left.\frac{\partial}{\partial x^i}\right|_p \mid i = 1, \dots, m \right\}$ is a basis of T_pM.

Proof. We may assume without loss of generality that $x(p) = 0$ and that $\mathcal{O}$ is star-shaped. Let $f \in C^\infty(U)$ and apply Lemma 3.7 with $h \coloneqq f \circ x^{-1}$. We obtain $f = f(p) + \sum_{i=1}^{m} x^i \, (g_i \circ x)$, where

$$g_i(0) = D_i(f \circ x^{-1})(0) = \left.\frac{\partial}{\partial x^i}\right|_p (f).$$

Thus for any derivation ξ, one has

$$\begin{aligned} \xi(f) &= \underbrace{\xi(f(p))}_{=0} + \sum_{i=1}^{m} \Big(\xi(x^i) g_i(0) + \underbrace{x^i(p)}_{=0} \xi(g_i \circ x) \Big) \\ &= \sum_{i=1}^{m} \xi(x^i) \left.\frac{\partial}{\partial x^i}\right|_p (f), \end{aligned}$$

where we used Corollary 3.4 and the assumption that $x(p) = 0$.

This shows that $\left\{ \left.\frac{\partial}{\partial x^i}\right|_p \mid i = 1, \dots, m \right\}$ spans T_pM. It remains to prove linear independence. For this we note that:

$$\begin{aligned} \left.\frac{\partial}{\partial x^i}\right|_p (x^j) &= \left.\frac{\partial}{\partial x^i}\right|_p (u^j \circ x) \\ &= D_i(u^j \circ x \circ x^{-1})(x(p)) \\ &= D_i u^j(x(p)) \\ &= \delta_i^j, \end{aligned} \tag{3.2}$$

where we used the fact that $Du^j = u^j$ as u^j is a linear function, together with (2.2). Thus if $\zeta \coloneqq \sum_{i=1}^{m} b^i \left.\frac{\partial}{\partial x^i}\right|_p = 0$ then feeding x^j to ζ gives $b^j = 0$. This shows linear independence, and thus completes the proof. $\blacksquare$

Remark 3.9. Suppose x and y are two charts about p, with corresponding coordinate systems (x^i) and (y^i). Taking $\xi = \frac{\partial}{\partial y^j}\big|_p$ in Proposition 3.8 tells us that

$$\frac{\partial}{\partial y^j}\bigg|_p = \sum_{i=1}^m \frac{\partial}{\partial y^j}\bigg|_p (x^i)\, \frac{\partial}{\partial x^i}\bigg|_p .$$

But unravelling the definitions,

$$\begin{aligned}\frac{\partial}{\partial y^j}\bigg|_p (x^i) &= D_j(x^i \circ y^{-1})(y(p)) \\ &= D_j(u^i \circ x \circ y^{-1})(y(p)),\end{aligned}$$

which is just the (i, j)th entry of the matrix $D(x \circ y^{-1})(y(p))$. Thus we have shown:

> The transition matrix from the basis
>
> $$\left\{ \frac{\partial}{\partial y^i}\bigg|_p \mid i = 1, \dots, m \right\} \quad \text{to the basis} \quad \left\{ \frac{\partial}{\partial x^i}\bigg|_p \mid i = 1, \dots, m \right\}$$
>
> is given by the matrix $D(x \circ y^{-1})(y(p))$.

We conclude this lecture by introducing partitions of unity, and using these to prove Lemma 3.2.

Definition 3.10. Let M be a smooth manifold. A **partition of unity** is a collection $\{\kappa_a \mid a \in A\}$ of smooth functions $\kappa_a \colon M \to \mathbb{R}$ such that:

(i) $0 \leq \kappa_a(p) \leq 1$ for all $p \in M$ and $a \in A$.

(ii) The collection $\{\operatorname{supp}(\kappa_a) \mid a \in A\}$ is locally finite, i.e. any $p \in M$ has a neighbourhood that intersects at most finitely many of $\operatorname{supp}(\kappa_a)$.

(iii) For all $p \in M$ one has

$$\sum_{a \in A} \kappa_a(p) = 1$$

(note by (ii) this sum only has finitely many non-zero terms for every p).

We say that a partition of unity $\{\kappa_a \mid a \in A\}$ is **subordinate** to an open cover $\{U_a \mid a \in A\}$ if $\operatorname{supp}(\kappa_a) \subset U_a$ for each $a \in A$.

Theorem 3.11 (Partitions of unity). *Let M be a smooth manifold. For any open cover of M, there exists a partition of unity subordinate to that cover.*

The proof of Theorem 3.11 is carried out in the bonus section below. Lemma 3.2 is an easy consequence of Theorem 3.11:

Proof of Lemma 3.2. Consider the open cover $\{U, M \setminus K\}$ of M. By Theorem 3.11 there exists a partition of unity $\{\kappa_U, \kappa_{M\setminus K}\}$. The function $\chi := \kappa_U$ has the properties we desire. ∎

Bonus Material for Lecture 3

In this bonus section we carry out the proof of Theorem 3.11. In fact, we will first establish a special case of Lemma 3.2 where the smaller set K is *compact* (instead of merely closed).

LEMMA 3.12 (Bump functions, the compact case). *Let M be a smooth manifold and let $K \subset U$ be subsets, where K is compact and U is open. Then there exists a smooth function $\chi \colon M \to \mathbb{R}$ such that:*

(i) $0 \le \chi(p) \le 1$ *for all* $p \in M$,

(ii) $\operatorname{supp}(\chi) \subset U$,

(iii) $\chi(p) = 1$ *for all* $p \in K$.

Proof. We prove the result in four steps.

1. We first prove that for any pair of real numbers $r < R$ there exists a smooth function $f \colon \mathbb{R} \to [0, 1]$ such that $f(t) = 1$ for $t \le r$, $f(t) = 0$ for all $t \ge R$, and $0 < f(t) < 1$ for all $t \in (r, R)$. For this, consider the function

$$h \colon \mathbb{R} \to \mathbb{R}, \qquad h(t) := \begin{cases} e^{-1/t}, & t > 0, \\ 0, & t \le 0. \end{cases}$$

A somewhat tedious computation shows that h is smooth. Our desired function f is then given by

Exercise: Enjoy.

$$f(t) := \frac{h(R-t)}{h(R-t) + h(t-r)}.$$

One can easily check this function f has the desired properties.

2. Now let us extend this to $\mathbb{R}^m$. Let $B_r \subset \mathbb{R}^m$ denote the open ball of radius r about the origin (so that $B_1 = B^m$). Then for any $0 < r < R$ there exists a smooth function $g \colon \mathbb{R}^m \to \mathbb{R}$ such that $g(p) = 1$ for all $p \in \overline{B}_r$, $g(p) = 0$ on $\mathbb{R}^m \setminus B_R$, and $0 < g(p) < 1$ for all $p \in B_R \setminus \overline{B}_r$. Indeed, the function $g(p) := f(\|p\|)$, where f is as in the previous step works.

3. Now let M be a smooth manifold, let $p \in M$, and let U be an arbitrary neighbourhood of p. Then we can choose a smaller neighbourhood $V \subset U$ of p with $\overline{V} \subset U$ that has the following property: there exists a smooth function $\chi \colon M \to \mathbb{R}$ such that $\chi(p) = 1$ for all $p \in \overline{V}$, $0 \le \chi(p) \le 1$ for all $p \in M$, and $\chi(p) = 0$ for all $p \in M \setminus U$. This follows from the previous step, by choosing an appropriate chart about p.

4. We now complete the proof. For each point $p \in K$, choose neighbourhoods $V_p \subset U_p$ such that $\overline{V}_p \subset K$ and $U_p \subset U$. Since K is compact, there are finitely many points $p_1, \dots, p_N$ such that $K \subset \bigcup_{i=1}^N V_{p_i}$. For each i, choose functions $\chi_i \colon M \to \mathbb{R}$ such that

Such sets exist as metrisable spaces are normal, cf. Definition 1.34.

$\chi_i(p) = 1$ for all $p \in \overline{V}_i$, $0 \leq \chi_i(p) \leq 1$ for all $p \in M$, and $\chi_i(p) = 0$ for all $p \in M \setminus U_i$. Now set

$$\chi \coloneqq 1 - \prod_{i=1}^{m} (1 - \chi_i) .$$

One easily checks this χ does the trick. ■

We now prove the following alternative version of Theorem 3.11. This proof assume you are familiar with paracompact space – see Definition 1.29.

THEOREM 3.13. *Let M be a smooth manifold. Let $\{U_a \mid a \in A\}$ be an open cover of M. There exists a locally finite refinement $\{V_b \mid b \in B\}$ and a partition of unity $\{\kappa_b \mid b \in B\}$ subordinate to $\{V_b \mid b \in B\}$ with the additional property that $\operatorname{supp}(\kappa_b)$ is a compact subset of M for every $b \in B$.*

Of course, the main content of the theorem is the existence of the partition of unity $\{\kappa_b \mid b \in B\}$ – the existence of the locally finite refinement $\{V_b \mid b \in B\}$ is just the very definition of paracompactness.

Proof. Paracompactness guarantees us the existence of a locally finite refinement $\{V_b \mid b \in B\}$. In fact, we can do a little better than this: we can find a locally finite refinement $\{V_b \mid b \in B\}$ together with another open cover $\{W_b \mid b \in B\}$ (with the same index set B) such that $\overline{W}_b$ is compact for each $b \in B$ and such that $\overline{W}_b \subset V_b$. This argument uses the fact that M is also locally compact (Definition 1.35). We won't dwell on the details as they not important to the main theme of the course.

We now apply Lemma 3.12 to each pair $\overline{W}_b \subset V_b$ to obtain a smooth function $\chi_b \colon M \to \mathbb{R}$ such that $0 \leq \chi_b(p) \leq 1$ for all $p \in M$, $\chi_b|_{\overline{W}_b} \equiv 1$, and $\operatorname{supp}(\chi_b) \subset V_b$ is compact. The desired partition of unity is then given by

$$\kappa_b \coloneqq \frac{\chi_b}{\sum_{b \in B} \chi_b}.$$

This completes the proof. ■

We conclude by proving Theorem 3.11

Proof of Theorem 3.11. Let $\{U_a \mid a \in A\}$ be an arbitrary open cover. Let $\{V_b \mid b \in B\}$ be a locally finite refinement and let $\{\kappa_b \mid b \in B\}$ be a partition of unity subordinate to $\{V_b \mid b \in B\}$, whose existence are guaranteed by Theorem 3.13. Choose a function $\beta \colon B \to A$ such that $V_b \subset U_{\beta(b)}$ for each $b \in B$. Now define

$$\kappa_a \coloneqq \sum_{b \in \beta^{-1}(a)} \kappa_b.$$

If $\beta^{-1}(a) = \emptyset$ this should be interpreted as the zero function. Then

$$\begin{aligned} \operatorname{supp}(\kappa_a) &= \overline{\bigcup_{b \in \beta^{-1}(a)} \{x \in M \mid \kappa_b(p) \neq 0\}} \\ &= \bigcup_{b \in \beta^{-1}(a)} \operatorname{supp}(\kappa_b) \subset U_a, \end{aligned}$$

where the second equality used the fact that $\{\text{supp}(\kappa_b) \mid b \in B\}$ is a locally finite. It is immediate that the collection $\{\text{supp}(\kappa_a) \mid a \in A\}$ is locally finite, and thus we conclude that $\{\kappa_a \mid a \in A\}$ is another partition of unity which is subordinate to our original cover $\{U_a \mid a \in A\}$. ■

Note however that κ_a need not have compact support.

LECTURE 4

The Derivative

Let us now finally define the derivative of a smooth map.

DEFINITION 4.1. Let M and N be smooth manifolds, and let $\varphi\colon M \to N$ be a smooth map. Fix $p \in M$ and $\xi \in T_pM$. We define a tangent vector $\zeta \in T_{\varphi(p)}N$ by setting

$$\zeta(f) \coloneqq \xi(f \circ \varphi), \qquad \forall\, f \in C^\infty(N).$$

It is clear ζ is a linear derivation of $C^\infty(N)$ at $\varphi(p)$, and hence an element of $T_{\varphi(p)}N$. Moreover if we denote ζ by $D\varphi(p)\xi$ then it is immediate that the map $\xi \mapsto D\varphi(p)\xi$ is a linear map. We call this linear map the **derivative of φ at p**.

The chain rule becomes essentially tautologous.

PROPOSITION 4.2 (The chain rule on manifolds). *Let M, N and L be smooth manifolds, and suppose $\varphi\colon M \to N$ and $\psi\colon N \to L$ are smooth maps. Then*

$$D(\psi \circ \varphi)(p) = D\psi(\varphi(p)) \circ D\varphi(p).$$

Proof. Take $\xi \in T_pM$ and $f \in C^\infty(L)$. Then

$$\begin{aligned}\big(D(\psi \circ \varphi)(p)\xi\big)(f) &= \xi(f \circ \psi \circ \varphi)\\ &= \big(D\varphi(p)\xi\big)(f \circ \psi)\\ &= D\psi(\varphi(p)) \circ \big(D\varphi(p)\xi\big)(f).\end{aligned}$$

The claim follows. ■

REMARK 4.3. You may wonder why the chain rule is so (suspiciously) easy to prove. After all, the Euclidean version (Proposition 1.7) is quite tricky. Does Proposition 4.2 give a shortcut to proving the Euclidean version? The answer is sadly no: indeed, we already used the Euclidean version at least twice (in Proposition 1.21 and Lemma 3.7), and hence any attempt to "prove" the Euclidean version via Proposition 4.2 would yield a circular argument.

Let us compute the map $D\varphi(p)$ in local coordinates.

LEMMA 4.4. *Let $\varphi\colon M \to N$ be a smooth map between two smooth manifolds, where M has dimension m and N has dimension n. Fix $p \in M$, and let (U, x) be a chart on M about p and (V, y) be a chart on N about $\varphi(p)$. Then the matrix of $D\varphi(p)$ with respect to the bases $\left\{ \frac{\partial}{\partial x^j}\big|_p \mid j = 1, \dots, m \right\}$ of T_pM and $\left\{ \frac{\partial}{\partial y^i}\big|_{\varphi(p)} \mid i = 1, \dots, n \right\}$ of $T_{\varphi(p)}N$ is given by the matrix $D(y \circ \varphi \circ x^{-1})(x(p))$.*

Proof. We compute

$$\begin{aligned} D\varphi(p)\left(\frac{\partial}{\partial x^j}\Big|_p\right) &= \sum_{i=1}^n D\varphi(p)\left(\frac{\partial}{\partial x^j}\Big|_p\right)(y^i)\,\frac{\partial}{\partial y^i}\Big|_{\varphi(p)} \\ &= \sum_{i=1}^n \frac{\partial}{\partial x^j}\Big|_p (y^i\circ\varphi)\,\frac{\partial}{\partial y^i}\Big|_{\varphi(p)} \\ &= \sum_{i=1}^n D_j(u^i\circ y\circ\varphi\circ x^{-1})(x(p))\,\frac{\partial}{\partial y^i}\Big|_{\varphi(p)} \end{aligned}$$

The number $D_j(u^i\circ y\circ\varphi\circ x^{-1})(x(p))$ is the (i,j)th entry of the matrix $D(y\circ\varphi\circ x^{-1})(x(p))$, and thus the proof is complete. ∎

Remark 4.5. Suppose $f\colon \mathcal{O}\subset\mathbb{R}^m\to\mathbb{R}^n$ is smooth. We now have two (!) definitions of the map $Df(p)$. We temporarily write the two maps as $Df(p)^{\text{calc}}$ and $Df(p)^{\text{man}}$. Thus $Df(p)^{\text{calc}}$ is a linear map $\mathbb{R}^m\to\mathbb{R}^n$; it is the matrix of partial derivatives, as at the beginning of Lecture 1. Meanwhile $Df(p)^{\text{man}}$ is a linear map $T_p\mathbb{R}^m\to T_{f(p)}\mathbb{R}^n$. In fact, these are the "same" map. Indeed, we apply Lemma 4.4 with x and y the respective identity maps. Then the (i,j)th entry of $Df(p)^{\text{man}}$ is given by

$$D_j(u^i\circ f)(p) = \frac{\partial f^i}{\partial u^j},$$

which is also the (i,j)th entry of $Df(p)^{\text{calc}}$. From now on we will drop the "calc" and "man" superscripts, and just call both maps $Df(p)$. It should be clear from the context which is meant.

A coordinate-free proof of this fact is given in Corollary 4.14 below.

We now give an entirely different way of defining tangent vectors. This approach is not quite as aesthetically pleasing as using derivations, but it has the advantage that it is easier to compute.

Suppose $\gamma\colon(a,b)\to\mathbb{R}^m$ is a smooth map. We usually write the coordinate on $\mathbb{R}=\mathbb{R}^1$ as t instead of u^1, and we denote the derivative of γ at a point t by $\gamma'(t)$. Writing $\gamma=(\gamma^1,\dots,\gamma^m)$, the vector $\gamma'(t)$ is just the row vector $((\gamma^1)'(t),\dots,(\gamma^m)'(t))$. Our aim now is to extend this to manifolds.

Definition 4.6. A **curve** in a smooth manifold M is a smooth map $\gamma\colon(a,b)\to M$, where we think of (a,b) as a 1-dimensional smooth manifold. Now fix $t\in(a,b)$. There are, a priori, two different ways we could define an element $\dot\gamma(t)$ of $T_{\gamma(t)}M$, which we will call the **velocity vector of γ at time t**.

(i) Firstly, we can define a derivation on $C^\infty(M)$ at $\gamma(t)$ by setting

$$\dot\gamma(t)(f) := (f\circ\gamma)'(t), \qquad f\in C^\infty(M). \tag{4.1}$$

(ii) Secondly, if we think of γ as a smooth map between manifolds then we can define a tangent vector $\dot\gamma(t)$ at $\gamma(t)$ via the derivative $D\gamma(t)$:

$$\dot\gamma(t) := D\gamma(t)\left(\frac{\partial}{\partial t}\Big|_t\right)\in T_{\gamma(t)}M. \tag{4.2}$$

The use of both a dot and a dash to denote derivatives in (4.1) is deliberate: a dot denotes a tangent vector in a manifold, whereas a dash denotes the normal derivative from calculus. See Definition 4.11 below.

To see that these two definitions agree, let x be a chart defined on a neighbourhood of $\gamma(t)$ with local coordinates (x^i). Let $\gamma^i := x^i \circ \gamma$ so that γ^i is a curve in $\mathbb{R}$. Applying Proposition 3.8 to (4.1), we see that

$$\dot\gamma(t) = \sum_{i=1}^{m} (\gamma^i)'(t) \, \frac{\partial}{\partial x^i}\Big|_{\gamma(t)}, \tag{4.3}$$

since

$$\dot\gamma(t)(x^i) = (x^i \circ \gamma)'(t) = (\gamma^i)'(t).$$

But similarly by applying Lemma 4.4 to (4.2) we see that this definition also gives the same formula (4.3) for $\gamma'(t)$.

Lemma 4.7. *Let M be a smooth manifold and let $\gamma, \delta \colon (-\varepsilon, \varepsilon) \to M$ be two smooth curves such that $\gamma(0) = \delta(0)$. Then $\dot\gamma(0) = \dot\delta(0)$ as elements of $T_{\gamma(0)}M$ if and only if for some (and hence any) chart (U, x) defined on a neighbourhood of $\gamma(0)$, we have*

$$(x \circ \gamma)'(0) = (x \circ \delta)'(0). \tag{4.4}$$

Proof. The stated condition is equivalent to requiring that $(\gamma^i)'(0) = (\delta^i)'(0)$ for each i, where $\gamma^i = x^i \circ \gamma$ and $\delta^i = x^i \circ \delta$. The claim follows from (4.3), since $\left\{ \frac{\partial}{\partial x^i}\big|_{\gamma(0)} \mid i = 1, \dots, m \right\}$ is a basis of $T_{\gamma(0)}M$. ■

What is less clear is that *every* tangent vector can be written as the velocity vector of a curve.

Proposition 4.8. *Let M be a smooth manifold of dimension m, let $p \in M$ and let $\xi \in T_pM$. There exists a smooth curve $\gamma \colon (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\dot\gamma(0) = \xi$.*

Proof. Choose a chart $x \colon U \to \mathcal{O} \subset \mathbb{R}^m$, where $\mathcal{O}$ is an open set containing 0 such that $x(p) = 0$. Write

$$\xi = \sum_{i=1}^{m} a^i \, \frac{\partial}{\partial x^i}\Big|_p,$$

where the a^i are real numbers. For sufficiently small $\varepsilon > 0$, the vector $(ta^1, \dots, ta^m)$ belongs to $\mathcal{O}$ for all $|t| < \varepsilon$. This means that if we define

$$\gamma \colon (-\varepsilon, \varepsilon) \to M, \qquad \gamma(t) := x^{-1}(ta^1, ta^2, \dots, ta^m),$$

then γ is well-defined, smooth, and satisfies $\gamma(0) = p$. Moreover (4.3) shows us that $\dot\gamma(0) = \xi$. ■

Remark 4.9. This tells us that we can make the following alternative definition of T_pM: a tangent vector at $p \in M$ is an equivalence class of smooth curves $\gamma \colon (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$, where $\gamma \sim \delta$ if and only if for some chart x centred about p, (4.4) holds.

Note however that this only works because we have *already* established that T_pM was a vector space with basis $\left\{ \frac{\partial}{\partial x^i}\big|_p \right\}$. If one wanted to *start* with this definition of T_pM, one would need to use Problem B.1 to endow T_pM with a vector space structure.

Let us examine how velocity vectors behave with respect to smooth maps.

Proposition 4.10. *Let $\varphi\colon M \to N$ be a smooth map between two smooth manifolds, and let $\gamma\colon (a,b) \to M$ be a smooth curve in M. Then $\delta := \varphi \circ \gamma$ is a smooth curve in N, and*

$$D\varphi(\gamma(t))\dot{\gamma}(t) = \dot{\delta}(t).$$

Proof. We will give two proofs, one for each of the two (equivalent) definitions (4.1) and (4.2) of $\dot{\gamma}(t)$. Of course these are really the same proof.

(i) *Proof using (4.1) as the definition of $\dot{\gamma}(t)$:* Take $f \in C^\infty(N)$. Then by the definition of $D\varphi(p)$ and (4.1)

$$\begin{aligned} D\varphi(\gamma(t))\dot{\gamma}(t)(f) &= \dot{\gamma}(t)(f \circ \varphi) \\ &= (f \circ \varphi \circ \gamma)'(t) \\ &= (f \circ \delta)'(t) \\ &= \dot{\delta}(t)(f) \end{aligned}$$

(ii) *Proof using (4.2) as the definition of $\dot{\gamma}(t)$:* For this we simply use the chain rule (Proposition 4.2):

$$\begin{aligned} D\varphi(\gamma(t))\dot{\gamma}(t) &= D\varphi(\gamma(t)) \circ D\gamma(t)\left(\frac{\partial}{\partial t}\Big|_t\right) \\ &= D(\varphi \circ \gamma)(t)\left(\frac{\partial}{\partial t}\Big|_t\right) \\ &= D\delta(t)\left(\frac{\partial}{\partial t}\Big|_t\right) \\ &= \dot{\delta}(t). \end{aligned}$$

This completes the proof (twice). ■

We now use our new definition of a tangent vector to prove that the tangent space of a vector space is canonically isomorphic to the vector space itself.

Definition 4.11. Let E be a vector space of dimension m, endowed with its standard smooth structure (cf. Example 1.14). Fix $p \in E$. Define the **dash-to-dot map**

$$\mathcal{J}_p\colon E \to T_pE, \qquad \mathcal{J}_p\xi := \dot{\gamma}(0), \qquad \text{where } \gamma(t) := p + t\xi.$$

Note that $\xi = \gamma'(0)$ (normal derivative), and hence $\mathcal{J}_p$ is the map

This explains the name "dash to dot".

$$\mathcal{J}_p\colon \gamma'(0) \mapsto \dot{\gamma}(0).$$

Lemma 4.12. *The dash-to-dot map is a canonical isomorphism.*

Proof. The smooth structure on E is determined taking a chart which is a linear isomorphism $\ell\colon E \to \mathbb{R}^m$, cf. Example 1.14. Let $\ell^i := u^i \circ \ell$ denote the local coordinates of such a chart. The map ℓ determines a basis $\{v^i\}$ of E via the equation $\ell v_i = e_i$. If one writes an arbitrary vector in E in terms of this basis as $\xi = \sum_{i=1}^{m} a^i v_i$ then $a^i = \ell^i(\xi)$. Now with $\gamma(t) := p + t\xi$ one has

$$\begin{aligned}
\mathcal{J}_p \xi &= \dot{\gamma}(0) \\
&= \sum_{i=1}^{m} \dot{\gamma}(0)(\ell^i) \frac{\partial}{\partial \ell^i}\Big|_p \\
&= \sum_{i=1}^{m} (\ell^i \circ \gamma)'(0) \frac{\partial}{\partial \ell^i}\Big|_p \\
&= \sum_{i=1}^{m} \ell^i(\xi) \frac{\partial}{\partial \ell^i}\Big|_p \\
&= \sum_{i=1}^{m} a^i \frac{\partial}{\partial \ell^i}\Big|_p .
\end{aligned}$$

This shows that the matrix of $\mathcal{J}_p$ with respect to the basis $\{v_i\}$ of E and $\left\{ \frac{\partial}{\partial \ell^i}\big|_p \right\}$ of T_pE is simply given by identity map, which in particular is an isomorphism. ■

The proof of Lemma 4.12 required us to fix a basis of E (i.e. to choose a chart ℓ) in order to prove that $\mathcal{J}_p$ was an isomorphism. But the definition of $\mathcal{J}_p$ did *not* require us to choose a basis of E. This explains the "canonical" in the statement of Lemma 4.12.

REMARK 4.13. If we go back to our original definition of tangent vectors as derivations, the dash-to-dot map is given by taking directional derivatives:

$$(\mathcal{J}_p \xi)(f) = Df(p)\xi, \qquad f \in C^\infty(E).$$

Compare to Example 2.8.

Remark 4.5 can now be expressed in a coordinate-free manner.

COROLLARY 4.14. *Suppose $f \colon \mathcal{O} \subset \mathbb{R}^m \to \mathbb{R}^n$ is smooth. Then using the notation from Remark 4.5, the following diagram commutes:*

This means that going clockwise is the same as going anticlockwise.

$$\begin{array}{ccc}
\mathbb{R}^m & \xrightarrow{\;Df(p)^{\text{calc}}\;} & \mathbb{R}^n \\
{\scriptstyle \mathcal{J}_p}\downarrow & & \downarrow{\scriptstyle \mathcal{J}_{f(p)}} \\
T_p\mathbb{R}^m & \xrightarrow[\;Df(p)^{\text{man}}\;]{} & T_{f(p)}\mathbb{R}^n
\end{array}$$

We conclude with the following statement, whose proof is deferred to Problem Sheet B.

LEMMA 4.15. *Let E and F be vector spaces and assume that $\ell \colon E \to F$ is a linear map. Then for any $p \in E$ the following diagram commutes:*

$$\begin{array}{ccc}
E & \xrightarrow{\;\ell\;} & F \\
{\scriptstyle \mathcal{J}_p}\downarrow & & \downarrow{\scriptstyle \mathcal{J}_{\ell p}} \\
T_pE & \xrightarrow[\;D\ell(p)\;]{} & T_{\ell p}F
\end{array}$$

LECTURE 5

The Tangent Bundle

We begin this lecture by defining the cotangent space of a manifold. We then move onto the tangent and cotangent bundles. We conclude by recalling the Euclidean versions of the Inverse and Implicit Function Theorems. These will be generalised to manifolds next lecture.

Definition 5.1. Let M be a smooth manifold of dimension m and let $p \in M$. We denote the dual vector space $(T_pM)^*$ by T_p^*M and call it the **cotangent space** of M at p.

Convention. We write a typical element of cotangent space with the symbols λ and η. We sometimes write λ_p to indicate that $\lambda \in T_p^*M$.

The cotangent space T_p^*M is another vector space of dimension m. Since elements of T_pM are linear derivations eating functions, the standard duality construction tells us that we can interpret elements of T_p^*M as functions eating linear derivations.

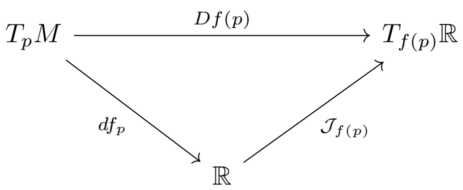
Example 5.2. Let M be a smooth manifold of dimension m and let $p \in M$. Let U be a neighbourhood of p and let $f \in C^\infty(U)$. Then f defines an element $df_p \in T_p^*M$ by

$$df_p(\xi) := \xi(f), \qquad \xi \in T_pM.$$

One calls df_p the **differential** of f at p.

Remark 5.3. Thus df_p is a linear function $T_pM \to \mathbb{R}$. In contrast, the derivative $Df(p)$ is a linear function $T_pM \to T_{f(p)}\mathbb{R}$. The two are related via the dash-to-dot map $\mathcal{J}_{f(p)} \colon \mathbb{R} \to T_{f(p)}\mathbb{R}$ in the sense that the following commutes:

Note: "derivative" and "differential" are two different words!



i.e. $Df(p) = \mathcal{J}_p \circ df_p$.

Since by definition $\mathcal{J}_{f(p)}(1) = \frac{\partial}{\partial t}\big|_{f(p)}$, this means that

$$Df(p)\xi = df_p(\xi)\, \frac{\partial}{\partial t}\Big|_{f(p)}, \qquad \forall\, \xi \in T_pM.$$

Proposition 5.4. *Let M be a smooth manifold of dimension m and let $p \in M$. Let (U, x) be a chart about p, with corresponding local coordinates (x^i). Then $\{dx_p^i\}$ is a basis of T_p^*M.*

Proof. We need only note that $\{dx_p^i\}$ is the dual basis to $\left\{\frac{\partial}{\partial x^i}\big|_p\right\}$ since

$$dx_p^j\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial}{\partial x^i}\Big|_p (x^j) = \delta_i^j,$$

by (3.2) from the last lecture. ∎

We now aim to "glue" the vector spaces T_pM together into one big manifold TM.

DEFINITION 5.5. Let M be a smooth manifold. The **tangent bundle** of M is the disjoint union of the tangent spaces:

$$TM = \bigsqcup_{p \in M} T_pM.$$

The tangent bundle is the prototypical example of a more general construction called a **vector bundle** over a smooth manifold. We will take up their study in Lecture 13.

We adopt a somewhat flexible notation for points in TM. If $\xi \in T_pM$ we write the corresponding point in TM either simply by ξ again, or as a pair (p, ξ). There is a map $\pi \colon TM \to M$ given by $\pi(p, \xi) = p$. We call π the **footpoint** map.

As it stands TM is only a set. Let us now prove it is actually a smooth manifold.

THEOREM 5.6. *Let M be a smooth manifold of dimension m. The smooth structure on M naturally induces a smooth structure on TM, making TM into a smooth manifold of dimension $2m$. Moreover the map $\pi \colon TM \to M$ is smooth.*

Proof. Let $\mathfrak{X} = \{x_a \colon U_a \to \mathcal{O}_a \mid a \in A\}$ be our smooth atlas on M. Write $x_a^i = u^i \circ x_a$ for the local coordinates of x_a. We build a chart $\tilde{x}_a \colon \pi^{-1}(U_a) \to \mathcal{O}_a \times \mathbb{R}^m$ by setting

$$\tilde{x}_a(p, \xi) = \left(x_a(p), \sum_{i=1}^m (dx_a^i)_p(\xi)\, e_i \right), \qquad p \in U_a,\ \xi \in T_pM.$$

We will prove that if $U_a \cap U_b \neq \emptyset$ then for all $z \in x_a(U_a \cap U_b)$ and $\zeta \in \mathbb{R}^m$, one has:

$$\tilde{x}_b \circ \tilde{x}_a^{-1}(z, \zeta) = \Big(x_b \circ x_a^{-1}(z), D(x_b \circ x_a^{-1})(z)\zeta \Big), \tag{5.1}$$

From this it follows from Proposition 1.17 that TM is a smooth manifold. To prove (5.1), write $\zeta = \sum_{j=1}^m c^j\, e_j$ and set $p := x_a^{-1}(z) \in U_a \cap U_b$. Then

$$\tilde{x}_a^{-1}(z, \zeta) = \left(p, \sum_{j=1}^m c^j \frac{\partial}{\partial x_a^j}\Big|_p \right).$$

Fix $1 \le i \le m$. We compute

$$\begin{aligned} (dx_b^i)_p \left(\sum_{j=1}^m c^j \frac{\partial}{\partial x_a^j}\Big|_p \right) &= \sum_{j=1}^m c^j\, (dx_b^i)_p \left(\frac{\partial}{\partial x_a^j}\Big|_p \right) \\ &= \sum_{j=1}^m c^j \frac{\partial}{\partial x_a^j}\Big|_p (x_b^i) \end{aligned}$$

By Remark 3.9, the number $\frac{\partial}{\partial x_a^j}\Big|_p (x_b^i)$ is the (i,j)th entry of the matrix $D(x_b \circ x_a^{-1})(z)$. Thus

$$\sum_{i=1}^m (dx_b^i)_p \left(\sum_{j=1}^m c^j \frac{\partial}{\partial x_a^j}\Big|_p \right) e_i = D(x_b \circ x_a^{-1})(z)\zeta,$$

and (5.1) is proved.

The right-hand side of (5.1) is a diffeomorphism by assumption. Thus

$$\tilde{\mathfrak{X}} = \left\{ \tilde{x}_a : \pi^{-1}(U_a) \to \mathcal{O}_a \times \mathbb{R}^m \mid a \in A \right\}$$

is a smooth atlas on TM. This proves that TM is a smooth manifold of dimension $2m$. To check that π is smooth, we simply observe that if $z \in \mathcal{O}_a$ and $\zeta \in \mathbb{R}^m$ then

$$x_a \circ \pi \circ \tilde{x}_a^{-1}(z, \zeta) = z,$$

which is obviously smooth. ■

We can use the tangent bundle to unify the derivatives $D\varphi(p)$ from Definition 4.1 into a single map.

Definition 5.7. Let $\varphi \colon M \to N$ be a smooth map between two smooth manifolds. Define the **derivative of** φ to be the map

$$D\varphi \colon TM \to TN, \qquad D\varphi(p, \xi) := (\varphi(p), D\varphi(p)\xi)\,.$$

On Problem Sheet C you will prove this map is smooth.

Definition 5.8. Let M be a smooth manifold. The **cotangent bundle** of M is the disjoint union of the cotangent spaces:

$$T^*M = \bigsqcup_{p \in M} T_p^*M.$$

As with TM, points in T^*M will sometimes be denoted by (p, λ), or λ_p, or sometimes just λ. We denote again by $\pi \colon T^*M \to M$ the **footpoint** map $\pi(p, \lambda) = p$.

On Problem Sheet B you will show that T^*M is also naturally a smooth manifold of twice the dimension of M.

We conclude this lecture by discussing the Inverse and Implicit Function Theorems. We state the Euclidean version of the Inverse Function Theorem, and use it to prove the manifold version of the Inverse Function Theorem, and the Euclidean version of the Implicit Function Theorem. Next lecture we will take this one step further and prove a version of the Implicit Function Theorem for manifolds.

We say that a smooth map $f \colon \mathcal{O} \subset \mathbb{R}^m \to \mathbb{R}^n$ has **rank** k at $p \in \mathcal{O}$ if the $n \times m$ matrix $Df(p)$ has rank k. We say that f has **maximal rank at** p if the rank of f at p is as large as it can be (which is thus equal to the minimum of m and n). If $m = n$ then f has maximal rank at p if and only if $Df(p)$ is invertible.

Theorem 5.9 (The Inverse Function Theorem). *Let $f \colon \mathcal{O} \subset \mathbb{R}^m \to \mathbb{R}^m$ be a smooth map, where $\mathcal{O}$ is open. Let $p \in \mathcal{O}$ and assume the matrix $Df(p)$ has maximal rank ($= m$). Then there exists a neighbourhood $\Omega \subset \mathcal{O}$ of p such that the restriction $f \colon \Omega \to f(\Omega)$ is a diffeomorphism.*

The theorem immediately carries over to manifolds. We say that a smooth map $\varphi \colon M \to N$ has **rank** k at a point p if the linear subspace $D\varphi(p)(T_pM)$ has dimension k inside of $T_{\varphi(p)}N$.

Theorem 5.10 (The Inverse Function Theorem for manifolds). *Let M and N be smooth manifolds of the same dimension m and suppose $\varphi\colon M \to N$ is a smooth map. Let $p \in M$ and assume that φ has maximal rank ($= m$) at p. Then there exists a neighbourhood W of p such that the restriction $\varphi\colon W \to \varphi(W)$ is a diffeomorphism.*

Proof. The assertion is purely local. Choose a chart $x\colon U \to \mathcal{O}$ on M at p and a chart $y\colon V \to \Omega$ on N at $\varphi(p)$ such that $\varphi(U) \subset V$. Since x and y are diffeomorphisms (cf. Example 1.20), the derivative of the map

$$y \circ \varphi \circ x^{-1}\colon x(U) \to y(V)$$

has rank m at $x(p)$. Thus by Theorem 5.9 there exists $\mathcal{O}_0 \subset \mathcal{O}$ such that $y \circ \varphi \circ x^{-1}|_{\mathcal{O}_0}$ is a diffeomorphism. Then using once more that x and y are diffeomorphisms, if $W := x^{-1}(\mathcal{O}_0)$ then $\varphi|_W\colon W \to \varphi(W)$ is also a diffeomorphism. ■

We now move onto the Implicit Function Theorem. We shall give a quick proof using the Inverse Function Theorem.

Theorem 5.11 (The Implicit Function Theorem). *Let $\mathcal{O}$ be a neighbourhood of 0 in $\mathbb{R}^m$ and suppose $f\colon \mathcal{O} \to \mathbb{R}^n$ is a smooth map such that $f(0) = 0$, and such that $Df(0)$ has maximal rank.*

(i) **The case $m \le n$:** *Let $\iota\colon \mathbb{R}^m \to \mathbb{R}^n$ denote the inclusion*

$$\iota(u^1, \dots, u^m) := (u^1, \dots, u^m, 0, \dots, 0). \tag{5.2}$$

Then there exists a chart y about 0 on $\mathbb{R}^n$ such that $y \circ f = \iota$ on a neighbourhood of 0 in $\mathbb{R}^m$.

(ii) **The case $m \ge n$:** *Let $\rho\colon \mathbb{R}^m \to \mathbb{R}^n$ denote the projection*

$$\rho(u^1, \dots, u^m) := (u^1, \dots, u^n). \tag{5.3}$$

Then there exists a chart x about 0 in $\mathbb{R}^m$ such that $f \circ x = \rho$ on a neighbourhood of 0 in $\mathbb{R}^m$.

Proof. We start with (i). The matrix $Df(0)$ has rank m. By rearranging the coordinate functions $f^i = u^i \circ f$ if necessary (this corresponds to composing f with a linear isomorphism $\mathbb{R}^n \to \mathbb{R}^n$, which is a diffeomorphism), we may assume that the $m \times m$ submatrix $\left(\frac{\partial f^i}{\partial u^j}(0)\right)_{1 \le i,j \le m}$ is invertible. Now define a map

$$\tilde{f}\colon \mathcal{O} \times \mathbb{R}^{n-m} \to \mathbb{R}^n$$

by

$$\tilde{f}(u^1, \dots, u^n) = f(u^1, \dots, u^m) + (0, \dots, 0, u^{m+1}, \dots, u^n).$$

Then $\tilde{f} \circ \iota = f$ and the derivative $D\tilde{f}(0)$ takes the following form:

$$D\tilde{f}(0) = \begin{pmatrix} \left(\frac{\partial f^i}{\partial u^j}(0)\right)_{1 \le i,j \le m} & 0 \\ * & \mathrm{id}_{\mathbb{R}^{n-m}} \end{pmatrix}$$

where $*$ denotes the other entries of $Df(0)$. Thus $\det D\tilde{f}(0) \neq 0$, and consequently $D\tilde{f}(0)$ has rank n. Thus by Theorem 5.9, there exists a neighbourhood $\mathcal{O}_0 \subset \mathcal{O} \times \mathbb{R}^{n-m}$ of the origin $0 \in \mathbb{R}^n$ such that $\tilde{f}\colon \mathcal{O}_0 \to \tilde{f}(\mathcal{O}_0)$ is a diffeomorphism. If y denotes the inverse to $\tilde{f}|_{\mathcal{O}_0}$ then $y \circ f = y \circ \tilde{f} \circ \iota = \iota$. This proves (i).

The proof of (ii) is very similar. This time we may assume that the submatrix $\left(\frac{\partial f^i}{\partial x^j}(0)\right)_{1\leq i,j\leq n}$ is invertible, and we define $\tilde{f}\colon \mathcal{O} \to \mathbb{R}^m$ by

$$\tilde{f}(u^1, \ldots, u^m) \coloneqq (f(u^1, \ldots, u^m), u^{n+1}, \ldots, u^m).$$

Then $f = \rho \circ \tilde{f}$ and the derivative $D\tilde{f}(0)$ takes the following form:

$$D\tilde{f}(0) = \begin{pmatrix} \left(\frac{\partial f^i}{\partial x^j}(0)\right)_{1\leq i,j\leq n} & * \\ 0 & \mathrm{id}_{\mathbb{R}^{m-n}} \end{pmatrix}$$

This is invertible, whence $\tilde{f}$ has a local inverse x, and $f \circ x = \rho \circ \tilde{f} \circ x = \rho$. ∎

LECTURE 6

Submanifolds

We begin this lecture by proving a version of the Implicit Function Theorem 5.11 for manifolds. We remind the reader that unless stated otherwise, M should always be assumed to have dimension m and N should always be assumed to have dimension n. As in the statement of Theorem 5.11, we must make a case distinction depending as to which of m and n is larger. Unlike in the Euclidean case, however, the two statements are not analogous to each other – as we will see, the case $m \leq n$ is straightforward, but the case $m \geq n$ is rather deeper.

We first deal with the case where $m \leq n$.

DEFINITIONS 6.1. Let $\varphi \colon M \to N$ be a smooth map.

- We say that φ is an **immersion** if the linear map $D\varphi(p) \colon T_pM \to T_{\varphi(p)}N$ is injective for every $p \in M$.
- If in addition φ itself is injective then we say that φ is an **injective immersion**.
- If in addition φ maps M homeomorphically onto $\varphi(M)$ (where $\varphi(M)$ is endowed with the subspace topology in N) we say that φ is an **embedding**.

Note an immersion can only exist when $m \leq n$.

REMARK 6.2. If M is compact, then an injective immersion $\varphi \colon M \to N$ is automatically an embedding, as you will prove on Problem Sheet C. However in the non-compact case, this need not be the case (see again Problem Sheet C). An immersion is always *locally* an embedding, as the next result shows.

The next result is the manifold version of part (i) of the Implicit Function Theorem 5.9.

PROPOSITION 6.3. *Suppose $\varphi \colon M \to N$ is an immersion. Then for any $p \in M$, there exists a neighbourhood U of p and a chart $y \colon V \to \Omega$ on N, where V is some neighbourhood of $\varphi(p)$ such that:*

(i) *One has:*

$$\varphi(U) \cap V = \left\{ q \in V \mid y^{m+1}(q) = \cdots = y^n(q) = 0 \right\}. \tag{6.1}$$

(ii) *$\varphi|_U$ is an embedding.*

Proof. The assertion is again local. Let $\iota \colon \mathbb{R}^m \to \mathbb{R}^n$ denote the inclusion, as in part (i) of the Implicit Function Theorem 5.11. Let x denote a chart on M with $x(p) = 0$ and let z denote a chart on N with $z(\varphi(p)) = 0$. Then $z \circ \varphi \circ x^{-1}$ has maximal rank at 0, and hence by part (i) of the Implicit Function Theorem there exists a chart $\tilde{y}$ on $\mathbb{R}^n$ about 0 and a neighbourhood $\mathcal{O}$ of 0 in $\mathbb{R}^m$ such that

$$\tilde{y} \circ z \circ \varphi \circ x^{-1}|_{\mathcal{O}} = \iota|_{\mathcal{O}}.$$

Set $U := x^{-1}(\mathcal{O})$ and set $y := \tilde{y} \circ z$. Then after restricting the domain if necessary, (6.1) holds. To prove the second statement, simply note that $\varphi|_U = y^{-1} \circ \iota \circ x|_U$ is the composition of embeddings, and thus is an embedding. ■

REMARK 6.4. If φ is an embedding then the set $\varphi(U)$ from Proposition 6.3 can be written as $\varphi(U) = \varphi(M) \cap W$ for some open set $W \subset N$. (This is just the definition of the subspace topology). Replacing V with $W \cap V$, (6.1) becomes

$$\varphi(M) \cap V = \left\{ q \in V \mid y^{m+1}(q) = \cdots = y^n(q) = 0 \right\}. \tag{6.2}$$

DEFINITION 6.5. Let M and N be manifolds with $M \subset N$ (as sets). We say that M is a **embedded submanifold** of N if the inclusion $M \hookrightarrow N$ is an embedding. If the inclusion is merely an immersion, we say that M is an **immersed submanifold**.

Note the inclusion is always injective!

If M is an embedded submanifold of N then Remark 6.4 tells us we can always choose charts on N that satisfy (6.2). We give such a chart a special name:

DEFINITION 6.6. Let M be an embedded submanifold of N. A **slice chart for M in N** is a chart $y \colon V \to \Omega$ on N such that

$$M \cap V = \left\{ q \in V \mid y^{m+1}(q) = \cdots = y^n(q) = 0 \right\}.$$

In fact, the existence of slice charts is an "if and only if" condition, in the sense that we can use slice charts to endow a subset with a smooth structure. The next result makes this more precise.

PROPOSITION 6.7. *Let N be a smooth manifold and suppose $M \subset N$ is a subset with the property that around every point $p \in M$ there exists a chart $y \colon V \to \Omega$ on N with $p \in V$ such that*

$$M \cap V = \left\{ q \in V \mid y^{m+1}(q) = \cdots = y^n(q) = 0 \right\}. \tag{6.3}$$

Then if we endow M with the subspace topology on N, M is a topological manifold of dimension m, and moreover it has a smooth structure that makes it into an embedded submanifold of N.

Proof. Let $\rho \colon \mathbb{R}^n \to \mathbb{R}^m$ denote the projection

Warning: We are still in the case $m \leq n$. Thus m and n have switched roles compared to the projection from (5.3).

$$\rho(u^1, \dots, u^n) := (u^1, \dots, u^m).$$

Fix $p \in M$ and let $y \colon V \to \Omega$ be a chart as in (6.3). Let $U := M \cap V$ and let $\mathcal{O} := \rho(y(U))$, and set $x := \rho \circ y|_U$. If M is given the subspace topology then x is a homeomorphism. If we do this at every point $p \in M$, we end up with a collection of maps for which the hypotheses of Proposition 1.17 are satisfied. Thus M is a smooth manifold of dimension m. Moreover the topology on M that Proposition 1.17

provides coincides with the subspace topology, since the maps x were already homeomorphisms in the subspace topology. Finally if $i\colon M \hookrightarrow N$ denotes the inclusion then with y, x as above, one has $y \circ i \circ x^{-1} = \iota$, where ι was defined in (5.2). Since ι is smooth, so is i. ■

REMARK 6.8. If $\varphi\colon M \to N$ is an injective immersion then M is diffeomorphic to an immersed submanifold of N — namely, $M \cong \varphi(M)$. The same is true for embeddings.

We now move to the case where $m \geq n$.

DEFINITIONS 6.9. Let $\varphi\colon M \to N$ be smooth. A point $p \in M$ is said to be a **regular point** of φ if φ has rank n at p. A point $p \in M$ is called a **critical point** if it is not a regular point. Similarly a point $q \in N$ is called a **regular value** if every point in $\varphi^{-1}(q)$ is a regular point. A point $q \in N$ is called a **critical value** if it is not a regular value.

Note this can only happen when $m \geq n$.

If $q \in N \setminus \varphi(M)$ then q is vacuously a regular value of φ.

We now state the manifold version of part (ii) of the Implicit Function Theorem 5.11. Despite the fact that this is only "half" of the Euclidean Implicit Function Theorem, this result is usually called "the" Implicit Function Theorem (and Proposition 6.3 doesn't get a name).

THEOREM 6.10 (The Implicit Function Theorem for manifolds). *Let $\varphi\colon M \to N$ be a smooth map and suppose $q \in N$ is a regular value of φ such that $L := \varphi^{-1}(q)$ is not empty. Then L is a topological manifold of dimension $m-n$. Moreover there exists a smooth structure on L which makes L into a smooth embedded submanifold of M.*

Unlike Proposition 6.3 this is a much deeper result, as the assertion is *not* local — there is no reason why L should be contained in the domain of a chart on M. This proof is deferred to the bonus section at the end of the lecture, since it is rather fiddly.

DEFINITION 6.11. A smooth map $\varphi\colon M \to N$ is called a **submersion** if every point of M is a regular point of φ, i.e. if $D\varphi(p)$ is surjective for every $p \in M$.

Thus if φ is a submersion then by the Implicit Function Theorem 6.10, every point $p \in M$ belongs to the $(m-n)$-dimensional embedded submanifold $\varphi^{-1}(\varphi(p))$.

DEFINITION 6.12. Let $\varphi\colon M \to N$ be a smooth map. Fix $p \in M$. We say that φ admits **smooth local sections** if for every $p \in M$ there exists a neighbourhood U of p and a neighbourhood V of $\varphi(p)$, together with a smooth map $\psi\colon V \to U$ such that

$$\varphi \circ \psi = \mathrm{id}, \qquad \text{on } V.$$

The motivation behind the name "section" will come in Lecture 15.

PROPOSITION 6.13. *Let $\varphi\colon M \to N$ be a submersion. Then φ is an open map which admits smooth local sections.*

Proof. Let $p \in M$. From the Implicit Function Theorem 5.11, we may choose a chart $x\colon U \to \mathcal{O}$ on M about p and a chart $y\colon V \to \Omega$ on N about $\varphi(p)$ such that $y \circ \varphi \circ x^{-1}$ is of the form

$$(u^1, \dots, u^m) \mapsto (u^1, \dots, u^n). \tag{6.4}$$

By shrinking the domains if necessary we may assume

$$\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2 \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$$

and that $\Omega = \mathcal{O}_1$. Define

$$\psi\colon V \to U, \qquad \psi(q) := x^{-1}\big(y(q), \pi_2(x(p))\big)$$

Then ψ is smooth and by (6.4) one has $\varphi \circ \psi = \mathrm{id}$ on V. The fact that φ is an open map is clear from the representation (6.4), since a small open cube $\{|u^i| < \varepsilon \mid i = 1, \dots, m\}$ is mapped onto the small open cube $\{|u^i| < \varepsilon, \mid i = 1, \dots, n\}$. ■

COROLLARY 6.14. *Let $\varphi\colon M \to N$ be a surjective submersion. Then φ is a quotient map.*

A surjective continuous map $f\colon X \to Y$ is a **quotient map** if for $V \subset Y$, one has V open $\Leftrightarrow f^{-1}(V)$ open.

We now identify the tangent space of a submanifold produced via the Implicit Function Theorem.

PROPOSITION 6.15. *Let $\varphi\colon M \to N$ be a smooth map and let $q \in N$ be a regular value of φ such that $L := \varphi^{-1}(q) \neq \emptyset$. Let $i\colon L \hookrightarrow M$ denote the inclusion. Then for all $p \in L$, one has*

$$Di(p)(T_pL) = \ker D\varphi(p).$$

Proof. By assumption both sides are linear subspaces of T_pM of dimension $m-n$, so it suffices to show that $Di(p)(T_pL) \subset \ker D\varphi(p)$. For this take $f \in C^\infty(N)$ and $\xi \in T_pL$. Then by the chain rule (Proposition 4.2), one has

$$\begin{aligned} D\varphi(p) \circ \big(Di(p)\xi\big)(f) &= \big(D(\varphi \circ \iota)(p)\xi\big)(f) \\ &= \xi(f \circ \varphi \circ i). \end{aligned}$$

But $f \circ \varphi \circ i \in C^\infty(L)$ is the constant function $p \mapsto f(q)$ and hence by Corollary 3.4 one has $\xi(f \circ \varphi \circ i) = 0$. The result follows. ■

Proposition 6.15 finally allows us to recover the "intuitive" definition of the tangent space for S^2 given in Figure 2.1 at the beginning of Lecture 2.

EXAMPLE 6.16. Let $f\colon \mathbb{R}^{m+1} \to \mathbb{R}$ be the map $f(p) = \|p\|^2 - 1$. It is straightforward to check that f is smooth and that the only critical point of f is $0 \in \mathbb{R}^{m+1}$. Thus $0 \in \mathbb{R}$ is a regular value of f, and so by the Implicit Function Theorem 6.10, $S^m = f^{-1}(0)$ is a smooth manifold of dimension m. This is the same smooth structure as the one given in Proposition 1.16. If we denote by $i\colon S^m \to \mathbb{R}^{m+1}$ the inclusion then (as you will check on Problem Sheet C), one has

$$Di(p)(T_pS^m) = \mathcal{J}_p(p^\perp), \tag{6.5}$$

where $\mathcal{J}_p\colon \mathbb{R}^{m+1} \to T_p\mathbb{R}^{m+1}$ is the dash-to-dot map from Definition 4.11, and

$$p^\perp := \left\{ q \in \mathbb{R}^{m+1} \mid \langle p, q \rangle = 0 \right\},$$

for $\langle \cdot, \cdot \rangle$ the standard Euclidean dot product. Now a moment's thought shows that (6.5) implies that the tangent space to S^m at a point p is the hyperplane tangent to S^m at p, as Figure 2.1 claimed.

We now state a version of Sard's Theorem valid for manifolds.

Theorem 6.17 (Sard's Theorem for Manifolds). *Let $\varphi\colon M \to N$ be a smooth map. The set of critical values of φ has measure zero and is nowhere dense. The set of regular values of φ is residual and thus dense in N. In particular, if $m < n$ then every point of M is necessarily a critical point of φ, and hence $N \setminus \varphi(M)$ is dense in N.*

A sketch of the proof of Theorem 6.17 is in the bonus section below.

We next discuss a generalisation of the Implicit Function Theorem in Euclidean spaces (Theorem 5.11), which will occasionally be useful. Suppose $\mathcal{O}$ is a neighbourhood of 0 in $\mathbb{R}^m$ and $f\colon \mathcal{O} \to \mathbb{R}^n$ is a smooth map. As Theorem 5.11 showed, if we assumed that the rank of f at 0 was maximal (and thus either equal to m or n, depending which was larger), then the rank of f was also maximal for all p near 0 too. Thus having maximal rank is an *open* condition.

If the rank is not maximal, then it can "jump", i.e. if f has (non-maximal) rank k at 0 then for p arbitrarily close to 0 the rank of f at p can be different to k. However if one adds as a hypothesis that the rank of f does *not* jump, then an analogous result to Theorem 5.11 holds. Here is a precise statement:

Theorem 6.18 (The Constant Rank Theorem). *Let $\mathcal{O}$ be a neighbourhood of 0 in $\mathbb{R}^m$ and suppose $f\colon \mathcal{O} \to \mathbb{R}^n$ is a smooth map such that $f(0) = 0$. Assume that f has constant rank k for all $p \in \mathcal{O}$, and let $\kappa\colon \mathbb{R}^m \to \mathbb{R}^n$ denote the map*

$$\kappa(u^1, \ldots, u^m) = (u^1, \ldots, u^k, 0, \ldots, 0). \tag{6.6}$$

There exists chart x about 0 on $\mathbb{R}^m$ and a chart y about 0 on $\mathbb{R}^n$ such that $y \circ f \circ x = \kappa$ on a neighbourhood of 0 on $\mathbb{R}^m$.

The proof is similar (albeit slightly messier) than Theorem 5.11, and we omit the details. Just as in Proposition 6.3, one can immediately translate this to a local statement about smooth maps between manifolds:

Corollary 6.19. *Let $\varphi\colon M \to N$ be a smooth map. Let $p \in M$ and assume there exists a neighbourhood of p such that φ has constant rank k on that neighbourhood. Then there exists a chart x on M about p and a chart y on N about $\varphi(p)$ such that $y \circ \varphi \circ x^{-1} = \kappa$, where κ is as in (6.6).*

Bonus Material for Lecture 6

In this bonus section we prove Theorem 6.10 and sketch the proof of Sard's Theorem 6.17.

Proof of Theorem 6.10. We prove the result in four steps.

1. Let us first fix some notation. Write $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n}$. Let ρ_1 and ρ_2 denote the two projections $\mathbb{R}^m \to \mathbb{R}^n$ and $\mathbb{R}^{m-n}$ respectively:

$$\rho_1(u^1, \dots, u^m) := (u^1, \dots, u^n), \qquad \rho_2(u^1, \dots, u^m) := (u^{n+1}, \dots, u^m),$$

and let $\jmath\colon \mathbb{R}^{m-n} \to \mathbb{R}^m$ denote the inclusion onto the last $m-n$ coordinates:

$$\jmath(u^1, \dots, u^{m-n}) = (0, \dots, 0, u^1, \dots, u^{m-n}).$$

Now let $y\colon V \to y(V) \subset \mathbb{R}^n$ denote a chart on N such that $y(q) = 0$. Fix a point $p \in L$ and let $x\colon U \to x(U) \subset \mathbb{R}^m$ denote a chart on M such that $x(p) = 0$. Then $y \circ \varphi \circ x^{-1}$ has maximal rank n at $0 \in \mathbb{R}^m$, and hence by part (ii) of the Implicit Function Theorem 5.11 there exists a chart z on $\mathbb{R}^m$, defined on an open ball $\mathcal{O}$ containing the origin such that

$$y \circ \varphi \circ x^{-1} \circ z|_{\mathcal{O}} = \rho_1|_{O}.$$

Shrinking $\mathcal{O}$ if necessary, we may assume $\mathcal{O} = \rho_1(\mathcal{O}) \times \rho_2(\mathcal{O}) \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$. Set $\Omega := \rho_2(O)$. Then

$$y \circ \varphi \circ x^{-1} \circ z \circ \jmath|_{\Omega} = \rho_1 \circ \jmath|_{\Omega} \equiv 0.$$

Thus if $\sigma := x^{-1} \circ z \circ \jmath|_{\Omega}$ then $\sigma(\Omega) \subset L$.

2. In this step we show that the inclusion $L \hookrightarrow M$ is a topological embedding in a neighbourhood of p. For this, it suffices to show that σ maps Ω homeomorphically onto a neighbourhood of p in L in the subspace topology. This means that we must prove:

$$\sigma(\Omega) = L \cap (x^{-1} \circ z)(\mathcal{O}). \tag{6.7}$$

It is clear that the left-hand side of (6.7) is contained in the right-hand side. Indeed, we have

$$\begin{aligned} \sigma(\Omega) &= (x^{-1} \circ z \circ \jmath)(\Omega) \\ &= (x^{-1} \circ z)\big(\mathcal{O} \cap \big(0 \times \mathbb{R}^{m-n}\big)\big) \\ &\subset L \cap (x^{-1} \circ z)(\mathcal{O}). \end{aligned}$$

To see the other direction, if $v \in L \cap (x^{-1} \circ z)(\mathcal{O})$ then $v = (x^{-1} \circ z)(u)$ for a unique $u \in \mathcal{O}$, and since

$$\begin{aligned} \rho_1(u) &= \big(y \circ \varphi \circ x^{-1} \circ z\big)(u) \\ &= y \circ \varphi(v) \\ &= 0, \end{aligned}$$

we can write $v = (0, v_0)$ for a unique $v_0 \in \Omega$. Then $v = \sigma(v_0)$. This proves the other inclusion, and hence establishes (6.7).

3. We now show that L is a smooth manifold. Using the notation from above, set $W := \sigma(\Omega)$ and let $w := \sigma^{-1}$. We shall show that $w\colon W \to \Omega$ can serve as a chart on L. More precisely, we claim that the collection of all such charts, as p ranges over L, determines a smooth structure on L. To see this, suppose p_1 was another point in L with corresponding chart $x_1\colon U_1 \to x_1(U_1) \subset \mathbb{R}^m$. Assume that $U \cap U_1 \neq \emptyset$. Let z_1 denote the corresponding diffeomorphism of $\mathbb{R}^m$, and define σ_1 and w_1 similarly. Then by assumption $x_1 \circ x^{-1}$ is a diffeomorphism where defined, and hence so is $\tau := z_1^{-1} \circ x_1 \circ x^{-1} \circ z$. Moreover from (6.7) we can write $\tau(0, u) = (0, \tau_1(u))$ for τ_1 a diffeomorphism defined on a neighbourhood of 0 in $\mathbb{R}^{m-n}$. Thus

$$w_1 \circ w^{-1} = \jmath^{-1} \circ \tau \circ \jmath = l_1$$

is a diffeomorphism where defined. This shows that we have built a smooth structure on L.

4. To complete the proof, we show that the inclusion $\iota\colon L \hookrightarrow M$ is smooth. For this we note that with x, w and z as above,

$$x \circ \iota \circ w^{-1} = x \circ \iota \circ \sigma = z \circ \jmath,$$

which is smooth. This completes the proof. ■

The Implicit Function Theorem also generalises to constant rank maps.

Theorem 6.20 (Constant Rank Implicit Function Theorem). *Suppose $\varphi\colon M \to N$ has constant rank k. Take $q \in \varphi(M)$ and set $L := \varphi^{-1}(q)$. Then L is a topological manifold of dimension $m - k$. Moreover there exists a smooth structure on L which makes L into a smooth embedded submanifold of M, and if $i\colon L \hookrightarrow M$ denotes the inclusion then for all $p \in L$, one has*

$$Di(p)(T_pL) = \ker D\varphi(p).$$

The proof of Theorem 6.20 proceeds in an analogous fashion to that of Theorem 6.10, only starting with Theorem 6.18 instead.

We conclude by giving a brief sketch of the proof of Sard's Theorem 6.17.

Proof of Sard's Theorem 6.17. The classical version of Sard's Theorem says that if $\mathcal{O} \subset \mathbb{R}^m$ is an open set and $f\colon \mathcal{O} \to \mathbb{R}^n$ is a smooth map, then the set of critical values of f has measure zero in $\mathbb{R}^n$. Since manifolds are second countable they can covered by countably many open sets that are diffeomorphic to balls in Euclidean spaces, cf. Proposition 1.32. As the countable union of measure zero sets is also of measure zero, the result follows. ■

A nice proof can be found in Chapter 3 of Milnor's classic textbook "Topology from a Differentiable Viewpoint".

Sard's Theorem is the main reason we require that manifolds have at most countably many components (cf. Proposition 1.32) – the theorem is false if this condition is not imposed.

LECTURE 7

The Whitney Theorems

In this lecture we will prove two famous theorems of Whitney. The first states that *every* smooth manifold can be embedded inside Euclidean space. Recall a continuous function $f\colon X \to Y$ between two topological spaces is **proper** if the preimage of any compact set in Y is compact in X. If X is compact and Y is Hausdorff then every continuous function is proper.

THEOREM 7.1 (The Strong Whitney Embedding Theorem). *Let M be a smooth manifold of dimension m. Then there exists a proper embedding $\varphi\colon M \to \mathbb{R}^{2m}$.*

Theorem 7.1 is a genuinely difficult result. It is much easier to prove that M always embeds in $\mathbb{R}^{2m+1}$ (this is sometimes called the "Weak Whitney Embedding Theorem"). This is still too hard for us, however, so we will prove this only for the special case of compact manifolds. We call this the "Baby Whitney Embedding Theorem".

THEOREM 7.2 (The Baby Whitney Embedding Theorem). *Let M be a compact smooth manifold of dimension m. Then there exists a (proper) embedding $\varphi\colon M \to \mathbb{R}^{2m+1}$.*

The "proper" is in parentheses, as this is automatic when M is compact.

Proof. We prove the result in four steps.

1. We begin by showing that M admits an embedding into *some* Euclidean space $\mathbb{R}^n$ (this method will typically produce a very large n). In the next step we will reduce n down to $2m + 1$. Since M is compact we can find a finite cover $\{V_1, \dots, V_k\}$ of open sets, with the property that there exist charts (U_i, x_i) for $i = 1, \dots, k$ with $\overline{V}_i \subset U_i$. Now let $\chi_i\colon M \to \mathbb{R}$ denote a bump function (whose existence is guaranteed by Lemma 3.2) such that $\chi_i(\overline{V}_i) \equiv 1$, $0 \le \chi_i(p) \le 1$ for all $p \in M$ and $\operatorname{supp}(\chi_i) \subset U_i$. Set $f_i = \chi_i\, x_i$, which we think of as a function from $M \to \mathbb{R}^m$ by extending it to be zero outside of U_i. Then define

$$\varphi\colon M \to \mathbb{R}^{(m+1)k}, \qquad \varphi(x) := \Big(f_1(x), \dots, f_k(x), \chi_1(x), \dots, \chi_k(x)\Big).$$

We claim that φ is an injective immersion. Since M is compact, it then follows from Problem C.3 that φ is an embedding. To see that φ is injective, suppose $\varphi(p) = \varphi(q)$. Since the sets $\{V_i\}$ cover M, there is some i such that $p \in V_i$, and hence $\chi_i(p) = 1$. Since $\varphi(p) = \varphi(q)$, we also have $\chi_i(q) = 1$, and thus $q \in \operatorname{supp}(\chi_i) \subset U_i$. Then also

$$x_i(p) = \chi_i(p)x_i(p) = f_i(p) = f_i(q) = \chi_i(q)x_i(q) = x_i(q).$$

But x_i is a diffeomorphism, and hence in particular injective. Thus $p = q$.

Finally, to check φ is an immersion, pick an arbitrary $p \in M$. Then $p \in V_i$ for some i. Since $\chi_i \equiv 1$ on a neighbourhood of p, we have $Df_i(p) = Dx_i(p)$, which is injective. Thus also $D\varphi(p)$ is injective. This completes the proof of the weak version we wished to prove, where we took $n = (m+1)k$.

2. Replacing M by $\varphi(M)$, we now have $M \subset \mathbb{R}^n$. Assume that $n > 2m+1$, otherwise there is nothing to prove. Think of $\mathbb{R}^{n-1}$ as sitting inside $\mathbb{R}^n$ as the hyperplane $\{(u^1, \dots, u^n) \mid u^n = 0\}$. Given $\xi \in \mathbb{R}^n \setminus \mathbb{R}^{n-1}$, let $\rho_\xi \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ denote the projection parallel to ξ, that is, the unique linear map with

$$\ker \rho_\xi = \mathbb{R}\cdot\xi.$$

We will look for unit vectors ξ with the property that

$$\rho_\xi|_M \colon M \to \mathbb{R}^{n-1}$$

is an embedding. Using Problem C.3 again, it suffices to show that $\rho_\xi|_M$ is an injective immersion. But what does that mean in this context? In words, saying that $\rho_\xi|_M$ is injective is saying that ξ is not parallel to any secant of M, that is,

Throughout this lecture, the norm $\|\cdot\|$ denotes the standard Euclidean norm. This is true both for points in M and for points in $TM \subset T\mathbb{R}^n = \mathbb{R}^{2n}$.

$$\xi \neq \frac{p-q}{\|p-q\|}, \qquad \forall\, p, q \in M. \tag{7.1}$$

The kernel of the linear map ρ_ξ is the line through ξ. Since ρ_ξ is linear, its derivative is the same linear map. Thus a tangent vector $\zeta \in T_pM$ lies in the kernel of $D\rho_\xi(p)$ if and only if ζ is parallel to ξ. We therefore see that ρ_ξ is an immersion if

$$\xi \neq \frac{\zeta}{\|\zeta\|}, \qquad \forall\, \zeta \in T_pM,\ \forall\, p \in M. \tag{7.2}$$

3. We will use Sard's Theorem 6.17 to prove a ξ exists such that both (7.1) and (7.2) are satisfied. For (7.1), consider the map

$$\psi \colon (M \times M) \setminus \Delta \to S^{n-1}, \qquad (p,q) \mapsto \frac{p-q}{\|p-q\|}.$$

Here Δ is the *diagonal* inside $M \times M$:

$$\Delta := \{(x,x) \mid x \in M\}.$$

Clearly ξ satisfies (7.1) if and only if ξ is *not* in the image of ψ. Note that $(M \times M) \setminus \Delta$ is an open set of $M \times M$, and thus $(M \times M) \setminus \Delta$ is a manifold of dimension $2m$ by Lemma 1.15 and Problem A.3. The map ψ is visibly smooth. Since $2m < n-1 = \dim S^{n-1}$, by Sard's Theorem 6.17 the image of ψ is nowhere dense in S^{n-1}. Thus in particular, any non-empty open set of S^{n-1} contains a point ξ satisfying (7.1).

Now we consider (7.2). It suffices to check that it holds for all vectors ζ of norm 1. To this end we focus on the *unit tangent bundle*

$$SM := \{(p,\zeta) \in TM \mid \|\zeta\| = 1\}.$$

We will come back to unit tangent bundles next semester when we discuss Riemannian geometry. To see this is a manifold, consider the map $h\colon T\mathbb{R}^n \to \mathbb{R}$ given by $h(p,\zeta) = \|\zeta\|^2$. It is easy to see that 1 is a regular value of $h|_{TM}$ and that $SM = h|_{TM}^{-1}(1)$. Thus by the Implicit Function Theorem 6.10, SM is a manifold of dimension $2m-1$. Moreover since M is compact so is SM. Since $M \subset \mathbb{R}^n$, we have

$$TM \subset M \times \mathbb{R}^n \subset \mathbb{R}^{2n} = T\mathbb{R}^n$$

See part (i) of Problem D.1 if you are worried about the identification $T\mathbb{R}^n \cong \mathbb{R}^{2n}$.

and similarly SM is identified with a subset of $M \times S^{n-1}$. Projecting onto the second factor, this gives us a map

$$SM \to M \times S^{n-1} \to S^{n-1}$$

which geometrically takes a unit vector based at a point in M and translates it to a unit vector based at the origin in $\mathbb{R}^n$. Using Sard's Theorem 6.17 again, the image of the composite map $SM \to S^{n-1}$ is nowhere dense. Since SM is compact, it follows that the complement – let us call it W – of the image is a dense open set in S^{n-1}. Thus W meets $S^{n-1} \cap \left(\mathbb{R}^n \setminus \mathbb{R}^{n-1}\right)$ in an non-empty open set W_0. From what we already proved, such a non-empty open set W_0 contains a vector ξ which is not in the image of ψ.

4. We now complete the proof. The choice of ξ found above gives us an embedding $\rho_\xi\colon M \to \mathbb{R}^{n-1}$. If $n-1 = 2m+1$ we are done, if not then $n-1 > 2m+1$, and the same argument again works to provide a new embedding in $\mathbb{R}^{n-2}$. By induction, we eventually obtain our desired embedding $M \to \mathbb{R}^{2m+1}$. ■

Remark 7.3. Extending Theorem 7.2 to cover all smooth manifolds (not just compact ones) is not that much more work. We emphasise though that the stronger result (Theorem 7.1, where $2m+1$ is reduced down to $2m$) is *much* harder.

This appears as bonus problem on Problem Sheet D.

Theorem 7.1 implies one could equivalently *define* a manifold as an embedded submanifold of Euclidean space.

Definition 7.4 (Alternative definition of a manifold). Let $m \le n$. A subset $M \subset \mathbb{R}^n$ is called a **smooth manifold of dimension** m if each point p in M has a neighbourhood V in $\mathbb{R}^n$ such that $M \cap V$ is diffeomorphic to an open set in $\mathbb{R}^m$.

This is how manifolds are defined in most "baby" courses on differential geometry.

In more detail, this means: for each point $p \in M$ there exists an open set $\mathcal{O} \subset \mathbb{R}^m$ and a neighbourhood $V \subset \mathbb{R}^n$ of p, together with an injective smooth map $\sigma\colon \mathcal{O} \to \mathbb{R}^n$ of maximal rank m everywhere such that $\sigma(\mathcal{O}) = M \cap V$ and $x := \sigma^{-1}|_{M\cap V}\colon M \cap V \to \mathcal{O}$ is continuous (where M is given the subspace topology of $\mathbb{R}^n$). One usually calls σ a **parametrisation** of M. The inverse x of σ is then a chart on M in the normal sense. Note that if $m = n$ then this forces M to be an open subset of $\mathbb{R}^n$, and hence if M is compact then one necessarily has $m < n$.

Remark 7.5. Definition 7.4 is superficially much simpler than our original definition (Definition 1.13)—there is no need to first define topological manifolds, or even mention metrisability and separability. The equivalence of the definitions follows from Theorem 7.1 and the existence of slice charts (Definition 6.6). Moreover it is immediate from Definition 7.4 that manifolds are metrisable, since any subset of a metric space inherits a metric that determines its subspace topology.

You might therefore reasonably ask: was there any point in the abstract definition? The answer is of course **"yes"**, as we will now try to explain.

An embedded submanifold of Euclidean space should really be thought of as a pair (M, φ), where M is an (abstract) smooth manifold and φ is a choice of embedding. However it is possible to embed a given manifold in many different ways, and moreover if you can embed M in $\mathbb{R}^n$ then you can also embed M in $\mathbb{R}^k$ for any $k \geq n$. A different choice of embedding can lead to dramatically different geometry (this will be particularly evident when we study *Riemannian geometry* next semester). Thus when proving results about embedded submanifolds, one always needs to ask the question: is this proof really a statement about the manifold itself, or does it depend on the embedding? This can often vastly complicate the proofs. The upshot is that having a more complicated definition leads to simpler proofs, and hence in the long run – since you only need to define things once but there are many theorems to prove! – it is better to work with the abstract definition whenever possible.

Still another reason to prefer the abstract definition is the following: One of the key applications of differential geometry in theoretical physics is Einstein's theory of *General Relativity*. Here one views the universe as 4-dimensional (curved) space-time. In the finite universe model, the spacial part of space-time is taken to be compact 3-dimensional hyperbolic manifold. Since (by definition) the universe is "everything", it doesn't make any sense *at all* to require the theory to begin by embedding the universe in a larger Euclidean space...

We now aim to prove another theorem, also due to Whitney, called the *Whitney Approximation Theorem*, that allows us replace a continuous map with a smooth one. We begin with the following statement, which says a continuous function from a manifold to a Euclidean space can be approximated arbitrarily well by a smooth one.

PROPOSITION 7.6. *Let M be a smooth manifold and let $h \colon M \to \mathbb{R}^n$ be a continuous function. Given any positive continuous function $\delta \colon M \to \mathbb{R}$, there exists a smooth function $f \colon M \to \mathbb{R}^n$ such that*

$$\|f(p) - h(p)\| < \delta(p), \qquad \forall\, p \in M.$$

Proof. Fix $p \in M$ and let U_p be a neighbourhood of p such that for all $q \in U_p$, one has

$$\delta(q) > \frac{1}{2}\delta(p), \qquad \|h(q) - h(p)\| < \frac{1}{2}\delta(p).$$

Such a neighbourhood exists as h and δ are assumed to be continuous.

Then in particular we have that

$$\|h(q) - h(p)\| < \delta(q), \qquad \forall\, q \in U_p.$$

The collection $\{U_p \mid p \in M\}$ is an open cover of M. Let $\{\kappa_p \mid p \in M\}$ be a partition of unity subordinate to this open cover and define

$$f \colon M \to \mathbb{R}^n, \qquad f(q) := \sum_{p \in M} \kappa_p(q) h(p).$$

Recall that the right-hand side is actually a finite sum at every point, since $\{\mathrm{supp}(\kappa_p)\}$ is locally finite, and hence f is smooth. Moreover since $\sum_p \kappa_p \equiv 1$ and $\mathrm{supp}(\kappa_p) \subset U_p$, one has for any $q \in M$ that

$$\begin{aligned}
\|f(q) - h(q)\| &= \Big\| \sum_{p \in M} \kappa_p(q) h(p) - h(q) \Big\| \\
&= \Big\| \sum_{p \in M} \kappa_p(q) h(p) - \sum_{p \in M} \kappa_p(q) h(q) \Big\| \\
&\le \sum_{p \in M} \kappa_p(q) \|h(q) - h(p)\| \\
&< \sum_{p \in M} \kappa_p(q) \delta(q) = \delta(q).
\end{aligned}$$

This completes the proof. ■

Our aim now is to improve Proposition 7.6 to the case where the target space is another manifold, not a Euclidean space. The "obvious" tactic (given that we just proved the Whitney Embedding Theorem) is to embed the target manifold in a Euclidean space, and then approximate via the result we just proved. Unfortunately this doesn't quite work, as even though the function f can be chosen to be very close to h, it may still be the case that f "misses" our newly embedded manifold (remember an embedded manifold is not an open subset unless it is of full dimension). Thus we need a way to correct this. We will do this by making use of *tubular neighbourhoods*, which will be defined shortly.

Definition 7.7. Let M be an embedded submanifold of $\mathbb{R}^n$. We define the **normal space to M at p** to be the $(n-m)$-dimensional subspace $\mathrm{Nor}_p M \subset T_p\mathbb{R}^n$ consisting of all vectors that are orthogonal to T_pM with respect to the Euclidean dot product. We define the **normal bundle** of M as the set

$$\mathrm{Nor}\, M := \{(p, \xi) \in T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \mid p \in M, \xi \in \mathrm{Nor}_p M\}.$$

On Problem Sheet C you are asked to prove that $\mathrm{Nor}\, M$ is an embedded n-dimensional submanifold of $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. We define a map

$$T \colon \mathrm{Nor}\, M \to \mathbb{R}^n, \qquad T(p, \xi) := p + \xi.$$

We emphasise that this only makes sense as M is embedded in $\mathbb{R}^n$. In general one cannot add points together on a manifold! The map T is smooth, since it is the restriction to $\mathrm{Nor}\, M$ of the addition map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. If O_M denotes the **zero section**:

The explanation for the name "zero section" will come in Lecture 10, when we discuss vector bundles.

$$O_M := \{(p, 0) \mid p \in M\}.$$

then one has

$$T(O_M) = M.$$

Thus it is reasonable to hope that a small neighbourhood of O_M in $\operatorname{Nor} M$ gets mapped under T to a small neighbourhood of M in $\mathbb{R}^n$. This motivates the following definition.

DEFINITION 7.8. A **tubular neighbourhood of** M is a neighbourhood U of M in $\mathbb{R}^n$ which is the diffeomorphic image under T of an open subset $V \subset \operatorname{Nor} M$ of the form

$$V = \big\{(p, \xi) \in \operatorname{Nor} M \mid \|\xi\| < \varepsilon(p)\big\}, \tag{7.3}$$

where $\varepsilon\colon M \to \mathbb{R}$ is a strictly positive continuous function.

It is a non-trivial fact that such neighbourhoods always exist:

THEOREM 7.9 (The Tubular Neighbourhood Theorem). *Every embedded submanifold $M \subset \mathbb{R}^n$ admits a tubular neighbourhood.*

The proof is deferred to the bonus section below.

REMARK 7.10. Next semester we will define another "tubular neighbourhood" associated to compact submanifold M of any Riemannian manifold N. This is more general than the construction discussed here, since N does not have to be equal to a Euclidean space.

DEFINITION 7.11. Let $Y \subset X$ be a subspace of a topological space. A **retraction** of X onto Y is a continuous map $r\colon X \to Y$ such that $r|_Y$ is the identity map on Y.

COROLLARY 7.12. *Let $M \subset \mathbb{R}^n$ be an embedded submanifold, and let U be a tubular neighbourhood of M. There exists a smooth map $r\colon U \to M$ which is both a retraction and a submersion.*

Proof. Let $T\colon V \subset \operatorname{Nor} M \to U$ be our tubular neighbourhood, and write $\pi\colon \operatorname{Nor} M \to M$ for the footpoint map that sends a pair (p, ξ) to p. Define $r\colon U \to M$ by $r := \pi \circ T^{-1}|_U$. Since $T|_V$ is a diffeomorphism and π is clearly a submersion, it follows that r is a submersion. Finally since $T(p, 0) = p$, we see that

$$r(p) = \pi \circ T^{-1}(p) = p,$$

and hence r is a retraction. ■

Recall that if $h_0, h_1\colon X \to Y$ are two continuous maps, we say they are **homotopic** if there exists a continuous map $H\colon X \times [0, 1] \to Y$ such that $H(\cdot, 0) = h_0$ and $H(\cdot, 1) = h_1$. We can now state and prove another result due to Whitney. We will use this result later on in the course when we discuss the *homotopy invariance of de Rham cohomology* in Lecture 23.

THEOREM 7.13 (The Whitney Approximation Theorem). *Let $h\colon M \to N$ be a continuous map between two smooth manifolds. Then h is homotopic to a smooth map $\varphi\colon M \to N$.*

Proof. By the Whitney Embedding Theorem 7.1, we may assume that N is a properly embedded submanifold of some Euclidean space $\mathbb{R}^n$. Let U be a tubular neighbourhood of N, and let $r\colon U \to N$ be a smooth submersive retraction (whose existence is guaranteed by Corollary 7.12). Given $p \in N$, let

$$0 < \varepsilon(p) := \sup\big\{\varepsilon \le 1 \mid B_\varepsilon(p) \subset U\big\},$$

where $B_\varepsilon(p)$ denotes the ball of radius ε about p in the Euclidean norm. We claim that ε is actually a continuous function. To see this let $p, q \in N$ and first suppose that $\|p-q\| < \varepsilon(p)$. Then for $\delta := \varepsilon(p) - \|p-q\|$, one has by the triangle inequality that $B_\delta(q) \subset B_{\varepsilon(p)}(p)$, and hence $\varepsilon(q) \ge \varepsilon(p) - \|p-q\|$. Thus if $\|p-q\| < \varepsilon(p)$ then

$$\varepsilon(p) - \varepsilon(q) \le \|p-q\|.$$

On the other hand, if $\varepsilon(p) \le \|p-q\|$ then since $\varepsilon(q) > 0$ by definition, one trivially also has

$$\varepsilon(p) - \varepsilon(q) \le \|p-q\|.$$

Reversing the roles of p and q shows that

$$\|\varepsilon(p) - \varepsilon(q)\| \le \|p-q\|,$$

which proves ε is continuous. Now define

$$\delta := \varepsilon \circ h\colon M \to \mathbb{R}.$$

Then δ is a continuous positive function, and hence by Proposition 7.6, there exists a smooth function $f\colon M \to \mathbb{R}^n$ such that

$$\|f(p) - h(p)\| < \delta(p), \qquad \forall\, p \in M.$$

Define

$$H\colon M \times [0,1] \to N, \qquad H(p,t) := r((1-t)h(p) + tf(p)).$$

This is well-defined due to our choice of function δ, which implies that $(1-t)h(p) + tf(p) \in U$ for all $t \in [0,1]$. Since r is the identity on $N \subset U$ and h takes values in N, we see that $H(\cdot, 0) = h$. Moreover if $\varphi := r \circ f$ then φ is smooth and $H(\cdot, 1) = \varphi$. This completes the proof. ∎

Bonus Material for Lecture 7

In this bonus section we make some additional remarks about the Strong Whitney Embedding Theorem 7.1, prove the Tubular Neighbourhood Theorem, and finally discuss an improvement to the Whitney Approximation Theorem.

Remarks 7.14.

(i) The Whitney Embedding Theorem is sharp in the sense that if $m = 2^k$ then $\mathbb{R}P^m$ cannot be embedded in $\mathbb{R}^{2m-1}$. This can be proved using characteristic classes. We will come back to this in Differential Geometry II.

(ii) There are various other versions of the Whitney Embedding Theorem. For instance, if M is a compact *orientable* smooth manifold of dimension m (we will define orientability in Lecture 24) then M embeds inside $\mathbb{R}^{2m-1}$. This does not contradict the previous statement, since for m even $\mathbb{R}P^m$ is not orientable.

(iii) In many cases the upper bound can be improved—for instance, we in Lecture 1 we saw that S^m embeds into $\mathbb{R}^{m+1}$. Another result (due to Haefliger) is that if M is a compact smooth manifold of dimension m whose homotopy groups $\pi_i M$ vanish for $i \leq k$ then if $2k+3 \leq m$ one can embed M in $\mathbb{R}^{2m-k}$. In general, if eM denotes the optimal n such that M embeds inside $\mathbb{R}^n$ then computing eM is an open problem for many manifolds M.

Let us now prove the Tubular Neighbourhood Theorem.

Proof of the Tubular Neighbourhood Theorem 7.9. We prove the result in four steps.

1. We will prove that $DT(p,0)$ is invertible at every point $(p,0) \in O_M$. Since $T|_{O_M} : O_M \to M$ is obviously a diffeomorphism, one sees that $DT(p,0)$ maps $T_{(p,0)}O_M \subset T_{(p,0)} \operatorname{Nor} M$ isomorphically onto T_pM. Secondly, if we restrict T to the fibre $\operatorname{Norm}_p M$, T just becomes the affine map $\xi \mapsto p + \xi$, and thus $DT(p,0)$ maps $T_{(p,0)} \operatorname{Norm}_p M$ isomorphically onto $\operatorname{Norm}_p M$ by Lemma 4.15.

Thus by the Inverse Function Theorem 5.10 we see that for each $p \in M$ there exists an $\varepsilon_p > 0$ such that if

$$U(p, \varepsilon_p) := \big\{(q,\xi) \in \operatorname{Nor} M \mid \|p-q\| < \varepsilon_p \text{ and } \|\xi\| < \varepsilon_p\big\}$$

then $T|_{U(p,\varepsilon_p)}$ is a diffeomorphism. To complete the proof we need to show that there is open set of the form (7.3) on which T is a global diffeomorphism.

2. Let $\varepsilon \colon M \to \mathbb{R}$ be the function that assigns to a point $x \in M$ the supremun of all $\varepsilon \leq 1$ such that T is a diffeomorphism on $U(x,\varepsilon)$. Then ε is strictly positive, as $\varepsilon(p) \geq \varepsilon_p$. We now claim that ε is actually a continuous function. This argument is essentially identical to the proof of the Whitney Approximation Theorem, but we give it again anyway. So suppose $p, q \in M$ and suppose that $\|p-q\| < \varepsilon(p)$. Then for $\delta := \varepsilon(p) - \|p-q\|$, one has by the triangle inequality that $U(y,\delta) \subset U(p,\varepsilon(p))$, and hence $\varepsilon(q) \geq \varepsilon(p) - \|p-q\|$. Thus if $\|p-q\| < \varepsilon(p)$ then

$$\varepsilon(p) - \varepsilon(q) \leq \|p-q\|.$$

On the other hand, if $\varepsilon(p) \leq \|p-q\|$ then since $\varepsilon(q) \geq 0$ by definition, one trivially also has

$$\varepsilon(p) - \varepsilon(q) \leq \|p-q\|.$$

Reversing the roles of p and q shows that

$$|\varepsilon(p) - \varepsilon(q)| \le \|p - q\|,$$

which proves ε is continuous.

3. Set

$$V := \left\{ (p, \xi) \in \operatorname{Nor} M \mid \|\xi\| < \frac{1}{2}\varepsilon(p) \right\}.$$

We claim that T is injective on V. Indeed, suppose (p, ξ) and (q, ζ) both belong to V and satisfy

$$p + \xi = T(p, \xi) = T(q, \zeta) = q + \zeta.$$

Without loss of generality, assume $\varepsilon(q) \le \varepsilon(p)$. Then

$$\begin{aligned} \|p - q| &= \|\xi - \zeta\| \\ &\le \|\xi\| + \|\zeta\| \\ &\le \frac{1}{2}\varepsilon(p) + \frac{1}{2}\varepsilon(p) \\ &= \varepsilon(p), \end{aligned}$$

where the first equality used $p + \xi = q + \zeta$. Thus both (p, ξ) and (q, ζ) belong to $U(p, \varepsilon(p))$. But on this set T is injective by construction. Thus $(p, \xi) = (q, \zeta)$ as required.

4. We complete the proof. Set $U := T(V)$. Then U is open as T is a local diffeomorphism. Since $T|_V$ is injective, we see that $T \colon V \to U$ is smooth bijection, and hence (as T is a local diffeomorphism), also a diffeomorphism. This completes the proof. ■

We conclude this lecture with a a couple of additional remarks about the Whitney Approximation Theorem. First, a definition.

Definition 7.15. Suppose M and N are smooth manifolds and $A \subset M$ is an arbitrary set. We say a map $\varphi \colon A \to N$ is **smooth on** A if it can be locally smoothly extended, i.e. if for every $p \in A$ there exists a neighbourhood U of p in M and a smooth map $\tilde{\varphi} \colon U \to N$ such that $\tilde{\varphi}|_{U \cap A} = \varphi$.

With a little bit more work, Theorem 7.13 can be improved to give the following statement:

Theorem 7.16. *Suppose $h \colon M \to N$ is a continuous map between two smooth manifolds. Suppose $A \subset M$ is a closed set and $h|_A$ is smooth in the sense of Definition 7.15. Then h is homotopic to a smooth map φ such that $\varphi|_A \equiv h|_A$. In fact, h and φ are homotopic via a homotopy H with the property that $H(p, t) = h(p)$ for all $t \in [0, 1]$.*

One can also play the same game with smooth homotopies. Two smooth maps $\varphi, \psi \colon M \to N$ are **smoothly homotopic** if there exists a smooth map $M \times [0, 1] \to N$ – note we are using Definition 7.15 again here to make sense of this – such that $H(\cdot, 0) = \varphi$ and $H(\cdot, 1) = \psi$.

Theorem 7.17 (The Homotopy Whitney Approximation Theorem). *If $\varphi, \psi \colon M \to N$ are two smooth maps between smooth manifolds which are homotopic (in the normal sense), then they are also smoothly homotopic. Moreover the given normal homotopy H from φ to ψ is stationary on some closed set A then the approximating smooth homotopy can also be chosen to be stationary on A.*

i.e. $H(p,t) = \varphi(p)$ for all $p \in A$ – note this implies $\varphi|_A \equiv \psi|_A$.

LECTURE 8

Vector Fields

In this lecture we will define **vector fields**, which are smooth sections of the tangent bundle. We first introduce the following standard notational convention, which will hold for the remainder of the course.

We deliberately chose to delay introducing this convention until now, so that you could all see how cumbersome proofs with multiple summation signs are (eg. Theorem 5.6), and thus fully appreciate the new convention!

The Einstein Summation Convention. If the same index appears *exactly* twice in any monomial, written once as an *upper* index and once as a *lower* index, then that term is understood to be summed over all possible values of that index. Here are two examples:

(i) If e_i denotes the standard ith basis vector in $\mathbb{R}^m$, then we write

$$v = a^i e_i \qquad \text{as an abbreviation for} \qquad v = \sum_{i=1}^m a^i e_i$$

(ii) If M is an m-dimensional smooth manifold, $p \in M$, and (x^i) are local coordinates about p, then for $\xi \in T_pM$ we write

$$\xi = a^i \frac{\partial}{\partial x^i}\Big|_p \qquad \text{as an abbreviation for} \qquad \xi = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i}\Big|_p$$

Here $\frac{\partial}{\partial x^i}$ is understood to have i as a lower index, despite the fact that x^i has i as an upper index, because it is on the bottom of a fraction.

This convention will vastly simplify equations throughout the course. For instance, when we start to talk about tensors, we will have cause to consider quantities which have local expressions such as

$$A = A^{ij}_{kl} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes dx^k \otimes dx^l,$$

which is much simpler than writing this abomination

$$A = \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^m \sum_{l=1}^m A^{ij}_{kl} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes dx^k \otimes dx^l.$$

The caveat is that in order for the convention to "work", the choice of whether to write a given quantity as an upper index or a lower index is *not* arbitrary.

DEFINITION 8.1. Let M be a smooth manifold and let $W \subset M$ be a non-empty open set (possibly equal to all of M). A **vector field** X on W is a smooth map $X\colon W \to TM$ (where we regard W as a smooth manifold in its own right) that satisfies the **section property**:

$$\pi(X(p)) = p, \qquad \forall\, p \in W, \tag{8.1}$$

where $\pi \colon TM \to M$ is the footpoint map. We denote by $\mathfrak{X}(W)$ the set of all vector fields on W.

Convention. Vector fields will typically be written with capital letters: X, Y and Z.

Equation 8.1 is equivalent to requiring that $X(p) \in T_pM$ for each $p \in W$. Thus a vector field can be thought of as a smoothly varying choice of tangent vector at each point.

Let us give various equivalent ways of expressing what smooth means in this context. Let $x \colon U \to \mathcal{O}$ be a chart on M, and suppose $X \colon U \to TM$ is *any* function satisfying the section property (8.1) (not necessarily smooth). Let $p \in U$. Since $\left\{ \frac{\partial}{\partial x^i}\big|_p \mid i = 1, \dots, m \right\}$ is a basis of T_pM, we can write

$$X(p) = X^i(p)\, \frac{\partial}{\partial x^i}\bigg|_p, \tag{8.2}$$

Note here we are using the Einstein Summation Convention to omit the $\sum_{i=1}^m$.

for some real numbers $X^i(p)$. If we do this for every point $p \in U$, we can think of the X^i as defining functions $X^i \colon U \to \mathbb{R}$. In general these functions X^i need not even be continuous, but as we will shortly see, if X is smooth (i.e. a vector field on U) then the X^i are actually smooth functions.

Here is yet another way to think about it. Suppose $f \in C^\infty(U)$, and let as before X denote any map $U \to TM$ satisfying the section property. Then for any given $p \in U$, thinking of $X(p)$ as a derivation of $C^\infty(U)$ at p, we can feed f to $X(p)$ to get a number $X(p)(f)$. This gives us a function $X(f) \colon U \to \mathbb{R}$:

$$X(f)(p) := X(p)(f), \qquad \forall\, p \in U. \tag{8.3}$$

Once again, if X is just any map satisfying the section property then $X(f)$ will not in general even be continuous. However if X is smooth (i.e. a vector field) then $X(f)$ is smooth.

Proposition 8.2. *Let M be a smooth manifold and let $W \subset M$ be a non-empty open set. Let $X \colon W \to TM$ be any function satisfying the section property* (8.1)*. Then the following are equivalent.*

(i) *X is a vector field on W.*

(ii) *If $x \colon U \to \mathcal{O}$ is any chart on M with $U \subset W$ then the functions X^i defined in* (8.2) *belong to $C^\infty(U)$.*

(iii) *If $V \subset W$ is any open set (possibly equal to all of W) and $f \in C^\infty(V)$ then the function $X(f)$ defined by* (8.3) *also belongs to $C^\infty(V)$.*

Proof. We begin with proving that (i) $\Leftrightarrow$ (ii). Let $p \in W$, and let $x \colon U \to \mathcal{O}$ be a chart about p. By definition, the function X^i defined in (8.2) is smooth if and only if $X^i \circ x^{-1}\colon \mathcal{O} \to \mathbb{R}$ is smooth in the normal sense. Note that by Proposition 3.8 and the definition of dx^i the function $X^i \circ x^{-1}$ can alternatively be written as

$$X^i \circ x^{-1} = dx^i \circ X \circ x^{-1} \tag{8.4}$$

The right-hand side of this equation should be understood as the function $q \mapsto dx^i_{x^{-1}(q)}\big(X(x^{-1}(q))\big)$ for $q \in \mathcal{O}$.

Now let us recall from the proof of Theorem 5.6 that a chart $x\colon U \to \mathcal{O}$ on M defines a chart $\tilde{x}\colon \pi^{-1}(U) \to O \times \mathbb{R}^m$ on TM by

$$\tilde{x}(p,\xi) = \big(x(p), dx^i_p(\xi)\, e_i\big), \qquad p \in U,\ \xi \in T_pM.$$

Note how prettier this formula is with the Einstein Summation Convention in effect.

By definition, X is smooth at p if and only if the composition

$$\tilde{x} \circ X \circ x^{-1}\colon \mathcal{O} \to \mathcal{O} \times \mathbb{R}^m$$

is smooth at $x(p)$. Explicitly this is the map

$$\mathcal{O} \to \mathcal{O} \times \mathbb{R}^m, \qquad q \mapsto \Big(q, dx^i_{x^{-1}(q)}\big(X(x^{-1}(q))\big)\, e_i\Big). \tag{8.5}$$

Applying (8.4) tells us that (8.5) can equivalently be written as

$$\mathcal{O} \to \mathcal{O} \times \mathbb{R}^m, \qquad q \mapsto \big(q, X^i(x^{-1}(q))e_i\big). \tag{8.6}$$

Thus (8.6) is smooth if and only if $X^i \circ x^{-1}$ is smooth for each $i = 1, \dots, m$. This proves (i) $\Leftrightarrow$ (ii).

Now let us prove (ii) $\Rightarrow$ (iii). Let $V \subset W$ and let $f \in C^\infty(V)$. Choose a chart $x\colon U \to \mathcal{O}$ with $U \subset V$. Then for $p \in U$, we have that

$$X(f)(p) = X^i(p)\, \frac{\partial}{\partial x^i}\Big|_p (f)$$

The function $p \mapsto \frac{\partial}{\partial x^i}\big|_p(f)$ is smooth – this is just the function $p \mapsto D_i(f \circ x^{-1})(x(p))$. By (ii) the X^i are also smooth functions, and hence $X(f)$ is a finite sum of the pointwise product of smooth functions and hence is smooth. Thus $X(f)$ is smooth on U. But since U was arbitrary and smoothness is a local property, it follows that $X(f)$ is smooth on all of V. This proves (iii).

Finally we note that (ii) is a special case of (iii): if $x\colon U \to \mathcal{O}$ is a chart about p with local coordinates (x^i), then the function X^i defined in (8.2) is simply the function $X(x^i)$, and thus (iii) implies X^i is smooth. This completes the proof. ∎

Warning: One must be careful with notation here: $X(x^i)$ is a function defined on X. Despite the suggestive notation, however, this is *not* the "composition" $X \circ x^i$. Indeed, the expression $X \circ x^i$ makes no sense at all, since x^i takes values in $\mathbb{R}$, and X cannot eat numbers. Thus

$$X(x^i) \neq X \circ x^i.$$

In contrast, the composition $X \circ x^{-1}$ from the left-hand side of (8.4) really does mean composition. Moreover one cannot "feed" x^{-1} to X to produce a function $X(x^{-1})$, since x^{-1} is not a smooth function on M.

The confusion could be avoided by simply declaring that we only use the notation $X(f)$ in the sense of a vector field eating a function, and only use the notation $X \circ x^{-1}$ to denote actual composition. In practice, however, whilst we will never write $X \circ f$ to denote the function $X(f)$, it is too cumbersome to only use composition notation for expressions such as $X \circ x^{-1}$. See for instance the right-hand sides of (8.5) and (8.6).

Thus in the future whenever you see the notation $X(w)$ for some object w, you should double check exactly what w is, before deciding on how to interpret $X(w)$.

Example 8.3. Suppose $x\colon U \to \mathcal{O}$ is a chart on M with local coordinates (x^i). Then we can think of $\frac{\partial}{\partial x^i}$ as defining a vector field on U via:

$$\frac{\partial}{\partial x^i}(p) := \frac{\partial}{\partial x^i}\Big|_p.$$

It is immediate from Proposition 8.2 that $\frac{\partial}{\partial x^i}$ is smooth.

We now introduce a notational convention that is both totally logical and somewhat confusing at the same time:

Definition 8.4. If $f \in C^\infty(U)$ then we denote the function $\frac{\partial}{\partial x^i}(f)$ from (8.3) with $X = \frac{\partial}{\partial x^i}$ by $\frac{\partial f}{\partial x^i}$. Thus $\frac{\partial f}{\partial x^i}$ is the function

$$\frac{\partial f}{\partial x^i}(p) := \frac{\partial}{\partial x^i}(p)(f) = \frac{\partial}{\partial x^i}\Big|_p(f) = D_i(f \circ x^{-1})(x(p)).$$

If our given manifold is an open subset of $\mathbb{R}^m$ and x is the identity chart with local coordinates (u^i) then the notation $\frac{\partial f}{\partial u^i}$ is consistent with the "usual" definition of partial derivative.

Let us now continue with the general case, where $W \subset M$ is any non-empty open subset. The space $\mathfrak{X}(W)$ is a real vector space under pointwise addition:

$$(X + Y)(p) := X(p) + Y(p), \qquad (cX)(p) := cX(p)$$

In fact, $\mathfrak{X}(W)$ forms a module over the ring $C^\infty(W)$ by defining

$$(fX)(p) := f(p)X(p), \qquad X \in \mathfrak{X}(W),\ f \in C^\infty(W).$$

In order for this to be well-defined, one needs to know that eg. $X + Y$ is smooth and fX is smooth. This however is immediate from Proposition 8.2.

REMARK 8.5. Pay attention to the ordering. If $X \in \mathfrak{X}(W)$ and $f \in C^\infty(W)$ then $X(f)$ belongs to $C^\infty(W)$ whereas fX belongs to $\mathfrak{X}(W)$!

We now extend Definition 3.1 to derivations that are not based at a point.

DEFINITION 8.6. Let M be a smooth manifold and let $W \subset M$ be a non-empty open set. A **derivation on** $C^\infty(W)$ is a linear map

$$\mathcal{X} \colon C^\infty(W) \to C^\infty(W)$$

satisfying the *derivation property*

$$\mathcal{X}(fg) = f\mathcal{X}(g) + g\mathcal{X}(f), \qquad \forall\, f, g \in C^\infty(W).$$

Let us temporarily denote by $\mathfrak{X}^{\mathrm{deriv}}(W)$ the set of derivations on W. Observe that $\mathfrak{X}^{\mathrm{deriv}}(W)$ is another module over $C^\infty(W)$. It follows from Proposition 3.3 that any vector field $X \in \mathfrak{X}(W)$ defines a derivation $\mathcal{X} \in \mathfrak{X}^{\mathrm{deriv}}(W)$ via

$$\mathcal{X}(f) \coloneqq X(f)$$

In fact, the converse is true.

PROPOSITION 8.7. *Let M be a smooth manifold and let $W \subset M$ be a non-empty open set. Then $\mathfrak{X}^{\mathrm{deriv}}(W)$ and $\mathfrak{X}(W)$ are isomorphic as modules over $C^\infty(W)$.*

Proof. Suppose $\mathcal{X}$ is a derivation on $C^\infty(W)$. Fix $p \in W$. Then $\mathcal{X}$ defines a derivation on $C^\infty(W)$ at p, which we suggestively write as $X(p)$, via the formula

$$X(p)(f) \coloneqq \mathcal{X}(f)(p), \qquad \forall\, f \in C^\infty(W).$$

Proposition 3.3 tells us that we can then think of X as defining a map $W \to TM$ via $p \mapsto X(p)$. We claim that X is smooth, and hence defines a vector field on W. For this by part (iii) of Proposition 8.2, we need only check that $X(f)$ is smooth for any $f \in C^\infty(W)$. But by construction $X(f) = \mathcal{X}(f)$, which is smooth by assumption. ■

From now on we will identify a vector field $X \in \mathfrak{X}(W)$ with the corresponding derivation $\mathcal{X}$ of $C^\infty(W)$ and write both with Latin letters X. We will also abandon the notation $\mathfrak{X}^{\mathrm{deriv}}(W)$ and just write $\mathfrak{X}(W)$. Our next goal is to turn $\mathfrak{X}(W)$ into an algebra, that is, to have a bilinear operation

$$\mathfrak{X}(W) \times \mathfrak{X}(W) \to \mathfrak{X}(W).$$

The naive guess would be to try composition of derivations:

$$X \circ Y \colon C^\infty(W) \to C^\infty(W), \qquad (X \circ Y)(f) \coloneqq X(Y(f)).$$

Unfortunately, this is *not* a derivation. Indeed, if we take $f, g \in C^\infty(W)$ and compute:

$$\begin{aligned}(X \circ Y)(fg) &= X\Big(fY(g) + gY(f)\Big) \\ &= \Big(f(X \circ Y)(g) + g(X \circ Y)(f)\Big) + \Big(X(f)Y(g) + X(g)Y(f)\Big)\end{aligned}$$

However, observe that the "error" term $X(f)Y(g) + X(g)Y(f)$ is symmetric in X and Y. This means that if we consider the **commutator**

$$[X, Y] := X \circ Y - Y \circ X$$

then the error term cancels, and thus $[X, Y]$ is a derivation. We have thus justified the following definition.

DEFINITION 8.8. Let $X, Y \in \mathfrak{X}(W)$. Then the commutator $[X, Y] := X \circ Y - Y \circ X$ is another derivation. We call $[X, Y]$ the **Lie bracket** of X and Y.

REMARK 8.9. **Warning:** A few authors define the Lie bracket with the *opposite* sign: $[X, Y] := Y \circ X - X \circ Y$. From a "high-level" point of view, this other sign convention is actually the "correct" one, but this requires a little bit of infinite-dimensional Lie group theory to understand, as we will explain at the end of Lecture 13. The convention we are using, namely $[X, Y] := X \circ Y - Y \circ X$, is consistent with the majority of the literature.

Notably, Joel Robbin and Dietmar Salamon use the other sign convention in their wonderful lecture notes.

The next proposition gives a formula for $[X, Y]$ in coordinates. The proof is deferred to Problem Sheet D.

PROPOSITION 8.10. *Let (U, x) be a chart on M with local coordinates (x^i) and let $X, Y \in \mathfrak{X}(U)$. Write $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^i \frac{\partial}{\partial x^i}$. Then*

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j},$$

where $\frac{\partial Y^j}{\partial x^i}$ and $\frac{\partial X^j}{\partial x^i}$ are the functions from Definition 8.4.

In order to explain the name, we need an algebraic definition.

DEFINITION 8.11. A (real) **Lie algebra** is a vector space $\mathfrak{g}$ endowed with a bilinear operation called the **Lie bracket**

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \qquad (v, w) \mapsto [v, w]$$

which in addition is antisymmetric, $[v, w] = -[w, v]$ and satisfies the **Jacobi identity**

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0, \qquad \forall\, u, v, w \in \mathfrak{g}.$$

Thus a Lie algebra is a non-associative algebra. The name "Lie" comes from the Norwegian mathematician Sophus Lie. It is traditional to write Lie algebras using fraktur symbols $\mathfrak{g}$ and $\mathfrak{h}$. The **dimension** of a Lie algebra $\mathfrak{g}$ is simply the dimension of $\mathfrak{g}$ as a vector space. If $\mathfrak{g}$ is a Lie algebra then a linear subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a **Lie subalgebra** if $[v, w] \in \mathfrak{h}$ for all $v, w \in \mathfrak{h}$.

Example 8.12. Here are some examples of Lie algebras:

(i) The cross product $[x, y] := x \times y$ makes $\mathbb{R}^3$ into a 3-dimensional Lie algebra.

(ii) The set $\mathrm{Mat}(n)$ of $n \times n$ matrices is an n^2-dimensional Lie algebra under the normal commutator $[A, B] := AB - BA$.

(iii) If V is any vector space then we can turn V into a (rather boring) Lie algebra by defining $[v, w] := 0$. Such an Lie algebra is called **abelian**.

You will probably not be surprised to learn we have just constructed another example:

Theorem 8.13. *Let M be a smooth manifold and let $W \subset M$ be an open set. Then $\mathfrak{X}(W)$ is a Lie algebra.*

Proof. The only thing left to check is the Jacobi identity. This is Problem D.3 on Problem Sheet D. ■

Remark 8.14. As long as $\dim M > 0$ then for any non-empty open subset W, $\mathfrak{X}(W)$ is an infinite-dimensional Lie algebra. To see this, we need only note that $\mathfrak{X}(W)$ is a module over $C^\infty(W)$, and $C^\infty(W)$ is an infinite-dimensional vector space (cf. Lemma 2.13).

We conclude this lecture by looking at how functions and vector fields can be "pushed forward" with a diffeomorphism.

Definition 8.15. Let $\varphi \colon M \to N$ be a diffeomorphism. We define an algebra homomorphism

Warning! Many authors use the notation φ_* for the derivative $D\varphi$.

$$\varphi_* \colon C^\infty(M) \to C^\infty(N), \qquad f \mapsto \varphi_*(f)$$

where

$$\varphi_*(f) := f \circ \varphi^{-1}.$$

The claim that φ_* is an algebra homomorphism is just the assertion that

$$\varphi_*(f+g) = \varphi_*(f)+\varphi_*(g), \qquad \varphi_*(fg) = \varphi_*(f)\varphi_*(g), \qquad \varphi_*(cf) = c\varphi_*(f)$$

for all $f, g \in C^\infty(M)$ and $c \in \mathbb{R}$, which is immediate from the definitions.

Definition 8.16. Suppose $\varphi \colon M \to N$ is a diffeomorphism and $X \in \mathfrak{X}(M)$. We define the **pushforward** vector field $\varphi_* X \in \mathfrak{X}(N)$ by defining

$$(\varphi_* X)(q) := D\varphi(\varphi^{-1}(q))X(\varphi^{-1}(q)).$$

To check this is well defined, we need to know that (a) $(\varphi_* X)(q) \in T_q N$ for each $q \in N$, which is obvious, and (b) that $\varphi_* X \colon N \to TN$ is smooth. The latter holds because it is simply the composition

$$N \xrightarrow{\varphi^{-1}} M \xrightarrow{X} TM \xrightarrow{D\varphi} TN$$

of smooth maps, and hence is smooth.

The map φ_* is again linear:

$$\varphi_*(X+Y) = \varphi_* X + \varphi_* Y, \qquad \forall\, X, Y \in \mathfrak{X}(M).$$

Moreover one has

$$\varphi_*(fX) = \varphi_*(f)\varphi_* X, \qquad \forall\, X \in \mathfrak{X}(M),\ \forall\, f \in C^\infty(M).$$

Remark 8.17. It may at first seem confusing that we have defined two different maps (one from functions to functions and one from vector fields to vector fields) and called them *both* φ_*. The reason for this will become clear when we discuss the **tensor algebra** $\mathcal{T}(M)$ of a manifold. Roughly speaking, the tensor algebra is a big direct sum:

$$\mathcal{T}(M) = \bigoplus_{h,k \geq 0} \mathcal{T}^{h,k}(M)$$

where $\mathcal{T}^{h,k}(M)$ denotes the *tensors of type* (h, k). As we will eventually see, a tensor of type $(0, 0)$ is simply a function, i.e.

$$\mathcal{T}^{0,0}(M) = C^\infty(M),$$

whereas a tensor of type $(1, 0)$ is a vector field:

$$\mathcal{T}^{1,0}(M) = \mathfrak{X}(M).$$

Given a diffeomorphism $\varphi\colon M \to N$, in Lecture 21 we will construct a single morphism

$$\varphi_*\colon \mathcal{T}(M) \to \mathcal{T}(N) \tag{8.7}$$

that preserves type, i.e.

$$\varphi_* \mathcal{T}^{h,k}(M) \subset \mathcal{T}^{h,k}(N).$$

The map φ_* from Definition 8.15 is the restriction of the morphism φ_* from (8.7) to $\mathcal{T}^{0,0}(M) \subset \mathcal{T}(M)$ and the map φ_* from Definition 8.16 is the restriction of the master φ_* from (8.7) to $\mathcal{T}^{1,0}(M) \subset \mathcal{T}(M)$. Thus it makes sense to denote them both by φ_*.

Definition 8.18. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras. A **Lie algebra homomorphism** is a linear map $\ell\colon \mathfrak{g} \to \mathfrak{h}$ which respects the Lie brackets, i.e.

$$[\ell v, \ell w] = \ell[v, w], \qquad \forall\, v, w \in \mathfrak{g},$$

where the left-hand side is the Lie bracket in $\mathfrak{h}$ and the right-hand side is the Lie bracket in $\mathfrak{g}$. A **Lie algebra isomorphism** is a bijective Lie algebra homomorphism whose inverse is also a Lie algebra homomorphism.

Proposition 8.19. *Let $\varphi\colon M \to N$ be a diffeomorphism. Then $\varphi_*\colon \mathfrak{X}(M) \to \mathfrak{X}(N)$ is a Lie algebra isomorphism.*

Proposition 8.19 is a special case of part (ii) of Problem D.6.

LECTURE 9

Flows

In this lecture we make contact with the theory of ordinary differential equations. A vector field defines an ordinary differential equation on a manifold, and just as in the Euclidean case, solutions to this ordinary differential equation exist (at least for a short time) and are unique. As with the Implicit Function Theorem, the local version of this statement follows readily from the corresponding Euclidean statement, but the global version is deeper. We begin by recalling two theorems from the theory of ordinary differential equations.

THEOREM 9.1 (Existence of solutions). *Let $\mathcal{O} \subset \mathbb{R}^m$ be open and let $f \colon \mathcal{O} \to \mathbb{R}^m$ be smooth. For any $p \in \mathcal{O}$ there exists a neighbourhood V of p and an open interval (a, b) with $a < 0 < b$, together with a smooth map $h \colon (a, b) \times V \to \mathcal{O}$ such that:*

(i) $h(0, q) = q$, *for all* $q \in V$,

(ii) *If we write*

$$\frac{d}{dt} h(t, q) := \lim_{s \to 0} \frac{h(t + s, q) - h(t, q)}{s}$$

then

$$\frac{d}{dt} h(t, q) = f(h(t, q)), \qquad \forall\, (t, q) \in (a, b) \times V.$$

We deliberately write $\frac{d}{dt}$ and not $\frac{\partial}{\partial t}$ to emphasise that this is ordinary differentiation.

Theorem 9.1 can be interpreted as follows. Suppose

$$\gamma = (\gamma^1, \dots, \gamma^m) \colon (a, b) \to \mathcal{O}$$

is a smooth curve. One calls γ an **integral curve** of $f = (f^1, \dots, f^m)$ if

$$(\gamma^i)'(t) = f^i \circ \gamma(t), \qquad \forall\, 1 \le i \le n. \tag{9.1}$$

Thus Theorem 9.1 tells us that integral curves $\gamma(t) = h(t, q)$ exist for arbitrary initial conditions $\gamma(0) = q$, and depend smoothly on their initial conditions. Moreover, they all locally exist for a common time (i.e. for every q in V, the integral curve with initial condition q lasts for all $t \in (a, b)$). Next, we address uniqueness of solutions.

THEOREM 9.2 (Uniqueness of solutions). *Let $\mathcal{O} \subset \mathbb{R}^m$ be open and let $f \colon \mathcal{O} \to \mathbb{R}^m$ be smooth. If $\gamma, \delta \colon (a, b) \to \mathcal{O}$ are two integral curves of f with $\gamma(t) = \delta(t)$ for some $t \in (a, b)$ then $\gamma \equiv \delta$.*

We will not prove either Theorem 9.1 or Theorem 9.2. They are both hopefully familiar to you from previous courses you took on ordinary differential equations. Instead, we will generalise them to manifolds.

DEFINITION 9.3. Let M be a manifold and let X be a vector field on M. Let $(a, b) \subset \mathbb{R}$ be an interval, and suppose $\gamma \colon (a, b) \to M$ is a smooth map. We say that γ is an **integral curve** of X if

$$\dot{\gamma}(t) = X(\gamma(t)), \qquad \forall\, t \in (a, b). \tag{9.2}$$

This definition is consistent with the usual one (9.1) in the special case where $M = \mathcal{O}$ is an open subset of $\mathbb{R}^m$. See Problem D.1. Before stating the next result, let us introduce a convention.

Definition 9.4. If M is a manifold and (a,b) is an interval then $(a,b) \times M$ is also a manifold. Given $p \in M$ we denote by $c_p \colon (a,b) \to (a,b) \times M$ the smooth curve in $(a,b) \times M$ defined by

$$c_p(t) := (t, p).$$

We denote by

$$\frac{\partial}{\partial t}\Big|_{(t,p)} := Dc_p(t)\left(\frac{\partial}{\partial t}\Big|_t\right) = \dot{c}_p(t) \tag{9.3}$$

the tangent vector in $T_{(t,p)}((a,b) \times M)$ obtained from the canonical generator $\frac{\partial}{\partial t}\big|_t \in T_t\mathbb{R}$. One can think of $(t,p) \mapsto \frac{\partial}{\partial t}\big|_{(t,p)}$ as defining a vector field on $(a,b) \times M$.

Exercise: Check this.

Theorem 9.5 (Local flow). *Let M be a smooth manifold and let $X \in \mathfrak{X}(M)$. For any $p \in M$ there exists a neighbourhood W of p and an interval (a,b) with $a < 0 < b$, together with a smooth map*

$$\Phi^{\mathrm{loc}} \colon (a,b) \times W \to M.$$

such that,

(i) $\Phi^{\mathrm{loc}}(0, q) = q$, *for all* $q \in W$.

(ii) *For all* $(t,q) \in (a,b) \times W$ *one has*

$$D\Phi^{\mathrm{loc}}(t,q)\left(\frac{\partial}{\partial t}\Big|_{(t,q)}\right) = X(\Phi^{\mathrm{loc}}(t,q)). \tag{9.4}$$

We call Φ^{loc} a **local flow** of X. We will shortly get rid of the “loc”.

Proof. Let $x \colon U \to \mathcal{O}$ be a chart around p with local coordinates (x^i). Let $\tilde{x} \colon \pi^{-1}(U) \to \mathcal{O} \times \mathbb{R}^m$ denote the corresponding chart on TM. Then we can write (cf. (8.5))

$$\tilde{x} \circ X \circ x^{-1} = (\mathrm{id}, f)$$

where $f \colon \mathcal{O} \to \mathbb{R}^m$ is smooth. Theorem 9.1 gives us a neighbourhood V of $x(p)$, an interval (a,b), and a smooth map $h \colon (a,b) \times V \to \mathcal{O}$ such that the two stated conditions hold. To complete the proof, set $W := x^{-1}(V)$ and define

Compare this to Problem D.1.

$$\Phi^{\mathrm{loc}}(t,q) := x^{-1} \circ h(t, x(q)), \qquad (t,q) \in (a,b) \times W.$$

That Φ^{loc} satisfies the two required conditions is immediate from the fact that h did. ∎

Remark 9.6. The condition (9.4) is simpler than it looks. Given $q \in W$, set $\gamma_q(t) := \Phi^{\mathrm{loc}}(t,q)$, so that $\gamma_q \colon (a,b) \to U$ is a curve in M. Then by definition

$$\dot{\gamma}_q(t) = D\Phi^{\mathrm{loc}}(t,q)\left(\frac{\partial}{\partial t}\Big|_{(t,q)}\right),$$

and so (9.4) asserts that γ_q is an integral curve of X.

A similar argument also proves the manifold version of Theorem 9.2:

Theorem 9.7. *Let M be a smooth manifold and let $X \in \mathfrak{X}(M)$. If $\gamma, \delta \colon (a,b) \to M$ are two integral curves of X with $\gamma(t) = \delta(t)$ for some $t \in (a,b)$ then $\gamma \equiv \delta$.*

Thanks to Theorem 9.7, it makes sense to talk about the maximal integral curve through a given point.

Definition 9.8. Let X be a vector field on M. Given a point $p \in M$, we denote by $\big(t^-(p), t^+(p)\big)$ the maximal interval around 0 on which the (unique by Theorem 9.7) integral curve $\gamma_p \colon \big(t^-(p), t^+(p)\big) \to M$ of X whose initial condition is $\gamma_p(0) = p$ is defined. We call γ_p the **maximal integral curve through** p.

Lemma 9.9. *Let X be a vector field on M. Fix $p \in M$ and $s \in (t^-(p), t^+(p))$ and set $q := \gamma_p(s)$. Then*

$$t^{\pm}(\gamma_p(s)) = t^{\pm}(p) - s, \qquad \forall\, s \in \big(t^-(p), t^+(p)\big). \tag{9.5}$$

and

$$\gamma_q(t) = \gamma_p(t+s), \qquad \forall\, t \in (t^-(q), t^+(q)). \tag{9.6}$$

Proof. This follows from uniqueness. The curve $t \mapsto \gamma_p(t+s)$ is an integral curve for X with initial condition q, and hence it is γ_q. Thus (9.6) follows, and hence so does (9.5). ■

We emphasise that $\big(t^-(p), t^+(p)\big)$ typically will be larger than the interval (a,b) given by Theorem 9.5 – indeed, by construction $\Phi^{\mathrm{loc}}(t,p)$ never leaves the open set U that the chart x was defined on. Thus whilst $\gamma_p(t) = \Phi^{\mathrm{loc}}(t,p)$ for small enough t, in general the curve γ_p could wander all over the manifold. Here is the global version of Theorem 9.5, which extends Φ to all of M so that the equality $\gamma_p(t) = \Phi(t,p)$ holds whenever the former is defined.

Theorem 9.10 (Maximal flow). *Let M be a smooth manifold and let $X \in \mathfrak{X}(M)$. There exists a unique open set $\mathcal{D}(X) \subset \mathbb{R} \times M$ and a unique smooth map $\Phi \colon \mathcal{D}(X) \to M$ such that*

(i) *For all $p \in M$ one has*

$$\mathcal{D}(X) \cap \big(\mathbb{R} \times \{p\}\big) = \big(t^-(p), t^+(p)\big) \times \{p\}.$$

(ii) *$\Phi(t,p) = \gamma_p(t)$ for all $(t,p) \in \mathcal{D}(X)$.*

We call Φ the **flow** of X. The proof is relegated to the bonus section at the end of this lecture, since it is rather fiddly.

We can also reverse the roles of t and p. Given $t \in \mathbb{R}$, set

$$M_t := \{p \in M \mid (t,p) \in \mathcal{D}(X)\}.$$

Then M_t is open in M and $M = \bigcup_{t>0} M_t$. Moreover there is a well-defined smooth map $\Phi_t \colon M_t \to M_{-t}$ given by

$$\Phi_t(p) := \Phi(t,p), \qquad p \in M_t$$

– the fact that Φ_t takes values in M_{-t} follows from Lemma 9.9. This map Φ_t is a diffeomorphism, since $\Phi_{-t}\colon M_{-t} \to M_t$ is an inverse. More generally, if $s, t \in \mathbb{R}$ then the domain of $\Phi_s \circ \Phi_t$ is contained in (though not necessarily equal to) M_{s+t}. If s and t have the same sign then we have equality. In any case, by Lemma 9.9 again one has $\Phi_s \circ \Phi_t = \Phi_{s+t}$ on the domain of $\Phi_s \circ \Phi_t$.

Restricting the maps Φ_t to open subsets of M is annoying. The next condition rules this out.

DEFINITION 9.11. A vector field X is **complete** if the set $\mathcal{D}(X)$ from Theorem 9.10 is all of $\mathbb{R} \times M$. Equivalently, a vector field is complete if either (a) its integral curves exist for all time or (b) the maps $\Phi_t := \Phi(t, \cdot)$ are all diffeomorphisms of the entire manifold M.

DEFINITION 9.12. We write $\mathrm{Diff}(M)$ for the set of diffeomorphisms $\varphi\colon M \to M$. Note that $\mathrm{Diff}(M)$ is actually a group under composition, where the identity element is just the identity map.

REMARK 9.13. Assume that M is compact. Then more is true: the group $\mathrm{Diff}(M)$ can itself be given a (Fréchet) manifold structure. We will say more about this in the bonus section to Lecture 13.

DEFINITION 9.14. A **one-parameter group of diffeomorphisms** is a smooth group homomorphism $\mathbb{R} \to \mathrm{Diff}(M)$. Writing this as $t \mapsto \Phi_t$, the group property tells us that

Here "smooth" should be interpreted as saying that $t \mapsto \Phi_t$ is a smooth map from the manifold $\mathbb{R}$ to the (infinite-dimensional) manifold $\mathrm{Diff}(M)$. In more down-to-earth language, this just means that $(t, p) \mapsto \Phi_t(p)$ is a smooth function $\mathbb{R} \times M \to M$.

$$\Phi_0 = \mathrm{id}, \qquad \Phi_{s+t} = \Phi_s \circ \Phi_t, \qquad \forall\, s, t \in \mathbb{R}.$$

If $\{\Phi_t\}$ is a one-parameter group of diffeomorphisms then we define its **infinitesimal generator** as the (necessarily complete) vector field

$$X(p) := D\Phi(0, p)\left(\frac{\partial}{\partial t}\Big|_{(0,p)}\right), \tag{9.7}$$

where we wrote $\Phi(t, p) := \Phi_t(p)$ and used the convention from (9.3). Then the flow of X is simply the one-parameter group Φ_t.

Theorem 9.10 and Lemma 9.9 thus give us:

PROPOSITION 9.15. *Let M be a smooth manifold. Then there is a bijective correspondence between one-parameter subgroups of diffeomorphisms and complete vector fields.*

EXAMPLE 9.16. Perhaps the easiest example of a *non-complete* vector field is given by taking $M = \mathbb{R}^2 \setminus 0$ and taking $X = \frac{\partial}{\partial u^1}$. If $(u^1, u^2) \in \mathbb{R}^2 \setminus 0$ then the flow line passing through (u^1, u^2) takes the form: $(u^1, u^2) \mapsto (t + u^1, u^2)$. It is then obvious that something must go wrong if you take $(u^1, u^2) = (-1, 0)$ and try and flow forwards – indeed, if the flow existed for all time then at time $t = 1$ you would fall out the manifold through the hole. . .

Exercise: Make this rigorous.

From now on we will switch between the notations $\Phi(t, p)$, $\Phi_t(p)$ and $\gamma_p(t)$ whenever convenient.

Here is an easy way to guarantee completeness.

Lemma 9.17. *Let X be a vector field on M. Assume there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset \big(t^-(p), t^+(p)\big)$ for all $p \in M$. Then X is complete.*

Proof. If not, there exists some $p \in M$ such that either $t^+(p) < \infty$ or $t^-(p) > -\infty$. Assume the former (the proof in the other case is almost identical). Choose a number t_0 such that $t^+(p) - \varepsilon < t_0 < t^+(p)$. Set $p_0 := \gamma_p(t_0)$. By assumption $\gamma_{p_0}(t)$ is defined for all $t \in (-\varepsilon, \varepsilon)$. Now consider the curve

$$\gamma(t) := \begin{cases} \gamma_p(t), & t^-(p) < t < t^+(p), \\ \gamma_{p_0}(t - t_0), & t_0 - \varepsilon < t < t_0 + \varepsilon. \end{cases}$$

These two definitions agree on the overlap, since

$$\begin{aligned} \gamma_{p_0}(t - t_0) &= \Phi_{t-t_0}(p_0) \\ &= \Phi_{t-t_0} \circ \Phi_{t_0}(p) \\ &= \Phi_t(p) \\ &= \gamma_p(t). \end{aligned}$$

This shows that γ is an integral curve for X with initial condition p which is defined on $\big(t^-(p), t_0 + \varepsilon\big)$. Since $t_0 + \varepsilon > t^+(p)$, this contradicts the maximality of $t^+(p)$. ∎

We define the **support** of a vector field X in exactly the same way as we define the support of a function:

$$\operatorname{supp}(X) := \overline{\{p \in M \mid X(p) \neq 0 \in T_pM\}}.$$

Corollary 9.18. *Let X be a vector field with compact support. Then X is complete.*

Proof. By Theorem 9.5 for each $p \in \operatorname{supp}(X)$ there exists a neighbourhood U_p of p and an interval $(-\varepsilon_p, \varepsilon_p)$ such that the flow is defined on $(-\varepsilon_p, \varepsilon_p) \times U_p$. Since $\operatorname{supp}(X)$ is compact, we may select finitely many points $p_1, \dots, p_k$ such that

$$\operatorname{supp}(X) \subset \bigcup_{i=1}^k U_{p_i}.$$

Now set

$$\varepsilon := \min_{i=1,\dots,k} \varepsilon_{p_i}.$$

Then for every $p \in \operatorname{supp}(X)$ one has $(-\varepsilon, \varepsilon) \subset \big(t^-(p), t^+(p)\big)$. Since X is identically zero on $M \setminus \operatorname{supp}(X)$, every integral curve of X starting at some point in $M \setminus \operatorname{supp}(X)$ is trivially defined for all $t \in \mathbb{R}$. Thus the hypotheses of Lemma 9.17 are satisfied, and the proof is complete. ∎

It is the constant curve.

Corollary 9.19. *If M is compact then every vector field on M is complete.*

Proof. If M is compact then certainly every vector field has compact support. ∎

We conclude with another variant on Lemma 9.17, which is sometimes more useful.

LEMMA 9.20. *Let X be a vector field on M. If the maximal domain of an integral curve γ_p is not all of $\mathbb{R}$, then the image of that curve cannot be contained in any compact subset of M.*

Proof. Assume for instance that $t^+(p) < \infty$ and that γ_p is contained in a compact set K. Choose a sequence t_n such that $t_i \to t^+(p)$ from below. By compactness, $\gamma_p(t_i)$ converges to some point p_0. By Theorem 9.5, a local flow Φ^{loc} of X is defined on $(-\varepsilon, \varepsilon) \times U$ for some $\varepsilon > 0$ and some neighbourhood U of p_0. Choose i large enough so that $\gamma_p(t_i) \in U$ and $t_i + \varepsilon > t^+(p)$. Then arguing just as in the proof of Lemma 9.17, the curve

$$\gamma(t) := \begin{cases} \gamma_p(t), & t^-(p) < t < t^+(p), \\ \Phi^{\mathrm{loc}}(t - t_i, \gamma_p(t_i)), & t_i - \varepsilon < t < t_i + \varepsilon. \end{cases}$$

is a well-defined integral curve of X starting at p, and thus contradicts the maximality of $t^+(p)$. ■

Bonus Material for Lecture 9

In this bonus section we prove Theorem 9.1.

Proof of Theorem 9.1. Note that (i) determines $\mathcal{D}(X)$ uniquely, and (ii) does the same for Φ. It remains therefore to show that $\mathcal{D}(X)$ is open and Φ is smooth. This however is somewhat trickier than it looks.

Fix $p \in M$ and let I denote the set of all $t \in \big(t^-(p), t^+(p)\big)$ for which there exists some neighbourhood of (t, p) contained in $\mathcal{D}(X)$ on which Φ is smooth. Since smoothness is an open condition, I is an open set. We will prove that I is nonempty and closed, whence it follows that $I = \big(t^-(p), t^+(p)\big)$.

Firstly, I is non-empty, since $0 \in I$ by Theorem 9.5. Now suppose $t_0 \in \overline{I}$. Set $p_0 := \gamma_p(t_0)$. We apply Theorem 9.5 at the point p_0 to obtain a local flow $\Phi^{\mathrm{loc}} \colon (a, b) \times U_0 \to M$ about p_0. Since t_0 belongs to the closure of I, we may choose $t_1 \in I$ close enough to t_0 such that $t_0 - t_1$ belongs to (a, b) and such that $\gamma_p(t_1)$ belongs to U_0 (here we are using the fact that γ_p is continuous at t_0 and that U_0 is a neighbourhood of p_0).

Since (a, b) is an interval, we can do a little better: we can choose an interval I_0 about t_0 such that $t - t_1 \in (a, b)$ for all $t \in I_0$. Finally, by continuity of Φ at (t_1, p), there exists a neighbourhood V of p such that $\Phi(\{t_1\} \times V) \subset U_0$.

We now claim that our original Φ is defined and smooth on all of $I_0 \times V$, so that in particular $t_0 \in I$. Indeed, if $t \in I_0$ and $q \in V$ then $t - t_1 \in (a, b)$ and $\Phi(t_1, q) \in U_0$. Thus $\Phi^{\mathrm{loc}}(t - t_1, \Phi(t_1, q))$ is defined and smooth. But the curve $s \mapsto \Phi^{\mathrm{loc}}(s - t_1, \Phi(t_1, q))$ is an integral curve of X which passes through $\Phi(t_1, q)$ at t_1. By uniqueness, this curve is $\Phi(t, q)$. Therefore

$$\Phi(t, q) = \Phi^{\mathrm{loc}}(t - t_1, \Phi(t_1, q))$$

is defined and smooth at (t, q).

We have thus shown that for all $p \in M$ and for all $t \in \bigl(t^-(p), t^+(p)\bigr)$, there exists a neighbourhood of (t, p) in M contained in $\mathcal{D}(X)$ on which Φ is smooth. Thus $\mathcal{D}(X)$ is open and $\Phi\colon \mathcal{D}(X) \to M$ is smooth. This completes the proof. ■

LECTURE 10

Lie Groups

We now move on to defining the **Lie derivative** associated to a vector field X. As with maps φ_* from the last lecture, we will actually give two definitions, one for the Lie derivative eating a function, and one for the Lie derivative eating a vector field. In Lecture 22, after we have discussed tensors (cf. Remark 8.17), we will unify these two definitions into a single Lie derivative that eats any tensor and spits out another tensor of the same type.

DEFINITION 10.1. Let $X \in \mathfrak{X}(M)$ with flow Φ_t. We define the **Lie derivative** of X to be the map

$$\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$$

given by

$$(\mathcal{L}_X f)(p) := \lim_{t \to 0} \frac{f \circ \Phi_t(p) - f(p)}{t}.$$

To see that this is well-defined (i.e. why the limit exists and defines a smooth function), we prove:

LEMMA 10.2. $\mathcal{L}_X f = X(f)$.

Proof. From the definitions one has

$$\begin{aligned} X(f)(p) &= X(p)(f) \\ &= \dot{\gamma}_p(0)(f) \\ &= (f \circ \gamma_p)'(0). \end{aligned}$$

But then clearly

$$\begin{aligned} (f \circ \gamma_p)'(0) &= \lim_{t \to 0} \frac{f \circ \gamma_p(t) - f(p)}{t} \\ &= \lim_{t \to 0} \frac{f \circ \Phi_t(p) - f(p)}{t}. \end{aligned}$$

∎

Now we define the Lie derivative on vector fields.

DEFINITION 10.3. Let $X \in \mathfrak{X}(M)$ have flow Φ_t. We define the **Lie derivative** of X to be the map

$$\mathcal{L}_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$$

by

$$(\mathcal{L}_X Y)(p) := \lim_{t \to 0} \frac{D\Phi_{-t}(\Phi_t(p)) Y(\Phi_t(p)) - Y(p)}{t}.$$

To see that this is well-defined (i.e. why the limit exists and defines a vector field) we prove:

PROPOSITION 10.4. *For $X, Y \in \mathfrak{X}(M)$ one has $\mathcal{L}_X Y = [X, Y]$.*

Proposition 10.4 also explains the name "Lie derivative". Before going any further, let us emphasise once more: the main point of the Lie derivative is that we will eventually extend this to an operator $\mathcal{L}_X : \mathcal{T}(M) \to \mathcal{T}(M)$ on the tensor algebra of M. For now though, we can simply think of the Lie derivative of giving more insight into the Lie bracket; an example of this is Proposition 10.6 below.

See Lecture 22.

The proof of Proposition 10.4 requires a preliminary lemma, which can be thought of as a manifold version of Lemma 3.7.

LEMMA 10.5. *Let $U \subset M$ be open and let $a < 0 < b$. Let $f : (a, b) \times U \to \mathbb{R}$ be a smooth function such that $f(0, p) = 0$ for all $p \in U$. Then there exists another smooth function $h : (a, b) \times U \to \mathbb{R}$ such that*

$$f(t, p) = th(t, p), \qquad \frac{\partial}{\partial t}\Big|_{(0,p)}(f) = h(0, p), \qquad \forall\, (t, p) \in (a, b) \times U.$$

Proof. Simply define

$$h(t, p) := \int_0^1 \frac{\partial}{\partial t}\Big|_{(st,p)}(f)\, ds.$$

Then h is smooth. To see that $f(t, p) = th(t, p)$ one considers the curve $\gamma(s) := f(st, p)$. Then

$$\begin{aligned} f(t, p) &= f(t, p) - f(0, p) \\ &= \gamma(1) - \gamma(0) \\ &= \int_0^1 \gamma'(s)\, ds. \end{aligned}$$

But by definition

$$\gamma'(s) = t\frac{\partial}{\partial t}\Big|_{(st,p)}(f).$$

This completes the proof. ■

We now prove Proposition 10.4.

Proof of Proposition 10.4. Fix $p \in M$. By Theorem 9.5, there exists $a < 0 < b$ and a neighbourhood U of p such that $(a, b) \times U \subset \mathcal{D}(X)$, the domain of Φ. Now fix $g \in C^\infty(M)$. We apply Lemma 10.5 to the function

$$f(t, y) := g(\Phi_t(y)) - g(y)$$

to obtain a function h, which, writing $h_t(q) := h(t, q)$, satisfies:

$$g \circ \Phi_t = g + th_t, \qquad h_0 = X(g),$$

where we used Lemma 10.2. Thus for another vector field Y we have

$$\begin{aligned} D\Phi_{-t}(\Phi_t(p))Y(\Phi_t(p))(g) &= Y(\Phi_t(p))(g \circ \Phi_{-t}) \\ &= Y(\Phi_t(p))(g - th_{-t}) \\ &= Y(g)(\Phi_t(p)) - tY(h_{-t})(\Phi_t(p)). \end{aligned}$$

We therefore have

$$\begin{aligned}(\mathcal{L}_X Y)(g)(p) &= \lim_{t\to 0} \frac{Y(g)(\Phi_t(p)) - (Y(g))(p)}{t} - \lim_{t\to 0} Y(h_{-t})(\Phi_t(p)) \\ &= \mathcal{L}_X (Y(g)(p) - Y(h_0)(p) \\ &= X(Y(g))(p) - Y(X(g))(p) \\ &= [X, Y](g)(p),\end{aligned}$$

where the second equality used the definition of the Lie derivative $\mathcal{L}_X$ applied to the function $Y(g)$ and the third equality used used Lemma 10.2. Since p and g were arbitrary, this completes the proof. ■

An application of Theorem 10.4 is the following result, whose proof is deferred to Problem Sheet E.

Proposition 10.6. *Let X and Y be vector fields on M with flows Φ_t and Ψ_t respectively. Then $[X, Y] \equiv 0$ if and only if the two flows commute, i.e. $\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t$ for all s, t small.*

We now move onto to our next major topic – *Lie groups* – which will take most of the next four lectures. Lie groups are important in many areas of mathematics (not just geometry!), including representation theory, harmonic analysis, differential equations and more. Lie groups also crop up naturally in physics. On the classical side one has Noether's theorem, which states that every smooth symmetry of a physical system has a corresponding conservation law. In high-energy particle physics, Lie groups are an important ingredient of *gauge theory*. We will come back to gauge theory in Differential Geometry II when we study connections on principal bundles.

Definition 10.7. A **Lie group** G is a smooth manifold that is also a group in the algebraic sense, with the property that the group multiplication

$$\mu\colon G \times G \to G, \qquad \mu(g, h) = gh,$$

and group inversion

$$i\colon G \to G, \qquad i(g) = g^{-1},$$

are both smooth maps.

Convention. We typically use the letters G, H for Lie groups. We also follow the standard group theory notation and use the letters g, h to denote points, and e for the identity element. This has two important consequences for our notational conventions:

Exception: If G is a matrix Lie group we use the letters A, B to denote matrices and denote the identity matrix by I.

- The dimension of a Lie group is *not* given by the corresponding lower-case letter. When necessary, the dimension will always be explicitly stated.
- When discussing Lie groups, we will never use the letters g, h to denote functions.

Definition 10.8. A **Lie group homomorphism** $\varphi\colon G \to H$ is a smooth map $G \to H$ which is also a group homomorphism. A **Lie group isomorphism** is a Lie group homomorphism which is also a diffeomorphism.

A bijective group homomorphism is necessarily a group isomorphism; hence a Lie group isomorphism is in particular a group isomorphism.

Examples 10.9. Here are some examples of Lie groups.

(i) $\mathbb{R}^m$ is a Lie group under addition.

(ii) $\mathbb{R} \setminus \{0\}$ is a Lie group under multiplication.

(iii) The set $\mathrm{GL}(m)$ of invertible $m \times m$ matrices is a Lie group under matrix multiplication. Indeed, it is a manifold of dimension m^2 (cf. Problem A.2). Multiplication is smooth because the matrix entries of a product AB are given by polynomials in the entries of A and B, and inversion is smooth by Cramer's rule.

(iv) If G is a Lie group and $H \subset G$ is an **open subgroup** (that is, a subgroup which is also an open set in G) then H naturally inherits a Lie group structure (cf. Proposition 1.15). Thus the set $\mathrm{GL}^+(m)$ of invertible matrices with positive determinant is a Lie group.

(v) The m-torus $T^m = \mathbb{R}^m / \mathbb{Z}^m$ is an abelian Lie group, where the group structure is induced by addition on $\mathbb{R}^m$. In fact, one can show that any compact abelian Lie group is (isomorphic to) a torus.

(vi) The same underlying smooth manifold can carry multiple Lie group structures. For instance, a different Lie group structure on $\mathbb{R}^3$ is given by

$$\mu(u, v) \coloneqq \big(u^1 + v^1, u^2 + v^2, u^3 + v^3 + u^1 v^2\big).$$

This Lie group is known as the **Heisenberg group**. In order to see that this does indeed define a group structure, we identify $\mathbb{R}^3$ with upper triangular 3×3 matrices:

$$(u^1, u^2, u^3) \qquad \longleftrightarrow \qquad \begin{pmatrix} 1 & u^1 & u^3 \\ 0 & 1 & u^2 \\ 0 & 0 & 1 \end{pmatrix}$$

The group multiplication μ corresponds to normal matrix multiplication.

(vii) Not all smooth manifolds can be made into Lie groups. For instance, S^m admits a Lie group structure only for $m = 0, 1$ and 3. The reason for this is briefly discussed in Remark 13.15.

Definition 10.10. Let G be a Lie group and let $g \in G$. We define diffeomorphisms

$$l_g \colon G \to G, \qquad r_g \colon G \to G,$$

called the **left translation** by g and the **right translation** by g respectively, defined by

$$l_g(h) \coloneqq gh, \qquad r_g \coloneqq (h) = hg.$$

To see that these maps are diffeomorphisms, note that l_g is the composition of the smooth map $h \mapsto (g, h)$ and the group multiplication $\mu \colon G \times G \to \mathbb{R}$. The inverse of l_g is $l_{g^{-1}}$.

Remark 10.11. Throughout this lecture we will almost exclusively work with left translations. This is purely a convention – everything we do could be reformulated (with appropriate modifications) to work with right translations instead.

Proposition 10.12. *Every Lie group homomorphism has constant rank.*

Proof. Let $\varphi\colon G \to H$ be a Lie group homomorphism. Fix $g \in G$. We show that φ has the same rank at g as it does at e. Since φ is a group homomorphism, the following commutes:

$$\begin{array}{ccc} G & \xrightarrow{\;l_g\;} & G \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ H & \xrightarrow[\;l_{\varphi(g)}\;]{} & H \end{array}$$

Indeed, if we fix $h \in G$ then

$$\varphi(l_g(h)) = \varphi(gh) = \varphi(g)\varphi(h) = l_{\varphi(g)}(\varphi(h)).$$

Since h was arbitrary this shows that the diagram commutes. Differentiating the equation $\varphi \circ l_g = l_{\varphi(g)} \circ \varphi$ at e and applying the chain rule for manifolds (Proposition 4.2) to obtain

$$D\varphi(g) \circ Dl_g(e) = Dl_{\varphi(g)}(e) \circ D\varphi(e). \tag{10.1}$$

Since l_g and $l_{\varphi(g)}$ are diffeomorphisms, both $Dl_g(e)$ and $Dl_{\varphi(g)}$ are linear isomorphisms. The claim follows. ■

Corollary 10.13. *A Lie group homomorphism is a Lie group isomorphism if and only if it is bijective.*

Proof. This follows immediately from Proposition 10.12 and Problem C.7. ■

Definition 10.14. Let G be a Lie group. A **Lie subgroup** of G is a subgroup H endowed with a topology and a smooth structure that simultaneously makes H into a Lie group and into an *immersed* submanifold of G.

In fact, *embedded* submanifolds are automatically Lie subgroups.

Proposition 10.15. *Let G be a Lie group, and let $H \subset G$ be a subgroup which is an embedded submanifold. Then H is a Lie subgroup.*

Proof. We need to check that H is a Lie group in its own right. Thus for instance we must show that the group multiplication $\mu\colon H \times H \to H$ is smooth. For this we need the following two facts:

- If $M \subset N$ is an immersed submanifold and $\varphi\colon N \to L$ is smooth then $\varphi|_M\colon M \to L$ is also smooth. (*Proof*: the inclusion map $\imath\colon M \to N$ is smooth by definition of an immersed submanifold, and $\varphi|_M = \varphi \circ \imath$.)

- If $M \subset N$ is an embedded submanifold and $\varphi\colon L \to N$ is a smooth map with $\varphi(L) \subset M$ then $\varphi\colon L \to M$ is also smooth. (*Proof*: This is immediate from the definition of the subspace topology.)

This property will be formalised in Definition 15.1.

Going back to the proof, from the first bullet point, $\mu|_{H\times H}\colon H \times H \to G$ is smooth. Since H is a subgroup, $\mu(H \times H) \subset H$. By the second bullet point, $\mu|_{H\times H}\colon H \times H \to H$ is smooth. A similar argument applies for inversion. ■

The following result is much deeper. It is not that difficult to prove, but it would take the entire lecture (and then some), so we will skip it.

Theorem 10.16 (The Closed Subgroup Theorem). *Let G be a Lie group and suppose H is any subgroup of G. The following are equivalent:*

(i) *H is a closed subgroup (i.e. H is a closed set in G).*

(ii) *H is an embedded submanifold of G.*

(iii) *H is an embedded Lie subgroup of G.*

Clearly (iii) implies (ii). Proposition 10.15 proved that (ii) implies (iii). On Problem Sheet E you are asked to prove that (ii) implies (i). The trickier bit is to show that (i) implies (ii), and this is what we will skip.

Definition 10.17. A **matrix Lie group** is a closed subgroup of $\mathrm{GL}(m)$.

The Closed Subgroup Theorem 10.16 tells us that any matrix Lie group is a Lie subgroup of $\mathrm{GL}(m)$, and hence a Lie group in its own right. Nevertheless, quoting the Closed Subgroup Theorem is overkill here, since one can typically reduce the problem to a simple application of the Implicit Function Theorem. We illustrate this principle using the example of the orthogonal group $\mathrm{O}(m)$.

Proposition 10.18. *The set of orthogonal matrices* $\mathrm{O}(m)$ *is a Lie subgroup of* $\mathrm{GL}(m)$ *of dimension* $\frac{1}{2}m(m-1)$.

Proof. By Proposition 10.15 we need only show that $\mathrm{O}(m)$ is an embedded submanifold. For this first consider the set $\mathrm{Sym}(m)$ of *symmetric* matrices. We can identify $\mathrm{Sym}(m) \cong \mathbb{R}^{\frac{m(m+1)}{2}}$, and thus $\mathrm{Sym}(m)$ is naturally a smooth manifold. Now consider the (obviously smooth) map $\varphi\colon \mathrm{GL}(m) \to \mathrm{Sym}(m)$ given by

$$\varphi(A) := AA^T,$$

where A^T denotes the transpose of A. Then $\mathrm{O}(m) = \varphi^{-1}(I)$, where I is the $m \times m$ identity matrix. Thus by the Implicit Function Theorem 6.10, we need only show that I is regular value of φ, whence $\mathrm{O}(m)$ is an embedded submanifold of $\mathrm{GL}(m)$ of dimension $m^2 - \frac{1}{2}m(m+1) = \frac{1}{2}m(m-1)$. For $A \in \mathrm{O}(m)$ one has $\varphi \circ r_A = \varphi$, and thus

$$D\varphi(A) \circ Dr_A(I) = D\varphi(I).$$

Since r_A is a diffeomorphism, it follows that the rank of φ at A is the same as the rank of φ at I. Thus we need only show that φ has maximal rank at I, i.e. that $D\varphi(I)$ is surjective.

Since $\mathrm{GL}(m)$ is an open subset of the vector space $\mathrm{Mat}(m)$, its tangent space at I is canonically identified with $\mathrm{Mat}(m)$ via the dash-to-dot map $\mathcal{J}_I$, and similarly $T_I \,\mathrm{Sym}(m) \cong \mathrm{Sym}(m)$. Thus $D\varphi(I)$ induces a canonical linear map $\ell \colon \mathrm{Mat}(m) \to \mathrm{Sym}(m)$ such that the following commutes

$$\begin{array}{ccc} \mathrm{Mat}(m) & \xrightarrow{\ \ell\ } & \mathrm{Sym}(m) \\ \downarrow{\scriptstyle \mathcal{J}_I} & & \downarrow{\scriptstyle \mathcal{J}_I} \\ T_I\,\mathrm{Mat}(m) & \xrightarrow[D\varphi(I)]{} & T_I\,\mathrm{Sym}(m) \end{array}$$

Since $\mathcal{J}_I$ is an isomorphism, it suffices to show that ℓ is surjective. Take $A \in \mathrm{Mat}(m)$ and compute

Sanity check: $I + tA$ belongs to $\mathrm{GL}(m)$ for t small enough, so this expression makes sense.

$$\begin{aligned} D\varphi(I)\mathcal{J}_I(A) &= \frac{d}{dt}\Big|_{t=0} \varphi(I + tA) \\ &= \frac{d}{dt}\Big|_{t=0} \big(I + t(A + A^T) + t^2 AA^T\big) \\ &= \mathcal{J}_I(A + A^T) \end{aligned}$$

Thus ℓ is the map $A \mapsto A + A^T$. This map is surjective, since if $S \in \mathrm{Sym}(m)$ then $\ell(\frac{1}{2}S) = S$. This completes the proof. ■

This technique works for all matrix Lie groups – the only trick is to find the right map φ. On Problem Sheet E you are asked to carry this out for the symplectic linear group.

LECTURE 11

The Lie Algebra of a Lie Group

Today we make the connection between Lie groups and Lie algebras. We begin by explaining how to produce a Lie algebra from a Lie group. The converse – producing a Lie group from a Lie algebra – is harder. This is the content of the famous *Lie Correspondence Theorem*, stated as Theorem 11.11 below. The proof of the Lie Correspondence Theorem will be given in Lecture 14.

DEFINITION 11.1. Let G be a Lie group. We define the **Lie algebra** of G, written as $\mathfrak{g}$, as the tangent space to G at the identity element e:

$$\mathfrak{g} := T_e G.$$

Of course, for this definition not to be completely insane, the Lie algebra of a Lie group better be a Lie algebra in the sense of Definition 8.11. Luckily this is indeed the case, as we will prove in Corollary 11.6 below.

CONVENTION. The standard convention is that the Lie algebra of a given Lie group is written with the corresponding lower case Fraktur letter. This convention will be used throughout, often without comment. Thus for instance if H is a Lie group, the symbol $\mathfrak{h}$ should always be understood to denote $T_e H$, even if this is not explicitly stated. A similar convention applies for matrix Lie groups: we write $\mathfrak{gl}(m)$ for the Lie algebra of $\mathrm{GL}(m)$, $\mathfrak{o}(m)$ for the Lie algebra of $\mathrm{O}(m)$, and so on.

EXAMPLES 11.2. Here are some examples of Lie algebras of Lie groups.

(i) The Lie algebra of $\mathrm{GL}(m)$ is $\mathfrak{gl}(m) \cong \mathrm{Mat}(m)$.

(ii) The Lie algebra of $\mathrm{O}(m)$ is

$$\mathfrak{o}(m) := \left\{ A \in \mathfrak{gl}(m) \mid A + A^T = 0 \right\}.$$

This follows from Proposition 10.18 together with Proposition 6.15.

(iii) The Lie algebra of T^m is $\mathbb{R}^m$. Indeed, for $m = 1$ this is clear, and for $m > 1$ this follows from Problem C.1. More generally, the Lie algebra of any abelian Lie group is an abelian Lie algebra (and the converse holds if the Lie group is connected), as you will prove on Problem Sheets E and G.

The key to proving that the Lie algebra of a Lie group is indeed a Lie algebra is the following concept.

DEFINITION 11.3. Let G be a Lie group. A vector field $X \in \mathfrak{X}(G)$ is said to be **left-invariant** if $(l_g)_* X = X$ for all $g \in G$. Equivalently, this means that

$$Dl_g(h)X(h) = X(gh), \qquad \forall\, g, h \in G.$$

We denote by $\mathfrak{X}_l(G) \subset \mathfrak{X}(G)$ the set of left-invariant vector fields.

It is immediate that $\mathfrak{X}_l(G)$ is a linear subspace of $\mathfrak{X}(G)$. In fact, much more is true: the Lie bracket of two left-invariant vector fields is again left-invariant:

PROPOSITION 11.4. *Let G be a Lie group and let $X, Y \in \mathfrak{X}_l(G)$. Then $[X, Y]$ also belongs to $\mathfrak{X}_l(G)$. Consequently $\mathfrak{X}_l(G)$ is a Lie subalgebra of $\mathfrak{X}(G)$.*

Proof. Fix $g \in G$. Then by Proposition 8.19 one has

$$(l_g)_*[X, Y] = [(l_g)_*X, (l_g)_*Y] = [X, Y].$$

Since g was arbitrary, the result follows. ■

The next result is the main step needed to show that $\mathfrak{g}$ is a Lie algebra.

THEOREM 11.5. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The evaluation map*

$$\mathrm{eval}_e \colon \mathfrak{X}_l(G) \to \mathfrak{g}, \qquad \mathrm{eval}_e(X) := X(e)$$

is a vector space isomorphism. Thus $\mathfrak{X}_l(G)$ is a vector space of the same dimension as G.

Proof. The map eval_e is clearly linear. If $\mathrm{eval}_e(X) = 0$ then X is identically zero, since for any $g \in G$ one has by left-invariance:

$$X(g) = Dl_g(e)X(e) = 0.$$

Thus we need only show that eval_e is surjective. For this, fix an arbitrary $\xi \in \mathfrak{g} = T_eG$. We define a map $X_\xi \colon G \to TG$ by

$$X_\xi(g) := Dl_g(e)\xi.$$

Then X_ξ certainly satisfies the section property (8.1), since $Dl_g(e)$ is a map $T_eG \to T_gG$. To show that X_ξ is a vector field, it suffices to show that $X_\xi(f)$ is smooth for any $f \in C^\infty(G)$. Fix such an f, and choose a smooth curve $\gamma \colon (-\varepsilon, \varepsilon) \to G$ such that $\gamma(0) = e$ and $\dot\gamma(0) = \xi$. Define a smooth map

$$\delta \colon (-\varepsilon, \varepsilon) \times G \to \mathbb{R}, \qquad \delta(t, g) := f\big(\mu(g, \gamma(t))\big),$$

where $\mu \colon G \times G \to G$ is the group multiplication. Write $\delta_g := \delta(\cdot, g)$; we think of δ_g as a family of smooth curves in $\mathbb{R}$ which depend smoothly on g. Following through the definitions, for any $g \in G$ one has

Thus $\delta_g = f \circ l_g \circ \gamma$.

$$\begin{aligned} X_\xi(f)(g) &= X_\xi(g)(f) \\ &= \big(Dl_g(e)\xi\big)(f) \\ &= \xi(f \circ l_g) \\ &= \dot\gamma(0)(f \circ l_g) \\ &= \dot\delta_g(0). \end{aligned}$$

Since $g \mapsto \dot{\delta}_g(0)$ is smooth, this shows that $X_\xi(f)$ is smooth. Since f was arbitrary, it follows from Proposition 8.2 that X_ξ is smooth, and hence a vector field. Next, we claim X_ξ is left-invariant. Indeed, if $g, h \in G$ then

$$\begin{aligned} Dl_g(h)X_\xi(h) &= Dl_g(h) \circ Dl_h(e)\xi \\ &= D(l_g \circ l_h)(e)\xi \\ &= Dl_{gh}(e)\xi \\ &= X_\xi(gh). \end{aligned}$$

Thus $X_\xi \in \mathfrak{X}_l(G)$. Since $\operatorname{eval}_e(X_\xi) = X_\xi(e) = \xi$, this shows eval_e is surjective. The proof is complete. ∎

Corollary 11.6. *Let G be a Lie group of dimension m. Then its Lie algebra is a Lie algebra (!) of dimension m.*

Proof. We need only define a Lie bracket on $\mathfrak{g}$. For this, using the notation from Theorem 11.5, we define

$$[\xi, \zeta] \coloneqq \operatorname{eval}_e\big([X_\xi, X_\zeta]\big), \qquad \xi, \zeta \in \mathfrak{g}.$$

This works by Theorem 11.5 and Proposition 11.4. ∎

For the next result, let us recall from Problem D.6 that if $\varphi \colon M \to N$ is a smooth map between manifolds, and $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ then we say that X and Y are **φ-related** if

$$D\varphi(p)X(p) = Y(\varphi(p)), \qquad \forall\, p \in M.$$

Proposition 11.7. *Let $\varphi \colon G \to H$ be a Lie group homomorphism between two Lie groups. Then $D\varphi(e) \colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

Proof. Let $\xi \in \mathfrak{g}$ and let $X_\xi \in \mathfrak{X}_l(G)$ denote the unique left-invariant vector field such that $X_\xi(e) = \xi$. Let $\zeta \coloneqq D\varphi(e)\xi$ and let $Y_\zeta \in \mathfrak{X}_l(H)$ denote the unique left-invariant vector field such that $Y_\zeta(e) = \zeta$. We claim that X_ξ and Y_ζ are φ-related. Indeed, by (10.1) one has

$$\begin{aligned} D\varphi(g)X_\xi(g) &= D\varphi(g) \circ Dl_g(e)\xi \\ &= Dl_{\varphi(g)}(e) \circ D\varphi(e)\xi \\ &= Dl_{\varphi(g)}(e)\zeta \\ &= Y_\zeta(\varphi(g)). \end{aligned}$$

Now by part (ii) of Problem D.6, if $\xi_1, \xi_2 \in \mathfrak{g}$ and $\zeta_i \coloneqq D\varphi(e)\xi_i$ then $[X_{\xi_1}, X_{\xi_2}]$ is φ-related to $[Y_{\zeta_1}, Y_{\zeta_2}]$. Evaluating both sides at e gives

$$D\varphi(e)[\xi_1, \xi_2] = [\zeta_1, \zeta_2].$$

This completes the proof. ∎

Suppose now H is a Lie subgroup of G. Let $\imath \colon H \hookrightarrow G$ denote the inclusion. Then since $D\imath(e) \colon \mathfrak{h} = T_eH \to \mathfrak{g} = T_eG$ is injective, we can regard $\mathfrak{h}$ as a linear subspace of $\mathfrak{g}$. A priori however, this identification might not respect the Lie brackets. Thanks to Proposition 11.7, however, it does:

Corollary 11.8. *Let $H \subset G$ be a Lie subgroup, and identify $\mathfrak{h}$ with its image in $\mathfrak{g}$. The Lie bracket on $\mathfrak{h}$ is simply the restriction of the Lie bracket on $\mathfrak{g}$ to $\mathfrak{h}$. Thus $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.*

Proof. Apply Proposition 11.7 with φ the inclusion (note that the roles of H and G have been reversed!) ∎

Let us go back to $\mathrm{GL}(m)$. We now have potentially two different Lie brackets on $\mathfrak{gl}(m) \cong \mathrm{Mat}(m)$: the one coming from Corollary 11.6, and the commutator bracket (cf. part (ii) of Example 8.12). The next result, whose proof is deferred to Problem Sheet E, tells us that these coincide.

Proposition 11.9. *The Lie bracket on $\mathfrak{gl}(m)$ is given by matrix commutation, i.e.*

$$[A, B] = AB - BA, \qquad \forall\, A, B \in \mathfrak{gl}(m).$$

Combining Corollary 11.8 and Proposition 11.9, we end up with:

Corollary 11.10. *Let G be a matrix Lie group. Then the Lie bracket on $\mathfrak{g}$ is given by matrix commutation.*

The Lie group-Lie algebra correspondence established in Corollary 11.8 goes both ways. This is the content of the next famous result. The proof will be given in Lecture 13 after we have proved the Frobenius Theorem.

Theorem 11.11 (The Lie Correspondence Theorem). *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ then there is a unique connected Lie subgroup H of G whose Lie algebra is $\mathfrak{h}$.*

We conclude this lecture by defining a special type of Lie subgroup.

Definition 11.12. Let G be a Lie group. A **one-parameter subgroup** of G is a Lie group homomorphism $\mathbb{R} \to G$.

Here $\mathbb{R}$ is thought of as a Lie group under addition.

It turns out that one-parameter subgroups are easy to classify. The starting point for this classification is the following statement.

The relation between Definition 9.14 and Definition 11.12 will be explored in Lecture 13.

Proposition 11.13. *Let G be a Lie group and let $X \in \mathfrak{X}_l(G)$. Then X is complete.*

Proof. We will show that there exists $\varepsilon > 0$ such that for all $g \in G$ the integral curve γ_g of X with initial condition is defined on at least $(-\varepsilon, \varepsilon)$. This implies the result, by Lemma 9.17.

Choose $\varepsilon > 0$ such that the integral curve γ_e is defined on $(-\varepsilon, \varepsilon)$. Fix $g \in G$ and let $\delta := l_g \circ \gamma_e$. Then $\delta(0) = g$, and moreover

$$\begin{aligned}
\dot{\delta}(t) &= \frac{d}{ds}\Big|_{s=t} l_g(\gamma_e(s)) \\
&= Dl_g(\gamma_e(t))\dot{\gamma}_e(t) \\
&= Dl_g(\gamma_e(t))X(\gamma_e(t)) \\
&= X(g\gamma_e(t)) \\
&= X(\delta(t)),
\end{aligned}$$

where the penultimate line used left-invariance. By uniqueness of integral curves (Theorem 9.7), it follows that $\delta \equiv \gamma_g$ where defined. Thus γ_g is defined on (at least) $(-\varepsilon, \varepsilon)$. The result follows. ■

NOTATION. For $\xi \in \mathfrak{g}$ we denote by $\gamma^\xi \colon \mathbb{R} \to G$ the integral curve of the left-invariant vector field X_ξ with initial condition $\gamma^\xi(0) = e$.

PROPOSITION 11.14. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then each curve $\gamma^\xi \colon \mathbb{R} \to G$ is a one-parameter subgroup. Moreover any one-parameter subgroup is of this form.*

Proof. Fix $\xi \in \mathfrak{g}$. To show that γ^ξ is a one-parameter subgroup we must show that

$$\gamma^\xi(s+t) = \gamma^\xi(s)\gamma^\xi(t)$$

for all $s, t \in \mathbb{R}$, where on the right-hand side we use multiplication in G. For this, fix $s \in \mathbb{R}$ and consider the curve

$$\delta(t) := \gamma^\xi(s)^{-1}\gamma^\xi(s+t).$$

Then $\delta(0) = e$, and by the chain rule

$$\begin{aligned}\dot\delta(t) &= Dl_{\gamma^\xi(s)^{-1}}(\gamma^\xi(s+t))\dot\gamma^\xi(s+t)\\ &= Dl_{\gamma^\xi(s)^{-1}}(\gamma^\xi(s+t))X_\xi(\gamma^\xi(s+t))\\ &= X_\xi\big(\gamma^\xi(s)^{-1}\gamma^\xi(s+t)\big)\\ &= X_\xi(\delta(t)).\end{aligned}$$

where the penultimate line used left-invariance again. Thus by uniqueness of integral curves, one must have $\delta(t) = \gamma^\xi(t)$.

Conversely, suppose γ is a one-parameter subgroup. Let $\xi := \dot\gamma(0) \in \mathfrak{g}$. We claim that $\dot\gamma(t) = X_\xi(\gamma(t))$. Since $\gamma(t+s) = \gamma(t)\gamma(s) = l_{\gamma(t)}(\gamma(s))$, we have

$$\begin{aligned}\dot\gamma(t) &= \frac{d}{ds}\Big|_{s=0}\gamma(t+s)\\ &= \frac{d}{ds}\Big|_{s=0} l_{\gamma(t)}(\gamma(s))\\ &= Dl_{\gamma(t)}(\gamma(0))\dot\gamma(0)\\ &= Dl_{\gamma(t)}(e)\xi\\ &= X_\xi(\gamma(t)).\end{aligned}$$

where the last line used the definition of X_ξ. Thus again by uniqueness of integral curves we have $\gamma \equiv \gamma^\xi$. This completes the proof. ■

We can play a similar game by replacing $\xi \in \mathfrak{g}$ with a scalar multiple $s\xi$.

LEMMA 11.15. *For any $s, t \in \mathbb{R}$ one has*

$$\gamma^\xi(st) = \gamma^{s\xi}(t).$$

Proof. First note as $Dl_g(e)$ is a linear map one has for any $g \in G$ that

$$X_{s\xi}(g) = Dl_g(e)(s\xi) = sDl_g(e)\xi = sX_\xi(g).$$

Thus $X_{s\xi} = sX_\xi$. Now by the chain rule

$$\frac{d}{dt}\Big|_{t=t_0} \gamma^\xi(st) = s\dot{\gamma}^\xi(st_0) = sX_\xi(\gamma^\xi(st_0)) = X_{s\xi}(\gamma^\xi(st_0)).$$

Thus $t \mapsto \gamma^\xi(st)$ is an integral curve of $X_{s\xi}$ with initial condition e, and hence by uniqueness of integral curves once more, one has $\gamma^\xi(st) \equiv \gamma^{s\xi}(t)$. ■

LECTURE 12

Smooth Actions of Lie Groups

We begin this lecture by introducing the **exponential map** of a Lie group G, which will be a smooth map $\exp\colon \mathfrak{g} \to G$. The reason for the name will become apparent in Proposition 12.7 – namely, the exponential map of a matrix Lie group is given by matrix exponentiation. As in the previous lecture, given $\xi \in \mathfrak{g}$, we denote by $\gamma^\xi\colon \mathbb{R} \to G$ the integral curve of X_ξ with initial condition $\gamma^\xi(0) = e$.

DEFINITION 12.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The **exponential map** is the map

$$\exp\colon \mathfrak{g} \to G, \qquad \xi \mapsto \gamma^\xi(1).$$

The following result is an immediate corollary of Proposition 11.14, Lemma 11.15, and Problem E.2.

PROPOSITION 12.2 (Properties of the exponential map). *The exponential map* $\exp\colon \mathfrak{g} \to G$ *satisfies:*

(i) $\exp((s+t)\xi) = \exp(s\xi)\exp(t\xi)$ *for all* $\xi \in \mathfrak{g}$ *and* $s, t \in \mathbb{R}$,

(ii) $\exp(-t\xi) = (\exp(t\xi))^{-1}$ *for all* $\xi \in \mathfrak{g}$ *and* $t \in \mathbb{R}$,

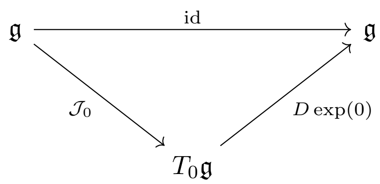
(iii) *The map* $t \mapsto \exp(t\xi)$ *is precisely the one-parameter subgroup* $\gamma^\xi(t)$.

(iv) *The flow* Φ^ξ_t *of* X_ξ *is given by* $\Phi^\xi_t = r_{\exp(t\xi)}$.

We now prove that the exponential map is smooth, and compute its derivative at the origin.

THEOREM 12.3. *The exponential map* $\exp\colon \mathfrak{g} \to G$ *is smooth. Moreover the following commutes:*

This is often imprecisely stated as $D\exp(0) = \mathrm{id}$.



Proof. We prove the result in three steps.

1. The space $G \times \mathfrak{g}$ is a smooth manifold by Problem A.3. We define a map $\widehat{X}\colon G \times \mathfrak{g} \to T(G \times \mathfrak{g})$ by

$$\widehat{X}(g, \xi) := \big(X_\xi(g), 0\big) \in T_gG \times T_\xi\mathfrak{g} \cong T_{(g,\xi)}(G \times \mathfrak{g}),$$

where the last isomorphism used Problem C.1. We claim that $\widehat{X}$ is a vector field on $G \times \mathfrak{g}$. It clearly satisfies the section property (8.1), and thus we need only check that $\widehat{X}$ is smooth. For this, suppose $f \in C^\infty(G \times \mathfrak{g})$ is smooth. Given $\xi \in \mathfrak{g}$, let $f_\xi := f(\cdot, \xi)\colon G \to \mathbb{R}$, so that f_ξ is a smooth function on G. Then by definition

$$\widehat{X}(f)(g, \xi) = X_\xi(f_\xi)(g).$$

The vector field X_ξ depends linearly on ξ by Theorem 11.5. Since linear functions are always smooth, $\xi \mapsto X_\xi$ is also smooth in ξ. The function f_ξ depends smoothly on ξ as f is smooth as a function of both g and ξ. Thus the expression $(g, \xi) \mapsto X_\xi(f_\xi)(g)$ depends smoothly on both g and ξ. Thus by Proposition 8.2, $\widehat{X}$ is indeed a vector field. Thus the flow $\widehat{\Phi}$ of $\widehat{X}$ is also smooth.

2. In this step we compute the flow $\widehat{\Phi}_t$ of $\widehat{X}$. Fix $(g, \xi) \in G \times \mathfrak{g}$. Define $\gamma \colon \mathbb{R} \to G \times \mathfrak{g}$ by

$$\gamma(t) := \big(g \exp(t\xi), \xi\big) = (l_g(\gamma^\xi(t)), \xi)$$

Then $\gamma(0) = (g, \xi)$ and

$$\begin{aligned} \dot{\gamma}(t) &= \big(Dl_g(\gamma^\xi(t))\dot{\gamma}^\xi(t), 0\big) \\ &= \big(Dl_g(\gamma^\xi(t))X_\xi(\gamma^\xi(t)), 0\big) \\ &= \big(X_\xi(g\gamma^\xi(t)), 0\big) \\ &= \widehat{X}(\gamma(t)). \end{aligned}$$

Thus γ is an integral curve of $\widehat{X}$. By uniqueness of integral curves, it follows that $\widehat{X}$ is complete and its flow is given by

$$\widehat{\Phi}_t(g, \xi) := \big(g \exp(t\xi), \xi\big).$$

In particular, $\widehat{\Phi}_1(e, \cdot) \colon \mathfrak{g} \to G \times \mathfrak{g}$ is smooth. This is the map $\xi \mapsto (\exp(\xi), \xi)$. Thus exp is smooth.

3. In this last step we compute $D\exp(0)$. Take $\xi \in \mathfrak{g}$. Then

$$\begin{aligned} D\exp(0)\mathcal{J}_0(\xi) &= \frac{d}{dt}\Big|_{t=0} \exp(t\xi) \\ &= \frac{d}{dt}\Big|_{t=0} \gamma^\xi(t) \\ &= \xi. \end{aligned}$$

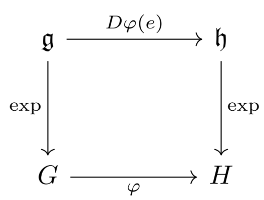
This completes the proof. ■

Corollary 12.4. *The exponential map is a diffeomorphism of some neighbourhood of the origin in $\mathfrak{g}$ onto its image in G.*

Proof. Since exp has maximal rank at 0 by Theorem 12.3, this follows immediately from the Inverse Function Theorem 5.10. ■

Now let us investigate how the exponential map behaves with respect to Lie group homomorphisms.

Proposition 12.5. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Let $\varphi \colon G \to H$ be a Lie group homomorphism. Then the following diagram commutes:*



Proof. If $\gamma\colon \mathbb{R} \to G$ is a homomorphism then since φ is a homomorphism so is $\varphi \circ \gamma\colon \mathbb{R} \to H$. Applying this with $\gamma(t) = \exp(t\xi)$ shows that $t \mapsto \varphi(\exp(t\xi))$ is a one-parameter subgroup of H. Set $\delta(t) := \varphi(\gamma(t))$. Then

$$\begin{aligned} \dot{\delta}(0) &= \frac{d}{dt}\Big|_{t=0} \varphi(\exp(t\xi)) \\ &= D\varphi(e) \circ D\exp(0)\mathcal{J}_0(\xi) \\ &= D\varphi(e)\xi, \end{aligned}$$

where we used the chain rule and Theorem 12.3. Thus by uniqueness of integral curves it follows that

$$\varphi(\exp(t\xi)) = \exp(tD\varphi(e)\xi),$$

which is what we wanted to prove. ∎

Applying Proposition 12.5 to an inclusion of a subgroup, as in Corollary 11.8 tells us:

COROLLARY 12.6. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $H \subset G$ a Lie subgroup with Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then the exponential map $\exp\colon \mathfrak{h} \to H$ is the restriction of $\exp\colon \mathfrak{g} \to G$ to $\mathfrak{h}$.*

The next result identifies the exponential map for $G = \mathrm{GL}(m)$.

PROPOSITION 12.7. *Let $A \in \mathfrak{gl}(m)$. Then the matrix exponential*

$$e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

converges and defines an element of $\mathrm{GL}(m)$. Moreover $A \mapsto e^A$ is the exponential map of $\mathrm{GL}(m)$.

The proof is deferred to Problem Sheet F. As with Corollary 11.10, this also allows us to characterise the matrix exponential for matrix Lie groups.

COROLLARY 12.8. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Then the exponential map $\exp\colon \mathfrak{g} \to G$ is given by matrix exponentiation: $\exp(A) = e^A$.*

Proof. Apply Corollary 12.6 and Proposition 12.7. ∎

This concludes our introductory treatment of Lie groups.

We now explore the manifold version of another key algebraic idea. If you think back to your introductory course on group theory, you will no doubt remember that one of the most important reasons to study groups is because we can let them act on sets. In fact, the notion of a group acting on a set is arguably more fundamental than that of a group itself, and certainly historically, the idea of a "transformation group" (i.e. an action of a group on a given set) predates that of an abstract group. One might even go so far as to say that groups are interesting precisely *because* of their actions.

Lie groups are no exception. Since we are working in the smooth category, we restrict our attention to smooth actions of Lie groups on manifolds.

Definition 12.9. Let G be a Lie group and let M be a manifold. A smooth map $\sigma \colon G \times M \to M$ satisfying

Here as usual $\sigma_g := \sigma(g, \cdot)$.

$$\sigma_{gh} = \sigma_g \circ \sigma_h \qquad \sigma_e = \mathrm{id}, \tag{12.1}$$

for all $g, h \in G$ is called **a smooth left action of G on M**. For fixed $g \in G$, this implies that $p \mapsto \sigma(g, p)$ is a diffeomorphism of M, which we denote by σ_g.

If one replaces (12.1) with

$$\sigma_{gh} = \sigma_h \circ \sigma_g \qquad \sigma_e = \mathrm{id},$$

one arrives at a **smooth right action of G on M**.

Remark 12.10. The difference between right actions and left actions is purely notational, since we can always convert one into the other. Indeed, if σ is a left action then we can define a right action $\tilde{\sigma}$ by setting $\tilde{\sigma}_g := \sigma_{g^{-1}}$, and conversely.

In the definitions that follow, by a **smooth action** we mean either a smooth left action or a smooth right action.

As you no doubt remember from algebra, there are various different types of group actions one can study. We define the **fixed point set of σ_g** as

$$\mathsf{fix}(\sigma_g) := \{p \in M \mid \sigma_g(p) = p\}.$$

and the **orbit** of p as

$$\mathsf{orb}_\sigma(p) := \{\sigma_g(p) \mid g \in G\}.$$

Finally the **stabiliser** of p is the set

$$\mathsf{stab}_\sigma(p) := \{g \in G \mid p \in \mathsf{fix}(\sigma_g)\}.$$

Definitions 12.11. Let $\sigma \colon G \times M \to M$ be a smooth action of G on M.

(i) We say that σ is an **effective** action if

$$\sigma_g = \mathrm{id} \qquad \Rightarrow \qquad g = e.$$

(ii) We say that σ is a **free** action if

Thus free $\Rightarrow$ effective.

$$\mathsf{fix}(\sigma_g) \neq \emptyset \qquad \Rightarrow \qquad g = e.$$

(iii) The action σ is said to be **transitive** if

$$\mathsf{orb}_\sigma(p) = M, \qquad \forall\, p \in M.$$

(iv) The action σ is said to be **proper** if the map

$$(\sigma, \mathrm{id})\colon G \times M \to M \times M \qquad (g, p) \mapsto (\sigma_g(p), p)$$

is a proper map in the topological sense.

i.e. the preimage of a compact set is compact.

An alternative characterisation of proper actions is the following statement, whose proof is left for you on Problem Sheet F.

LEMMA 12.12. *Let σ be a smooth action of G on M. Then σ is proper if and only if for every pair of points $p, q \in M$ and every sequence $p_k \to p$, if (g_k) is a sequence in G such that $\sigma_{g_k}(p_k) \to q$ then (g_k) admits a convergent subsequence. Moreover if $g_k \to g$ then $\sigma_g(p) = q$.*

EXAMPLES 12.13. Here are some examples of smooth actions.

(i) Let G be a Lie group. We can think of G acting on itself via both left and right translations (which are left and right actions respectively). These actions – denoted by l and r – are obviously both free and transitive. In fact, they are also proper, as you will prove on Problem Sheet F.

(ii) A Lie group can also act on itself via **conjugation**. Define

$$c_g(h) := ghg^{-1},$$

so that $c_g = l_g \circ r_{g^{-1}} = r_{g^{-1}} \circ l_g$. This left action is not proper unless G is compact. It is also never free or transitive (unless $G = \{e\}$). If G is commutative it also fails to be effective.

(iii) A smooth left action σ of a Lie group G on a vector space V is called a **representation** of G on V if σ_g is a linear map for each $g \in G$. Thus one can think of $g \mapsto \sigma_g$ as Lie group homomorphism $G \to \mathrm{GL}(V)$. The study of such actions is called *Representation Theory*, and is a major field of mathematics in itself. In the context of representation theory, an effective representation is usually called a **faithful representation**.

In fact, if σ is continuous then σ is automatically smooth – this can be proved using Problem F.10.

(iv) As a special case of the preceding example, if we take $G = \mathrm{GL}(V)$ and $\sigma = \mathrm{id}$, we obtain the **canonical representation** of $\mathrm{GL}(V)$ on V.

(v) The canonical representation of $\mathrm{GL}(m)$ on $\mathbb{R}^m$ restricts to define a transitive left action of $\mathrm{O}(m)$ on S^{m-1}.

Exercise: Prove this.

It will be useful in Lecture 16 when discussing fibre bundles to know that we can always convert an action into an effective action.

PROPOSITION 12.14. *Let σ be an action of a Lie group G on M. Let*

$$H := \{g \in G \mid \sigma_g = \mathrm{id}\}.$$

Then H is a normal subgroup of G. If $K := G/H$ then σ induces an action $\bar{\sigma}$ of K on M such that

$$\{\sigma_g \mid g \in G\} = \{\bar{\sigma}_k \mid k \in K\}, \qquad (12.2)$$

where both sides are thought of as subgroups of $\mathrm{Diff}(M)$.

Proof. Suppose $h \in H$ and $g \in G$. Then

$$\begin{aligned}\sigma_{ghg^{-1}} &= \sigma_g \circ \sigma_h \circ \sigma_{g^{-1}} \\ &= \sigma_g \circ \mathrm{id} \circ \sigma_{g^{-1}} \\ &= \mathrm{id}\,.\end{aligned}$$

Thus H is normal and the quotient group $K := G/H$ is well defined. We identify an element $k \in K$ with the coset $kH \subset G$. We define

$$\bar{\sigma}_k := \sigma_g,$$

where g is any element of kH. This is well-defined, since if g_1 is another element of kH then $g_1 g^{-1} \in H$ and hence

$$\sigma_{g_1} = \sigma_{g_1 g^{-1} g} = \sigma_{g_1 g^{-1}} \circ \sigma_g = \sigma_g.$$

The action $\bar{\sigma}$ is obviously effective, and (12.2) holds by definition. ∎

It is also convenient to regard $p \in M$ as fixed and consider the map $\sigma(\cdot, p)$. We denote this map by σ^p and call it an **orbit map**:

$$\sigma^p \colon G \to M, \qquad \sigma^p(g) := \sigma(g, p). \tag{12.3}$$

Thus by definition

$$\operatorname{im} \sigma^p = \mathsf{orb}_\sigma(p).$$

Immediately from the definition we obtain:

LEMMA 12.15. *Let σ by a smooth action of G on M. Then*

(i) *The action is free if and only if σ^p is injective for all $p \in M$.*

(ii) *The action is transitive if and only if σ^p is surjective for all $p \in M$.*

DEFINITION 12.16. Suppose $\sigma \colon G \times M \to M$ and $\tau \colon G \times N \to N$ are two smooth actions of the same Lie group G. A smooth map $\varphi \colon M \to N$ is said to be (σ, τ)**-equivariant** if the following commutes for all $g \in G$:

$$\begin{array}{ccc} M & \xrightarrow{\ \varphi\ } & N \\ {\scriptstyle \sigma_g}\downarrow & & \downarrow{\scriptstyle \tau_g} \\ M & \xrightarrow[\ \varphi\ ]{} & N \end{array}$$

If the actions σ and τ are clear from the context, we simply say that φ is **equivariant**.

EXAMPLES 12.17. Here are two examples of equivariant maps.

(i) Let G and H be Lie groups and $\varphi \colon G \to H$ a Lie group homomorphism. Define a left action σ of G on H by

$$\sigma_g(h) := \varphi(g)h.$$

Then φ is (l, σ) equivariant, where l is the action of G on itself by left translations.

(ii) Let σ be a smooth left action of G on M. Then for every point $p \in M$, the orbit map $\sigma^p \colon G \to M$ is (l, σ)-equivariant, where l is the action of G on itself by left translations. If instead σ is a smooth right action on M, then σ^p is (r, σ)-equivariant.

The next result is analogous to Proposition 10.12.

PROPOSITION 12.18. *Suppose $\sigma \colon G \times M \to M$ and $\tau \colon G \times N \to N$ are two smooth actions such that σ is transitive. Then any (σ, τ)-equivariant smooth map $\varphi \colon M \to N$ has constant rank.*

Proof. Fix $p, q \in M$. We show that the rank of φ at p is the same as the rank of φ at q. Since σ is transitive, there exists $g \in G$ such that $\sigma_g(p) = q$. We differentiate the equality $\varphi \circ \sigma_g = \tau_g \circ \varphi$ at p to obtain

$$D\varphi(q) \circ D\sigma_g(p) = D\tau_g(\varphi(p)) \circ D\varphi(p).$$

Since σ_g and τ_g are both diffeomorphisms, $D\sigma_g(p)$ and $D\tau_g(\varphi(p))$ are linear isomorphisms. The result follows. ∎

COROLLARY 12.19. *Let σ by a smooth action of G on M. Fix $p \in M$.*

(i) *If σ^p is injective then σ^p is an immersion, and thus $\mathsf{orb}_\sigma(p)$ is an immersed submanifold.*

(ii) *If σ^p is surjective then σ^p is a submersion, and thus σ^p is a quotient map which admits smooth local sections.*

Proof. By part (ii) of Examples 12.17 the orbit maps σ^p are equivariant, and hence by Proposition 12.18 they have constant rank. The result follows from Problem C.7. ∎

The last sentence of (ii) uses Proposition 6.13.

LECTURE 13

Homogeneous Spaces

In this lecture we state the *Quotient Manifold Theorem*, which tells us that the quotient space of a proper free action is naturally a smooth manifold. We use this to define **homogeneous spaces**. Finally we investigate the **adjoint** representation of a Lie group.

PROPOSITION 13.1. *Let σ be a proper action of G on M. Then for every $p \in M$, the orbit $\mathsf{orb}_\sigma(p)$ is an embedded submanifold.*

Proof. The orbit map σ^p is (l, σ)-equivariant (cf. part (ii) of Examples 12.17), and hence by Proposition 12.18 has constant rank, say n. By Corollary 6.19 there exists a neighbourhood U of $e \in G$ and an open set $\mathcal{O} \subset \mathbb{R}^n$, together with an immersion $\psi \colon \mathcal{O} \to G$ such that $\sigma^p \circ \psi \colon \mathcal{O} \to M$ is an embedding with

When the action is free – which is the main case of interest – one does not need to quote the (unproven) Corollary 6.19, and can instead use Corollary 12.19.

$$\sigma^p(\psi(\mathcal{O})) = \sigma^p(U). \tag{13.1}$$

Since $\sigma^p \circ \psi$ is an embedding there exists by Proposition 6.3 a slice chart (W, y) about p such that

$$\sigma^p(\psi(\mathcal{O})) \cap W = \left\{ q \in W \mid y^{n+1}(q) = \cdots = y^m(q) = 0 \right\},$$

and hence by (13.1) we have

$$\sigma^p(U) \cap W = \left\{ q \in W \mid y^{n+1}(q) = \cdots = y^m(q) = 0 \right\}. \tag{13.2}$$

We now claim there exists an open set $V \subset W$ containing p such that

$$\mathsf{orb}_\sigma(p) \cap V = \sigma^p(U) \cap V. \tag{13.3}$$

One always has $\supseteq$ in (13.3), so if no such neighbourhood V exists this means we can find a sequence $(p_k) \in \mathsf{orb}_\sigma(p)$ such that $p_k \to p$ and $p_k \notin \sigma^p(U)$. Let $g_k \in G$ be such that $\sigma_{g_k}(p) = p_k$. Then by Lemma 12.12, up to passing to a subsequence one has $g_k \to g$ for some $g \in G$, and moreover $\sigma_g(p) = p$. Since $r_g(U)$ is a neighbourhood of g in G, we must have $g_k \in r_g(U)$ for all k large. Thus

$$p_k = \sigma_{g_k}(p) \in \sigma^p(r_g(U))$$

for all large k. But by equivariance and the definition of an orbit

$$\sigma^p(r_g(U)) = \sigma^{\sigma_g(p)}(U) = \sigma^p(U).$$

This contradicts the claim that $p_k \notin \sigma^p(U)$ for all k, and hence (13.3) is proved. Combining (13.2) and (13.3) tells us that

$$\mathsf{orb}_\sigma(p) \cap V = \left\{ q \in V \mid y^{n+1}(q) = \cdots = y^m(q) = 0 \right\}.$$

This shows that $\mathsf{orb}_\sigma(p)$ is embedded in a neighbourhood of p. However the same argument also works for an arbitrary point $p_0 \in \mathsf{orb}_\sigma(p)$, since $\mathsf{orb}_\sigma(p_0) = \mathsf{orb}_\sigma(p)$ by definition. Thus we have produced slice charts at every point of $\mathsf{orb}_\sigma(p)$, and thus the result follows from Proposition 6.7. ∎

The hypotheses of Proposition 13.1 do *not* imply that the orbit $\mathsf{orb}_\sigma(p)$ is diffeomorphic to G – for instance, consider the trivial action $\sigma_g \equiv \mathrm{id}$. If however we additionally assume the action is free, then the orbits are all diffeomorphic to G.

COROLLARY 13.2. *Let σ be a free and proper action of G on M. Then for every point $p \in M$ the orbit map $\sigma^p \colon G \to M$ is an embedding, and thus* $\dim \mathsf{orb}_\sigma(p) = \dim G$.

Proof. Since $\mathsf{orb}_\sigma(p)$ is embedded, the orbit map is smooth as a map $G \to \mathsf{orb}_\sigma(p)$ (cf. the second bullet point in the proof of Proposition 10.15). The map $\sigma^p \colon G \to \mathsf{orb}_\sigma(p)$ is always surjective, and if the action is free, it is also injective by part (i) of Lemma 12.15. Thus σ^p is a bijective smooth map of constant rank, and hence by Problem C.7 it is a diffeomorphism. ■

The stabilisers are also always embedded Lie subgroups of G.

PROPOSITION 13.3. *Let σ be a smooth action of G on M. Then for every $p \in M$ the set $\mathsf{stab}_\sigma(p)$ is an embedded Lie subgroup of G.*

Proof. It is clear $\mathsf{stab}_\sigma(p)$ is a closed subgroup. The claim now follows from the Closed Subgroup Theorem 10.16. ■

We now move onto the Quotient Manifold Theorem.

DEFINITION 13.4. Let σ be a smooth action of G on M. Define an equivalence relation on M by saying that $p \sim q$ if and only if $\mathsf{orb}_\sigma(p) = \mathsf{orb}_\sigma(q)$. Denote the quotient space by M/G and let $\rho \colon M \to M/G$ denote the projection. We call M/G the **quotient space** of M by the G action.

This notation is somewhat imprecise, since the space depends on the choice of action σ. Where necessary we write $M/_\sigma G$ to indicate this dependence.

We endow M/G with the quotient topology. In general this topology can be very badly behaved, but if the action is proper then it is at least Hausdorff.

LEMMA 13.5. *Suppose σ is a proper smooth action of G on M. Then the quotient space M/G is Hausdorff.*

Proof. The quotient map ρ is an open map for the quotient topology, as if $U \subset M$ is open then

$$\rho^{-1}(\rho(U)) = \bigcup_{g \in G} \sigma_g(U)$$

is open in G. Since $(\sigma, \mathrm{id}) \colon G \times M \to M \times M$ is proper, its image – call it C – is a closed subset of $M \times M$. Now suppose $p, q \in M$ are such that $\rho(p) \neq \rho(q)$. This means that $(p, q) \notin C$. Since C is closed, there exist neighbourhoods U and V of p and q respectively such that $(U \times V) \cap C = \emptyset$. Then $\rho(U)$ and $\rho(V)$ are open neighbourhoods of $\rho(p)$ and $\rho(q)$ in M/G such that $\rho(U) \cap \rho(V) = \emptyset$. Thus M/G is Hausdorff. ■

In fact, more is true. We conclude this lecture with the following theorem.

Theorem 13.6 (The Quotient Manifold Theorem). *Let σ be a smooth action of G on M which is both proper and free. Then the quotient space M/G admits the structure of a topological manifold of dimension $\dim M - \dim G$. Moreover there exists a unique smooth structure on M/G such that the quotient map $\rho\colon M \to M/G$ is a smooth submersion.*

Just as with Theorem 11.11, the proof of Theorem 13.6 requires the Frobenius Theorem, and hence the proof is deferred until Lecture 14.

Corollary 13.7. *Let H be a closed subgroup of G. Then the space G/H of left cosets of H in G admits the structure of a topological manifold of dimension $\dim G - \dim H$. Moreover there exists a unique smooth structure on G/H such that the quotient map $\rho\colon G \to G/H$ is a smooth submersion.*

Recall by the Closed Subgroup Theorem 10.16 this is equivalent to asking that H be an embedded Lie subgroup of G.

Proof. By Problem F.5 the right action $r\colon H \times G \to G$ is both free and proper. The statement now follows from the Quotient Manifold Theorem. ■

Corollary 13.7 does not assert that G/H is a Lie group. Indeed, G/H is not even a group! If H is a *normal* subgroup however then G/H is a group, and in fact in this case:

Proposition 13.8. *Let G be a Lie group and let H be a closed normal subgroup. Then the smooth manifold G/H with its natural group structure is a Lie group.*

Proof. The map $\rho\colon G \to G/H$ is given by $\rho(g) = gH$ (the coset). The multiplication on G/H is given by

This is only well-defined when H is normal.

$$\bar{\mu}(g_1H, g_2H) := g_1g_2H.$$

To check this is smooth, fix $g_1, g_2 \in G$ and let ψ_1 and ψ_2 be smooth local sections of ρ near g_1H and g_2H respectively. Then near the point $(g_1H, g_2H) \in G/H \times G/H$, one has

These exist by Proposition 6.13.

$$\bar{\mu} = \rho \circ \mu \circ (\psi_1, \psi_2),$$

where $\mu\colon G\times G \to G$ is the multiplication on G. Thus near (g_1H, g_2H), $\bar{\mu}$ is the composition of smooth maps, and hence is smooth. Since g_1H and g_2H were arbitrary, $\bar{\mu}$ is smooth everywhere. A similar argument works for the inversion map. ■

Definition 13.9. A **homogeneous space** is a smooth manifold M which is diffeomorphic to a smooth manifold of the form G/H, where G is a Lie group, H is a closed subgroup, and G/H is given the smooth structure from Corollary 13.7.

Many manifolds are homogeneous spaces (we will shortly see some examples). The key tool used to prove a given manifold is a homogeneous space is Theorem 13.12 below, which needs a few preliminary definitions.

Definition 13.10. Let σ be a smooth action of G on M. A point $p \in M$ is called a **stationary point** of σ if

$$p \in \bigcap_{g \in G} \mathsf{fix}(\sigma_g), \qquad \text{that is,} \qquad \mathsf{stab}_\sigma(p) = G.$$

Proposition 13.11. *Let σ be a smooth left action of G on M. Assume that p is a stationary point of σ. Then the map*

$$\tau \colon G \to \mathrm{GL}(T_pM), \qquad \tau(g) := D\sigma_g(p)$$

is a Lie group homomorphism (i.e. a representation).

Proof. Let us first check τ is a homomorphism. For this, observe

$$\tau(gh) = D\sigma_{gh}(p) = D(\sigma_g \circ \sigma_h)(p) = \tau(g)\tau(h).$$

The smoothness issue is a little more delicate. At first glance, it would appear obvious – σ is smooth, and hence so is $D\sigma$. But herein lies the problem: $\mathrm{GL}(T_pM) \cong \mathrm{GL}(m)$ has its own topology, and a priori it is not clear how smoothness of σ helps.

By definition of the topology on $\mathrm{GL}(T_pM) \cong \mathrm{GL}(m)$, it suffices to show that for a fixed $\xi \in T_pM$ the map

$$g \mapsto D\sigma_g(p)\xi \tag{13.4}$$

is smooth as a map $G \to T_pM$. Since T_pM is an embedded submanifold of TM, it suffices to show that (13.4) is smooth as a map $G \to TM$. This however is the composition

The projection map $\pi \colon TM \to M$ is a smooth submersion (Theorem 5.6). Thus $T_pM = \pi^{-1}(p)$ is an embedded submanifold by the Implicit Function Theorem 6.10.

$$G \xrightarrow{Z} TG \times TM \cong T(G \times M) \xrightarrow{D\sigma} TM,$$

where Z is the smooth map $g \mapsto \big((g, 0), (p, \xi)\big)$ and the second map is the canonical identification coming from Problem C.1. This is the composition of smooth maps, and hence is smooth. ■

In general an action may have no stationary points. But if we restrict attention to the stabiliser group of a point (which is another smooth Lie group by Proposition 13.3) then that point is automatically stationary.

Theorem 13.12. *Let σ be a transitive left smooth action of G on M. Fix $p \in M$ and let $H := \mathsf{stab}_\sigma(p)$. Let $\rho \colon G \to G/H$ denote the quotient map, and endow G/H with the smooth structure from Corollary 13.7. Define*

$$\varphi \colon G/H \to M, \qquad \varphi(gH) := \sigma_g(p).$$

Then φ is a diffeomorphism, and hence M is a homogeneous space.

Proof. First observe that φ is well defined, since if $h \in H$ then $\sigma_{gh}(p) = \sigma_g \circ \sigma_h(p) = \sigma_g(p)$. We claim that φ is a bijection. Surjectivity follows from transitivity of σ. If $\varphi(g_1H) = \varphi(g_2H)$ then $\sigma_{g_1^{-1}g_2}(p) = p$, whence $g_1^{-1}g_2 \in H$ and thus $g_1H = g_2H$. This shows injectivity.

To show that φ is smooth in a neighbourhood of a point gH, it suffices to show that $\varphi \circ \rho$ is smooth near g. Indeed, if $\varphi \circ \rho$ is smooth at g and $\psi \colon G/H \to G$ is a smooth local section of ρ at gH then $\varphi = (\varphi \circ \rho) \circ \psi$ is the composition of smooth maps. Now observe that $\varphi \circ \rho = \sigma \circ i_p$, where $\imath_p \colon G \to G \times M$ is the smooth map $g \mapsto (g, p)$. Thus $\varphi \circ \rho$ is the composition of smooth maps, and hence is smooth.

We now show that φ is a diffeomorphism. We will do this by showing that φ has constant rank. Left translation induces a transitive smooth left action $\bar{l}$ of G on G/H:

$$\bar{l}_g(g_1 H) \coloneqq g g_1 H.$$

The map φ is $(\bar{l}, \sigma)$-equivariant, since

$$\begin{aligned} \varphi \circ \bar{l}_g(g_1 H) &= \varphi(g g_1 H) \\ &= \sigma_{g g_1}(p) \\ &= \sigma_g \circ \sigma_{g_1}(p) \\ &= \sigma_g \circ \varphi(g_1 H). \end{aligned}$$

Thus by Proposition 12.18 the map φ has constant rank, and then by Problem C.7 φ is a diffeomorphism. ■

Theorem 13.12 tells us that we can define a homogeneous space as a smooth manifold that admits a transitive Lie group action. We emphasise however that a given smooth manifold can sometimes be made into a homogeneous space in multiple ways.

Let us return to part (v) of Examples 12.13 and see how this fits into the homogeneous space picture.

EXAMPLE 13.13. The Lie group $\mathrm{GL}(m)$ acts on $\mathbb{R}^m$. This in itself is not very interesting, but observe the action of $\mathrm{O}(m) \subset \mathrm{GL}(m)$ restricts to a *transitive* action on $S^{m-1} \subset \mathbb{R}^m$ by elementary linear algebra. Moreover the isotropy subgroup of $e_m = (0, 0, \dots, 0, 1) \in S^{m-1}$ is given by those matrices $A \in \mathrm{O}(m)$ of the form

$$A = \begin{pmatrix} \begin{pmatrix} & & \\ & B & \\ & & \end{pmatrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \begin{matrix} 0 & \cdots & 0 \end{matrix} & 1 \end{pmatrix}$$

where $B \in \mathrm{O}(m-1)$. We conclude that S^{m-1} is the homogeneous space

$$S^{m-1} \cong \mathrm{O}(m) / \mathrm{O}(m-1).$$

The same argument works to show that

$$S^{m-1} \cong \mathrm{SO}(m) / \mathrm{SO}(m-1).$$

EXAMPLE 13.14. Let $\mathrm{U}(m) \subset \mathrm{GL}(m; \mathbb{C})$ denote the unitary group and $\mathrm{SU}(m) \subset \mathrm{U}(m)$ the special unitary group. If we regard S^{2m-1} as the unit sphere in $\mathbb{C}^m$ then a similar argument shows that

$$S^{2m-1} \cong \mathrm{U}(m) / \mathrm{U}(m-1) \qquad \text{and} \qquad S^{2m-1} \cong \mathrm{SU}(m) / \mathrm{SU}(m-1).$$

Since SU(1) is just the 1×1 identity matrix, taking $m = 2$ shows that S^3 is diffeomorphic to SU(2), and hence S^3 can be given a Lie group structure.

Remark 13.15. Not all smooth manifolds admit the structure of a Lie group. For instance, S^m admits a Lie group structure only for $m = 0, 1$ or $m = 3$. For $m = 0$ this is trivial. For $m = 1$, this was part (vi) from Examples 10.9 above, and we just did the case of S^3 in Example 13.14. The proof that no other sphere admits a Lie group structure is quite tricky, but roughly speaking proceeds as follows: suppose S^m admits a Lie group structure for $n > 1$. Since S^m is simply connected for $m > 1$, the Lie group structure is necessarily non-abelian. Next, one can show that any compact non-abelian Lie group G carries a natural closed but not exact bi-invariant differential 3-form. Thus $H^3(G; \mathbb{R}) \neq 0$. For S^m this forces $n = m = 3$.

This will be an exercise on one of the Problem Sheets next semester.

Next we return to part (ii) of Examples 12.13.

Definition 13.16. Let G be a Lie group and let c denote the conjugation action of G on itself. The identity e is a stationary point of this action, and hence by Proposition 13.11 we obtain a Lie group homomorphism $G \to \mathrm{GL}(\mathfrak{g})$. This is called the **adjoint representation** and is denoted by

$$\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g}).$$

We usually write $\mathrm{Ad}(g) = \mathrm{Ad}_g \colon \mathfrak{g} \to \mathfrak{g}$.

We can then go one step further and differentiate Ad. This requires us to look at the Lie algebra of $\mathrm{GL}(\mathfrak{g})$, which we write as

$$\mathfrak{gl}(\mathfrak{g}) = \{\text{all linear maps } \mathfrak{g} \to \mathfrak{g}\}.$$

Definition 13.17. The derivative of the adjoint representation is denoted by

$$\mathrm{ad} := D(\mathrm{Ad})(e) \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}).$$

We usually write $\mathrm{ad}(\xi) = \mathrm{ad}_\xi \colon \mathfrak{g} \to \mathfrak{g}$.

By Proposition 11.7 the map ad is a Lie algebra homomorphism. Moreover Proposition 12.5 gives us a commutative diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \mathrm{ad}\ } & \mathfrak{gl}(\mathfrak{g}) \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\ \mathrm{Ad}\ ]{} & \mathrm{GL}(\mathfrak{g}) \end{array}$$

The map ad has a pleasing description. The proof of the next result is deferred to Problem Sheet F.

Proposition 13.18. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then for $\xi, \zeta \in \mathfrak{g}$ one has $\mathrm{ad}_\xi(\zeta) = [\xi, \zeta]$.*

Bonus Material for Lecture 13

In this bonus section we survey one the most important infinite-dimensional Lie groups: namely, the diffeomorphism group of a compact manifold.

Let M be a compact manifold. The group $\mathrm{Diff}(M)$ can itself be given a Fréchet manifold structure. Under this Fréchet manifold structure, one can show that composition

$$\mu\colon \mathrm{Diff}(M) \times \mathrm{Diff}(M) \to \mathrm{Diff}(M), \qquad \mu(\varphi, \psi) := \varphi \circ \psi \tag{13.5}$$

is smooth. Similarly the map $\varphi \mapsto \varphi^{-1}$ is smooth. This means that $\mathrm{Diff}(M)$ is an **infinite-dimensional Fréchet Lie group**.

A Fréchet manifold is a weaker and less useful concept than that of a Banach manifold. The difference is that a Fréchet manifold is locally modelled on a Fréchet space rather than a Banach space. The reason they are less useful is that the Inverse and Implicit Function Theorems are valid for Banach manifolds, but not for Fréchet manifolds.

Sadly we have no choice in the matter. Even if we wanted to work with lower regularity, whilst the space of C^k-diffeomorphisms $C^k(M, M)$ does have a nice Banach manifold structure, it is *not* a Lie group. Indeed, with μ as in (13.5), whilst the map

$$\mu(\cdot, \psi)\colon C^k(M, M) \to C^k(M, M), \qquad \varphi \mapsto \varphi \circ \psi.$$

is smooth, the map

$$\mu(\varphi, \cdot)\colon C^k(M, M) \to C^k(M, M), \qquad \psi \mapsto \varphi \circ \psi.$$

is not even continuous!

Exercise: Why?

In any case, if we give $\mathrm{Diff}(M)$ its Fréchet smooth structure, then one can show that

$$T_{\mathrm{id}}\mathrm{Diff}(M) = \mathfrak{X}(M),$$

(as one would expect, the tangent space to an infinite-dimensional manifold is itself infinite-dimensional).

We now go through a few of the Lie-theoretic concepts we have studied, and see how they fit into the infinite dimensional picture:

(i) One-parameter subgroup of $\mathrm{Diff}(M)$ are precisely one-parameter groups of diffeomorphisms in the sense of Definition 11.12, i.e. paths $t \mapsto \Phi_t$ such that $\Phi_0 = \mathrm{id}$ and $\Phi_{s+t} = \Phi_s \circ \Phi_t$. Denote by X the infinitesimal generator of $\{\Phi_t\}$, defined in (9.7) If we adopt the notation from introduced before Proposition 11.14 then the curve $t \mapsto \Phi_t$ is the maximal integral curve γ^X.

(ii) The exponential map $\exp\colon \mathfrak{X}(M) \to \mathrm{Diff}(M)$ assigns to a vector field X its flow Φ_t – this is well-defined by Corollary 9.19.

(iii) The conjugation action

$$c_\varphi(\psi) := \varphi \circ \psi \circ \varphi^{-1}$$

gives rise to the adjoint map

$$\mathrm{Ad}_\varphi \colon \mathfrak{X}(M) \to \mathfrak{X}(M)$$

which one easily sees is given by

$$\mathrm{Ad}_\varphi(X) = \varphi_\star X.$$

(iv) If we differentiate this to get $\mathrm{ad}\colon \mathfrak{X}(M) \to \mathfrak{gl}(\mathfrak{X}(M))$, we find that

$$\mathrm{ad}_X(Y) = \mathcal{L}_Y X = [Y, X], \qquad \forall\, X, Y \in \mathfrak{X}(M),$$

Equation (iv) is somewhat problematic, since this sign error would appear to *contradict* Proposition 13.18!

Of course there is no actual contradiction, since this is all a matter of conventions. What we have learnt is:

If we want to think of $\mathfrak{X}(M)$ as the Lie algebra of the infinite-dimensional Lie group $\mathrm{Diff}(M)$ then the Lie bracket should have been defined with the *opposite sign convention*:

$$[X, Y] \coloneqq \mathcal{L}_Y X.$$

Some brave authors do indeed define the Lie bracket of vector fields in this way. Nevertheless we have chosen the "incorrect" sign convention so as to be consistent with the vast majority of the literature.

LECTURE 14

Distributions and Integrability

In this lecture we introduce *distributions* on manifolds and prove the local version of the famous *Frobenius Theorem*. The global version of this theorem – which will be proved next lecture – is the cornerstone of an area of differential geometry called *foliation theory*. This semester we will use the (global) Frobenius Theorem to prove the Lie Correspondence Theorem 11.11 and the Quotient Manifold Theorem 13.6. Next semester, we will use the Frobenius Theorem to show that flat connections on vector bundles have trivial restricted holonomy groups.

We begin with the following preliminary result.

PROPOSITION 14.1. *Let M be a smooth manifold and $W \subset M$ a non-empty open set. Suppose $X_1, \dots, X_l \in \mathfrak{X}(W)$ are vector fields such that*

(i) *There exists $p \in W$ such that the vectors $X_i(p)$ are all linearly independent in T_pM (and thus necessarily $l \leq m$)*

(ii) *For all i, j one has $[X_i, X_j] \equiv 0$.*

Then there exists a chart (U, x) about p with $U \subset W$ such that

$$\frac{\partial}{\partial x^i} = X_i|_U, \qquad \forall\, 1 \leq i \leq l.$$

An immediate corollary is the following extension of Problem D.2.

COROLLARY 14.2. *Let M be a smooth manifold and $W \subset M$ a non-empty open set. Let $X \in \mathfrak{X}(W)$ and suppose $X(p) \neq 0$ for some $p \in W$. Then there exists a chart (U, x) about p with $U \subset W$ such that $X|_U = \frac{\partial}{\partial x^1}$.*

Proof of Proposition 14.1. We prove the result in two steps. The first step reduces the problem to $\mathbb{R}^m$. That this is possible should be clear from the statement, since the assertion is visibly local.

1. If $x \colon U \to \mathcal{O}$ is any chart about p then the map $x_* \colon \mathfrak{X}(U) \to \mathfrak{X}(\mathcal{O})$ satisfies

$$x_* \left(\frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial u^i}, \qquad \forall\, 1 \leq i \leq m.$$

where (u^i) are the canonical local coordinates on $\mathcal{O} \subset \mathbb{R}^m$. Since x_* is an isomorphism, it is sufficient to find such a chart x so that

$$x_*(X_i|_U) = \frac{\partial}{\partial u^i}, \qquad \forall\, 1 \leq i \leq l. \tag{14.1}$$

Now let $y \colon U \to \mathcal{O}$ be an arbitrary chart about p such that $y(p) = 0$. We will modify y to produce a chart x satisfying (14.1). Let $Y_i \in \mathfrak{X}(O)$ denote the unique vector field such that

$$y_*(X_i|_U) = Y_i.$$

Since the X_i are linearly independent at p, the Y_i are linearly independent at 0. Thus there exists a linear isomorphism $\lambda\colon \mathbb{R}^m \to \mathbb{R}^m$ that maps $\mathcal{J}_0^{-1}(Y_i(0))$ to the standard basis vector e_i for each $1 \le i \le l$. Set $\tilde{y} := \lambda \circ y$ and set $\tilde{Y}_i = \tilde{y}_*(X_i|_U)$. Then

$$\tilde{Y}_i(0) = \frac{\partial}{\partial u^i}\Big|_0 \qquad \forall\, 1 \le i \le l. \tag{14.2}$$

We emphasise this identity only holds at the point 0. The aim now is to construct a local diffeomorphism h defined on a neighbourhood $V \subset \mathbb{R}^m$ about 0 such that $h(0) = 0$ and such that on V

$$h_*(\tilde{Y}_i) = \frac{\partial}{\partial u^i} \qquad \forall\, 1 \le i \le l. \tag{14.3}$$

Then setting $x := h \circ \tilde{y}$ one has where defined that

$$x_*(X_i) = h_* \circ \tilde{y}_*(X_i) = h_*(\tilde{Y}_i) = \frac{\partial}{\partial u^i}.$$

2. In this second step we construct such an h. Note by Proposition 8.19 that the vector fields $\tilde{Y}_i$ satisfy $[\tilde{Y}_i, \tilde{Y}_j] \equiv 0$. Let Φ^i_t denote the flow of $\tilde{Y}_i$. For a sufficiently small neighbourhood Ω of 0 in $\mathbb{R}^m$ there is a well defined smooth function $f\colon \Omega \to \mathbb{R}^m$ given by the following somewhat improbable looking formula:

$$f(u^1, \dots, u^m) := \left(\Phi^1_{u^1} \circ \dots \circ \Phi^l_{u^l}\right)(0, \dots, 0, u^{l+1}, \dots, u^m).$$

Let $g \in C^\infty(\mathbb{R}^m)$ and fix $q = (a^1, \dots, a^m) \in \Omega$. We first consider what $Df(q)$ does to $\frac{\partial}{\partial u^1}\big|_q$. Namely:

$$\begin{aligned} Df(q)\left(\frac{\partial}{\partial u^1}\Big|_q\right)(g) &= \frac{\partial}{\partial u^1}\Big|_q (g \circ f) \\ &= \lim_{t\to 0} \frac{\left(g \circ \Phi^1_{a^1+t} \circ \dots \circ \Phi^k_{u^k}\right)(0, \dots, 0, a^{l+1}, \dots, a^m) - (g \circ f)(q)}{t} \\ &= \lim_{t\to 0} \frac{g \circ \Phi^1_t(f(q)) - g(f(q))}{t} \\ &= \tilde{Y}_1(f(q))(g). \end{aligned}$$

Since g and q were arbitrary, this shows that $f_*\big(\frac{\partial}{\partial u^1}\big) = \tilde{Y}_1$. Since the Lie brackets vanish, using induction and Proposition 10.6 we have for any $1 \le i \le l$ that

$$\Phi^1_{u^1} \circ \dots \circ \Phi^i_{u^i} \circ \dots \circ \Phi^l_{u^l} = \Phi^i_{u^i} \circ \dots \circ \Phi^1_{u^1} \circ \dots \circ \Phi^l_{u^l},$$

and thus exactly the same argument shows that

$$f_*\left(\frac{\partial}{\partial u^i}\right) = \tilde{Y}_i, \qquad \forall\, 1 \le i \le l. \tag{14.4}$$

In particular, since $\tilde{Y}_i(0) = \frac{\partial}{\partial u^i}\big|_0$ by (14.2),

$$Df(0)\left(\frac{\partial}{\partial u^i}\Big|_0\right) = \frac{\partial}{\partial u^i}\Big|_0, \qquad \forall\, 1 \le i \le l, \tag{14.5}$$

In fact, we claim that (14.5) holds for all $1 \le i \le m$, and not just $1 \le i \le l$. To see this take $l < i \le m$ and observe with g as above that

$$\begin{aligned} Df(0)\left(\frac{\partial}{\partial u^i}\Big|_0\right)(g) &= \frac{\partial}{\partial u^i}\Big|_0 (g \circ f) \\ &= \lim_{t \to 0} \frac{g(0, \dots, 0, t, 0 \dots, 0) - g(0)}{t} \\ &= \frac{\partial}{\partial u^i}\Big|_0 (g). \end{aligned}$$

This shows that (14.5) holds for $l < i \le m$ as well, and hence $Df(0)$ is the identity. Thus by the Inverse Function Theorem 5.9 there exists a neighbourhood $V \subset \Omega$ containing 0 such that $f|_V$ is a diffeomorphism. Set $h := f|_V^{-1}$. Then (14.4) implies that the diffeomorphism h satisfies (14.3) and the proof is complete. ■

We now introduce the notion of a distribution.

Definition 14.3. Let M be a smooth manifold of dimension m, and let $l \le m$. A **distribution Δ on M of dimension** l is a choice of l-dimensional linear subspace $\Delta_p \subset T_pM$ for each $p \in M$ that varies smoothly with p in the following sense: For each point $p \in M$ there exists a neighbourhood U of p and l vector fields $X_1, \dots, X_l \in \mathfrak{X}(U)$ such that

$$\Delta_q = \operatorname{span}_{\mathbb{R}}\{X_1(q), \dots, X_l(q)\}, \qquad \forall\, q \in U.$$

The simplest example is $l = 1$.

Example 14.4. A vector field X is **non-vanishing** if $X(p) \ne 0$ for all $p \in M$. A non-vanishing vector field X defines a one-dimensional distribution by setting $\Delta_p := \operatorname{span}_{\mathbb{R}}\{X(p)\}$ for each $p \in M$.

Remark 14.5. Not every manifold admits such a vector field. Indeed, if m is even then every vector field on S^m vanishes in at least one point. This is the so-called "Hairy Ball Theorem", which you will be asked to prove later on in the course. In fact, the Hairy Ball Theorem is a purely topological result, and thus the smoothness assumption is not necessary: if m is even then any continuous map $S^m \to TS^m$ satisfying the section property 8.1 must vanish somewhere. This can be proved by applying the Whitney Approximation Theorem 7.13 to the smooth case, but it is also easy to show using some basic algebraic topology.

Definition 14.6. Let Δ be an l-dimensional distribution on M, and suppose $L \subset M$ is an l-dimensional immersed submanifold. We say that L is an **integral manifold** of Δ if

$$D\iota(p)\, T_pL = \Delta_p, \qquad \forall\, p \in L,$$

where $\iota \colon L \hookrightarrow M$ is the inclusion.

In the one-dimensional case, integral manifolds always exist about every point. Indeed, suppose Δ is a one-dimensional distribution. Given any $p \in M$ there exists a neighbourhood W of p and a vector field $X \in \mathfrak{X}(W)$ such that $\Delta_q = \operatorname{span}_{\mathbb{R}}\{X(q)\}$ for all $q \in W$. Since in particular $X(p) \neq 0$, by Corollary 14.2 there exists a chart (U, x) about p with $U \subset W$ such that $X|_U = \frac{\partial}{\partial x^1}$. Then the set

$$L := \{q \in U \mid x^2(q) = \cdots = x^m(q) = 0\}$$

is an embedded one-dimensional submanifold of M by Proposition 6.7. Moreover the proof of Proposition 6.7 shows that $D\iota(q)(T_q L) = \frac{\partial}{\partial x^1}\big|_q$ for all $q \in L$, where $\iota \colon L \hookrightarrow M$ is the inclusion. Thus L is an integral manifold of Δ containing p.

The case $l = 1$ of Theorem 14.13 shows that every connected integral manifold of Δ contained in U is of this form.

For higher dimensional distributions, integral manifolds need not exist. Here is an example.

Example 14.7. Consider the distribution on $\mathbb{R}^3$ spanned by the vector field

$$X := \frac{\partial}{\partial u^1} + u^2 \frac{\partial}{\partial u^3}, \qquad Y := \frac{\partial}{\partial u^2}.$$

We claim this distribution has no integral manifolds through the origin. Indeed, suppose such an L existed. Then (as the picture indicates), one would have $T_{(0,0,0)}L$ equal to the (u^1, u^2)-plane. But now suppose $\gamma \colon S^1 \to L$ is a closed curve in L that circles round the u^3-axis. Since γ is tangent to Δ, one readily sees that the u^3-component of γ is an increasing function. But then γ endlessly spirals upwards, and hence cannot close up – contradiction.

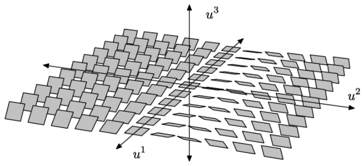


Figure 14.1: The standard contact distribution on $\mathbb{R}^3$. (Taken from Wikipedia.)

Example 14.7 is the starting point for the field of geometry called **contact geometry**. In general a **contact distribution** on a manifold is a distribution which is "maximally" non-integrable. Such a manifold is necessarily odd-dimensional. Contact manifolds are the odd-dimensional cousins of **symplectic manifolds**. Sadly we won't have time to study either contact or symplectic manifolds in this course.

We now formulate a condition that rules out such "pathologies".

Definition 14.8. Let Δ be a distribution on M and let X be a vector field on M. We say that X **belongs to** Δ if $X(p) \in \Delta_p$ for each $p \in M$. Let $\mathfrak{X}(\Delta, M) \subset \mathfrak{X}(M)$ denote the set of vector fields belonging to Δ.

Since Δ_p is a linear subspace of T_pM for each $p \in M$, $\mathfrak{X}(\Delta, M)$ is a linear subspace of the infinite-dimensional vector space $\mathfrak{X}(M)$.

Definition 14.9. A distribution Δ is said to be **integrable** if $\mathfrak{X}(\Delta, M)$ is a Lie subalgebra of $\mathfrak{X}(M)$, that is

$$X, Y \in \mathfrak{X}(\Delta, M) \qquad \Rightarrow \qquad [X, Y] \in \mathfrak{X}(\Delta, M)$$

Some authors use the word *involutive* instead of integrable to describe a distribution satisfying the conditions of Definition 14.9

Here are two conditions that guaranteee integrability.

Lemma 14.10. *Let Δ be an l-dimensional distribution on M. Suppose for every $p \in M$ there exists a neighbourhood U of p and l vector fields $X_1, \dots, X_l \in \mathfrak{X}(\Delta, U)$ such that Δ is spanned by the X_i over U and such that $[X_i, X_j] \in \mathfrak{X}(\Delta|_U, U)$ for all $1 \le i, j \le l$. Then Δ is integrable.*

Proof. Let $p \in M$ and let X and Y be vector fields that belong to Δ. Choose a neighbourhood U of p for which there exist vector fields spanning Δ as in the hypotheses of the Lemma. Then on U we can write

$$X|_U = f^i \, X_i, \qquad Y|_U = g^i \, X_i$$

for some smooth functions $f^i, g^i \colon U \to \mathbb{R}$. By Problem D.5 one has on U that

If you are worried why these functions are smooth, see Remark 20.10.

$$\begin{aligned}[X,Y]|_U &= [f^i X_i, g^j X_j] \\ &= f^i g^j [X_i, X_j] + f^i X_i(g^j) X_j - g^j X_j(f^i) X_i.\end{aligned}$$

Since $[X_i, X_j](q) \in \Delta_q$ for all $q \in U$, this shows that $[X,Y]$ belongs to Δ for every point in U. Since p was arbitrary, it follows that $[X,Y]$ belongs to Δ. ■

LEMMA 14.11. *Let Δ be a distribution on M. Assume that for every $p \in M$ there exists an integral manifold L_p of Δ with $p \in L_p$. Then Δ is integrable.*

Proof. Let X and Y belong to Δ. Fix an arbitrary point $p \in M$, and let $\iota_p \colon L_p \hookrightarrow M$ denote the inclusion. In the language of Problem D.7, X and Y are tangent to L_p. By part (iii) of Problem D.7, $[X,Y]$ is also tangent to L_p, or equivalently, $[X,Y](p) \in D\iota_p(p)(T_pL_p) = \Delta_p$. Since p was arbitrary, we conclude $[X,Y]$ belongs to Δ. ■

A more difficult result states that the converse to Lemma 14.11 holds.

NOTATION. Let $\mathbb{I}^l := (-1,1)^l$ denote the l-dimensional open unit cube, and write an element of $\mathbb{I}^l$ as a tuple $a = (a^1, \dots, a^l)$.

DEFINITION 14.12. A **shifted slice** in M of dimension l is an embedded submanifold of the form

$$L(a) := \left\{ p \in U \mid x^{l+1}(p) = a^1, \dots, x^m(p) = a^{m-l} \right\},$$

for some element $a = (a^1, \dots, a^{m-l}) \in \mathbb{I}^{m-l}$.

Thus the difference between a shifted slice and a normal slice is that instead of requiring the last $m - l$ coordinates to all be zero, we merely require them to be some fixed element in $\mathbb{I}^{m-l}$. Shifted slices are no more general than normal slices; nevertheless, they are a useful bookkeeping tool.

THEOREM 14.13 (The Local Frobenius Theorem). *Let M be a smooth manifold and let Δ be an integrable l-dimensional distribution on M. Then for every $p \in M$ there exists a chart $x \colon U \to \mathbb{I}^m$ with $x(p) = 0$ and such that for any $a \in \mathbb{I}^{m-l}$, the shifted slice*

$$L(a) := \left\{ q \in U \mid x^{l+1}(q) = a^1, \dots, x^m(q) = a^{m-l} \right\}$$

is an integral manifold of Δ. Moreover any connected integral manifold of Δ contained in U is contained in such a shifted slice.

Proof. Once again, the statement is purely local, so by arguing as in Step 1 of Proposition 14.1, we may assume that $M = \mathbb{R}^m$, $p = 0$, and Δ_0 is spanned by the vectors $\frac{\partial}{\partial u^i}\big|_0$ for $i = 1, \dots, l$. We argue in three steps.

Exercise: Fill in these details!

1. Write $\mathbb{R}^m = \mathbb{R}^l \times \mathbb{R}^{m-l}$. Let ρ_1 and ρ_2 denote the two projections $\mathbb{R}^m \to \mathbb{R}^l$ and $\mathbb{R}^{m-l}$ respectively:

$$\rho_1(u^1, \dots, u^m) := (u^1, \dots, u^l), \qquad \rho_2(u^1, \dots, u^m) := (u^{l+1}, \dots, u^m),$$

Let

$$\delta_q := D\rho_1(q)|_{\Delta_q} \colon \Delta_q \to T_q\mathbb{R}^l$$

Then $q \mapsto \delta_q$ is a smooth family of linear maps, whose domain ranges smoothly with q. By assumption δ_0 is an isomorphism. Since being invertible is an open condition, it follows that there is a neighbourhood W of 0 in $\mathbb{R}^m$ such that δ_q is an isomorphism for all $q \in W$.

This is because $q \mapsto \det \delta_q$ is a continuous function.

Thus up to possibly shrinking W, there exist unique vector fields $X_i \in \mathfrak{X}(\Delta, W)$ that are ρ_1-related to $\frac{\partial}{\partial u^i}$ for $i = 1, \dots, l$. By part (ii) of Problem D.6 one has that $[X_i, X_j]$ is ρ_1-related to $[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}]$. By Proposition 8.10, $[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}] = 0$, and thus $[X_i, X_j]$ is ρ_1-related to the zero vector field. Now since Δ is integrable, $[X_i, X_j]$ belongs to Δ, and since $D\rho_1(q)|_{\Delta_q} = \delta_q$ is injective for $q \in W$, it follows that $[X_i, X_j] = 0$.

This is the only place in the proof where we use integrability of Δ!

2. Thus we can apply Proposition 14.1 to obtain a chart $x\colon U \to \mathbb{I}^m$ defined on $U \subset W$ such that $X_i|_U = \frac{\partial}{\partial x^i}$. Now let

$$\varphi := \rho_2 \circ x \colon U \to \mathbb{I}^{m-l}$$

Then φ is a smooth surjective submersion, and thus by the Implicit Function Theorem 6.10, for any $a \in \mathbb{I}^{m-l}$, the set $L(a) := \varphi^{-1}(a)$ is an embedded submanifold of M, and any $q \in U$ belongs to a unique $L(a)$ – namely, $a = \varphi(q)$. Moreover by Proposition 6.15, if we denote by $\iota\colon L(a) \hookrightarrow U$ the inclusion then for any $q \in L(a)$ one has

$$\begin{aligned} D\iota(q)\, T_qL(a) &= \ker D\varphi(q) \\ &= \left\{ \xi \in T_qU \mid \xi(x^i) = 0 \text{ for } i = l+1, \dots, m \right\} \\ &= \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x^i}\bigg|_q \mid 1 \le i \le l \right\} \\ &= \Delta_q. \end{aligned}$$

3. It remains to prove the last sentence of the theorem. Suppose L is an arbitrary integral manifold of Δ contained in U. Then for any $q \in L$ and $\xi \in T_qL$, one has $(D\iota(q)\xi)(x^i) = 0$ for $i = l+1, \dots, m$. Thus $D(x^i \circ \iota)(q)$ is the zero map for each $i = l+1, \dots, m$, and hence $q \mapsto x^i(\iota(q))$ is a locally constant function. If L is connected, then it is constant, and thus L is contained in a single shifted slice. This completes the proof. ∎

LECTURE 15

Foliations and the Frobenius Theorem

In this lecture we will globalise the Local Frobenius Theorem 14.13, and then us the Global Frobenius Theorem to prove the two outstanding results from Lectures 11 and 13: the Lie Correspondence Theorem 11.11 and the Quotient Manifold Theorem 13.6.

We begin with an observation which will be useful later. The integral manifolds produced by the Local Frobenius Theorem are always embedded, despite the fact that Definition 14.6 only required them to be immersed. This is because we have only looked at integral submanifolds contained in some (small) set U. In general, integral manifolds do not always have to be embedded. Nevertheless, the Local Frobenius Theorem 14.13 shows that that an arbitrary integral manifold retains one of the important properties of embedded submanifolds. We now formalise the condition used in the second bullet point in the proof of Proposition 10.15 as a definition.

Definition 15.1. Let $L \subset M$ be an immersed submanifold. We say that L is **weakly embedded** if for every smooth manifold N and every smooth map $\varphi\colon N \to M$ such that $\varphi(N) \subset L$, the map φ is also smooth as a map $N \to L$.

Thus embedded submanifolds are automatically weakly embedded. Integral manifolds are too, as the following result shows.

Proposition 15.2. *Let Δ be an integrable distribution on a smooth manifold M. Every integral manifold L of Δ is a weakly embedded submanifold of M.*

Proof. Assume that $\varphi\colon N \to M$ is a smooth map such that $\varphi(N) \subset L$. Fix a point $p \in N$. By the Local Frobenius Theorem there exists a chart $x\colon U \to \mathbb{I}^m$ with $x(\varphi(p)) = 0$ such that all connected integral submanifolds of Δ contained in U are contained in shifted slices

$$L(a) = \left\{ q \in U \mid x^{l+1}(q) = a^1, \dots, x^m(q) = a^{m-l} \right\},$$

for $a \in \mathbb{I}^{m-l}$. Now consider $L \cap U$. Since U is open, this is another immersed submanifold, and hence – by definition of a manifold – has at most countably many connected components. Each such component is then a connected integral submanifold of Δ contained in U, and so by the Local Frobenius Theorem is contained in some shifted slice. Thus there are countably many $a_k \in \mathbb{I}^{m-l}$ such that

A topological space with uncountably many components can never be separable.

$$L \cap U \subset \bigcup_k L(a_k). \tag{15.1}$$

Now choose a chart (V, y) on N about p such that V is connected and $\varphi(V) \subset L \cap U$. Then the function

$$f \coloneqq x \circ \varphi \circ y^{-1}\colon y(V) \to \mathbb{I}^m$$

is smooth. Write $f^i := u^i \circ f$ as usual so that $f = (f^1, \dots, f^m)$. Then by (15.1) the last $(m-l)$ functions $(f^{l+1}, \dots, f^m)$ can only take values in the countable set $\{a_k\}_{k=1}^{\infty}$, and therefore they are locally constant. Since V is connected, they are constant. Thus $\varphi(V)$ is contained in a single shifted slice $L(a_k)$. Since $L \cap L(a_k)$ is an open subset of L that is embedded in M, it follows that $\varphi|_V : V \to L \cap L(a_k)$ is smooth. Thus also the composition $\varphi|_V : V \to L \cap L(a_k) \hookrightarrow L$ is smooth. Since p was an arbitrary point of N the claim follows. ∎

We now globalise Theorem 14.13. We begin with a definition.

Definition 15.3. Let M be a smooth manifold. An **l-dimensional foliation** $\mathcal{F}$ of M is a partition of M into l-dimensional connected immersed submanifolds, called the **leaves** of the foliation, such that:

(i) The collection of tangent spaces to the leaves defines a distribution Δ on M.

(ii) Any connected integral manifold of Δ is contained in a leaf of $\mathcal{F}$.

Each leaf L of $\mathcal{F}$ is called a **maximal integral manifold** of Δ. One says that the distribution Δ is **induced by** $\mathcal{F}$.

Here is the global version of Theorem 14.13.

Theorem 15.4 (The Global Frobenius Theorem). *Let Δ be an integrable distribution on M. Then Δ is induced by a foliation.*

Proof. Let Δ be an integrable distribution. By the Local Frobenius Theorem 14.13 for any point $p \in M$ there is a chart $x : U \to \mathbb{I}^m$ such that the slices

$$L(a) := \left\{ q \in U \mid x^{l+1}(q) = a^1, \dots, x^m(q) = a^{m-l} \right\} \tag{15.2}$$

for $a \in \mathbb{I}^{m-l}$ are integral manifolds of Δ. Since M is a separable metric space, we may choose a countable collection $(U_k, x_k)_{k \in \mathbb{N}}$ of charts whose domains U_k form an open cover of M. Now let $\mathcal{L}$ denote the collection of all slices $L(a)$ of the form (15.2) for all of the charts x_k. Define an equivalence relation on $\mathcal{L}$ by declaring that $L \sim L'$ if there exists a finite sequence $L = L_0, L_1, \dots, L_h = L'$ such that $L_i \cap L_{i+1} \neq \emptyset$ for $i = 0, \dots, h-1$.

Suppose $L \subset U_i$ is a shifted slice such that $L \cap U_j \neq \emptyset$. Since U_j is open, $L \cap U_j$ is a manifold, and hence $L \cap U_j$ has at most countably many components (we already used this argument in the proof of Proposition 15.2). It follows that each equivalence class can only contain countably many shifted slices $L \in \mathcal{L}$.

Now fix one such equivalence class $[L]$ and enumerate the countably many elements belonging to L as $\{K_i\}_{i \in \mathbb{N}}$, with $K_i \cap K_{i+1} \neq \emptyset$ for each i. For fixed $j \in \mathbb{N}$, if

$$\tilde{L}_j := \bigcup_{i=1}^{j} K_i$$

then $\tilde{L}_j$ is a finite union of connected embedded integral manifolds of Δ , and thus is itself an connected embedded integral manifold of Δ.

This means that if we set

$$\tilde{L} := \bigcup_{j=1}^{\infty} \tilde{L}_j$$

then $\tilde{L}$ is a union of an increasing sequence of connected embedded integral manifolds of Δ, and hence is itself a connected *immersed* integral manifold of Δ. If $[L] \neq [L']$ are two distinct equivalence classes then the corresponding unions $\tilde{L}$ and $\tilde{L}'$ are disjoint. Since by definition any connected integral manifold of Δ is contained in such a union, this shows that the set of these unions form a foliation of M which is induced by Δ. This completes the proof. ■

Exercise: Why?

We now provide the promised proofs of the Lie Algebra Correspondence Theorem 11.11 and the Quotient Manifold Theorem 13.6. For convenience, we restate both results here.

THEOREM 15.5 (The Lie Correspondence Theorem). *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ then there is a unique connected Lie subgroup H of G whose Lie algebra is $\mathfrak{h}$.*

Proof. Given $g \in G$, let Δ_g denote the subspace of T_gG given by the set of all vectors of the form $X_\xi(g)$, where $X_\xi \in \mathfrak{X}_l(G)$ is a left-invariant vector field such that $\xi = X_\xi(e) \in \mathfrak{h} \subset \mathfrak{g}$. Thus

$$\Delta_g := \{Dl_g(e)\xi \mid \xi \in \mathfrak{h}\}.$$

To see that Δ really is a distribution, note that if $\{\xi_i\}$ is a basis of $\mathfrak{h}$ then the left-invariant vector fields $\{X_{\xi_i}(g)\}$ span Δ_g at every point $g \in G$. Moreover since $\mathfrak{h}$ is a Lie subalgebra, $[\xi_i, \xi_j] \in \mathfrak{h}$ for each i, j and thus $[X_{\xi_i}, X_{\xi_j}] = X_{[\xi_i,\xi_j]}$ belongs to Δ for every i, j. Thus by Lemma 14.10 it follows that Δ is integrable. By the Global Frobenius Theorem 15.4, Δ induces a foliation of G. Let H denote the leaf containing e. For any $g_1 \in G$ we have $Dl_{g_1}(g)(\Delta_g) = \Delta_{g_1 g}$ by construction, and hence Dl_{g_1} leaves the distribution invariant. Thus l_{g_1} permutes the leaves of the foliation, i.e. it maps the leaf passing through g diffeomorphically onto the leaf passing through $g_1 g$. In particular, if $h \in H$ then $l_{h^{-1}}$ maps H to the leaf containing e, which is just H again. Thus $l_{h^{-1}}(H) = H$, which proves that H is a subgroup. It remains to prove that the multiplication map $m \colon H \times H \to H$ is smooth. We know that the multiplication $m \colon H \times H \to G$ is smooth and $m(H \times H) \subset H$. Thus by Proposition 15.2, m is also smooth as a map $H \times H \to H$. This complete the proof. ■

REMARK 15.6. This proof also shows that every Lie subgroup H of a Lie group G is weakly embedded.

THEOREM 15.7 (The Quotient Manifold Theorem). *Let σ be a smooth action of G on M which is both proper and free. Then the quotient space M/G admits the structure of a topological manifold of dimension $\dim M - \dim G$. Moreover there exists a unique smooth structure on M/G such that the quotient map $\rho \colon M \to M/G$ is a smooth submersion.*

The proof is non-examinable, and hence is delayed by one more line. . .

Bonus Material for Lecture 15

. . . to here.

Proof. We prove the result in five steps. Let

$$m := \dim M, \qquad l := \dim G.$$

We already know from Lemma 13.5 that M/G is Hausdorff. Moreover in the proof of Lemma 13.5 we showed that ρ is an open map. Thus if $\{B_i\}$ is a countable basis for the topology on M then $\{\rho(B_i)\}$ is a countable basis for the quotient topology on M/G. By Proposition 1.32 if we can show that M/G is locally Euclidean, it will follow that M/G is a topological manifold. In fact, we will directly construct a smooth atlas on M/G.

1. We now start the construction of a smooth atlas on M/G. For $p \in M$, let

$$\Delta_p := T_p \mathsf{orb}_\sigma(p)$$

denote the tangent space of the orbit. By Corollary 13.2 the subspace Δ_p has dimension l. We will show that Δ is an l-dimensional distribution on M. The idea is similar to the previous proof: let $\{\xi_i\}$ denote a basis for $\mathfrak{g}$. Define a vector field Y_i on M by

$$Y_i(p) := D\sigma^p(e)\xi_i \in T_pM.$$

The flow of Y_i is given by $\sigma_{\exp t\xi}$. By construction, Y_i belongs to Δ for every $p \in M$. Since $\dim \mathfrak{g} = \dim \Delta$, it follows that $\{Y_i\}$ span Δ everywhere. Thus Δ is a distribution. Since every point in M is contained in an integral manifold, Lemma 14.11 implies that this distribution is integrable. By the Global Frobenius Theorem 15.4, Δ induces a foliation of M. By Problems F.6 and F.7 the leaf of the foliation containing p is the connected component of $\mathsf{orb}_\sigma(p)$ containing p. Thus as in the previous proof, we see that σ_g permutes the leaves of this foliation, i.e. it sends the leaf through p to the leaf through $\sigma_g(p)$.

If G is connected, then the leaf is simply the orbit $\mathsf{orb}_\sigma(p)$.

2. Fix $p \in M$. In this step we apply the Local Frobenius Theorem 14.13. This provides us with a chart $x\colon U \to \mathbb{I}^m$ about p, where $\sigma(p) = 0$, such that each shifted slice

$$\left\{q \in U \mid x^{l+1}(q) = a^1, \dots x^m(q) = a^{m-l}\right\}$$

for $a = (a^1, \dots, a^{m-l}) \in \mathbb{I}^{m-l}$ is contained in an orbit. Set $V := \rho(U) \subset M/G$. Let

$$K := \{q \in U \mid x^1(q) = \dots = x^l(q) = 0\},$$

so that K is a connected embedded submanifold of M. Consider now the restriction of the action σ to $G \times K$:

$$\sigma\colon G \times K \to M.$$

Since $\dim K = m - l$, $G \times K$ has dimension m. Note that $\sigma|_{\{e\}\times K}$ is just the inclusion $\imath\colon K \hookrightarrow M$. Under the identification $T_{(e,p)}(G \times K) \cong \mathfrak{g} \oplus T_pK$ given by Problem C.1 the differential of σ at a point (e, p) is given by

$$D\sigma(e,p)(\xi,\zeta) = D\sigma^p(e)\xi + D\imath(p)\zeta, \qquad \xi \in \mathfrak{g},\ \zeta \in T_pM.$$

Since K is an immersed submanifold, $D\imath(p)$ is injective. By Corollary 13.2 the map $D\sigma^p(g)$ is an isomorphism. Thus $D\sigma(g, p)$ is injective linear map between two vector spaces of the same dimension, and hence is a linear isomorphism. Thus the Inverse Function Theorem 5.10 implies that there exists a neighbourhood W of (e, p) in $G \times K$ such that $\sigma|_W$ is a diffeomorphism.

3. In this step we prove that, up to replacing U with a smaller neighbourhood of p if necessary, the map $\rho|_K\colon K \to V$ is in fact homeomorphism. We begin by showing $\rho|_K$ is a bijective.

- **Surjective:** This is clear, since K intersects every shifted slice in U, so that $\rho(K) = \rho(U) = V$.

- **Injective:** We argue by contradiction: if the claim is false then we can find two sequences $(p_k), (q_k)$ in U such that $p_k \to p$ and $q_k \to p$ and

  Explicitly: if there exists no neighbourhood $U_0 \subset U$ of p such that $\rho|_{K\cap U_0}$ is injective, then ...

$$p_k \neq q_k, \quad \text{but} \quad \rho(p_k) = \rho(q_k), \qquad \forall\, k \in \mathbb{N}.$$

  Since $\rho(p_k) = \rho(q_k)$ the points p_k, q_k lie in the same orbit. Thus there exists $g_k \in G$ such that $q_k = \sigma_{g_k}(p_k)$. By Lemma 12.12, up to passing to a subsequence the sequence g_k converges to some $g \in G$. Then $\sigma_g(p) = p$, and hence as the action is free we must have $g = e$. For sufficiently large k, we therefore have

$$(e, p_k) \text{ and } (g_k, q_k) \text{ in } W, \qquad \sigma(e, p_k) = \sigma(g_k, q_k).$$

  This contradicts our assumption that σ is a diffeomorphism (and thus in particular injective) on W.

Thus $\rho|_{K_0}$ is bijective, as claimed. Since ρ is an open map, it follows that $\rho|_{K_0}$ is a homeomorphism.

4. We are now ready to construct our smooth atlas, and thus prove that M/G is a smooth manifold. With K as in the previous step, we define the homeomorphism

$$y \coloneqq \rho_2 \circ x \circ \rho|_K^{-1}\colon V \to \mathbb{I}^{m-l}.$$

The existence of y shows that M/G is locally Euclidean, and hence M/G is a topological manifold of dimension $m - l$. We will take y as our chart on M/G around $\rho(p)$. To show that the collection of charts (V, y) form a smooth atlas on M/G, we must check that the transition functions are smooth. So suppose (V_1, y_1) and (V_2, y_2) are constructed above with $V_1 \cap V_2 \neq \emptyset$ (with corresponding sets K_i, U_i and charts x_i). We must show

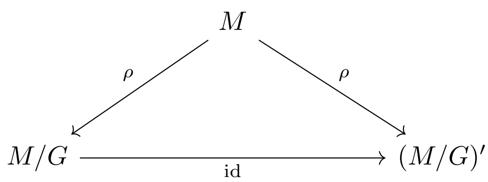
$$y_2 \circ y_1^{-1} \colon y_1(V_1 \cap V_2) \to y_2(V_1 \cap V_2)$$

is smooth. If $U_1 \cap U_2 \neq \emptyset$ then the claim is basically obvious, since the composition $x_2^{-1} \circ x_1$ is smooth where defined. For the general case, suppose $q_1 \in U_1$ and $q_2 \in U_2$ are such that $\rho(q_1) = \rho(q_2)$. Thus there exists $g \in G$ such that $\sigma_g(q_1) = q_2$. Let $\tilde{x}_1 = x_1 \circ \sigma_g^{-1}$, and let $\tilde{y}_1$ be defined accordingly. Then the argument above shows that $y_2 \circ \tilde{y}_1^{-1}$ is smooth. However by expanding the definitions one sees that actually $\tilde{y}_1 = y_1$ near $\rho(q_1)$, and hence $y_2 \circ y_1^{-1}$ is smooth near $\rho(q_1)$. Since $\rho(q_1) = \rho(q_2)$ was an arbitrary point of the intersection $V_1 \cap V_2$, it follows that $y_2 \circ y_1^{-1}$ is smooth.

5. We have now shown that M/G is a smooth manifold. The map $\rho\colon M \to M/G$ is smooth, since with the notation as above,

$$y \circ \rho \circ x^{-1} = \rho_2,$$

which is smooth. It remains to show that this is the unique smooth structure on M/G for which $\rho\colon M \to M/G$ is a smooth submersion. Suppose $(M/G)'$ is the same topological manifold, but endowed with a different smooth atlas for which ρ is a smooth submersion. We claim that $\mathrm{id}\colon M/G \to (M/G)'$ is a diffeomorphism:



Fix $p \in M$. By Proposition 6.13 there exists a neighbourhood U of p and a neighbourhood V of $\rho(p)$ together with a smooth (with respect to the smooth structure on M/G) map $\psi\colon V \to U$ such that $\rho \circ \psi = \mathrm{id}_V$. Thus the identity map $\mathrm{id}|_U \colon U \subset M/G \to U \subset (M/G)'$ is smooth. Since p was arbitrary, $\mathrm{id}\colon M/G \to (M/G)'$ is smooth. Reversing the roles of M/G and $(M/G)'$ shows that $\mathrm{id}\colon (M/G)' \to M/G$ is also smooth, and hence a diffeomorphism. Thus the smooth atlases on M/G and $(M/G)'$ both define the same smooth structure. This completes the proof. ■

LECTURE 16

Bundles

In this lecture we define the general notion of a **fibre bundle**. This is, roughly speaking, a space that locally looks like a product. Whilst fibre bundles are important in many areas of topology, they are slightly too vague to be useful for us. We therefore quickly specialise to the two special types of fibre bundles used in differential geometry: **vector bundles** and **principal bundles**. The study of such bundles will make up the majority of the rest of the course.

DEFINITIONS 16.1. Let E, M and L be smooth manifolds, and suppose $\pi \colon E \to M$ is a smooth surjective map. We say that $\pi \colon E \to M$ is a **fibre bundle over** M **with fibre** L if for every point $p \in M$ there exists a neighbourhood U of p and a smooth map

$$\varepsilon \colon \pi^{-1}(U) \to L$$

such that

$$(\pi, \varepsilon) \colon \pi^{-1}(U) \to U \times L$$

is a diffeomorphism. We call ε a **bundle chart** for E. A **bundle atlas** on E is any collection $\{(U_a, \varepsilon_a) \mid a \in A\}$ of bundle charts such that the sets U_a form an open cover of M.

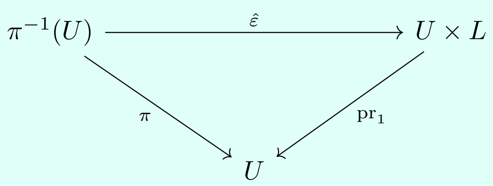
There are no compatibility conditions in the definition of a bundle atlas. This is because all the spaces involved are already assumed to be manifold.

We call E the **total space** of the bundle, M the **base space**, and L the **fibre**. We use the notation $L \to E \xrightarrow{\pi} M$ to denote a fibre bundle E over M with fibre L. When no confusion is possible we shorten the notation $L \to E \xrightarrow{\pi} M$ to simply E.

This is just notation: the arrow $L \to E$ does not represent any one particular map.

We should really say "smooth fibre bundle", but since we won't ever have cause in this course to look at non-smooth fibre bundles, we omit the adjective smooth.

If $\varepsilon \colon \pi^{-1}(U) \to L$ is a bundle chart, it is convenient to denote by $\hat{\varepsilon}$ the map (π, ε). Thus $\hat{\varepsilon}$ is a diffeomorphism such that the following commutes:



We call $\hat{\varepsilon}$ a **local trivialisation** of E. Thus there is a one-to-one correspondence between local trivialisations and bundle charts. We will use this notation without further comment for the rest of the course.

Principal bundles are an exception to this; see Definition below.

Convention. The total space of a fibre bundle will usually be denoted by E or F. As with Lie groups, this means that the dimension of such a total space is *not* written with the corresponding lower case letter. However in this case there is no confusion, since if $L \to E \xrightarrow{\pi} M$ is a fibre bundle then the existence of local trivialisations force

$$\dim E = m + l.$$

We normally write points in fibre bundles with the letters u and v.

Examples 16.2. Here are two examples:

(i) The simplest example of a fibre bundle is the product manifold $E = M \times L$ with $\pi \colon M \times L \to M$ the first projection. In this case we can take U to be all of M and define $\hat{\varepsilon} \colon M \times L \to M \times L$ to be the identity map. More generally, any fibre bundle E which is globally diffeomorphic to $M \times L$ is called a **trivial bundle**.

(ii) A **sphere bundle** is a fibre bundle $S^l \to E \xrightarrow{\pi} M$. Sphere bundles are particularly important in algebraic topology. On Problem Sheet G you will show that the Klein bottle is an S^1-bundle over S^1.

Definition 16.3. Given a fibre bundle $L \to E \xrightarrow{\pi} M$, we set $E_p := \pi^{-1}(p)$ for $p \in M$ and call E_p the **fibre** over p.

If (U, ε) is a bundle chart on E, then for $p \in U$ we denote by $\varepsilon_p \colon E_p \to L$ the restriction of ε to the fibre E_p. These maps are diffeomorphisms, as the next lemma shows.

Lemma 16.4. *Let $L \to E \xrightarrow{\pi} M$ be a fibre bundle. Then π is a surjective submersion, and each fibre E_p is an embedded submanifold of E diffeomorphic to L.*

Proof. Fix $p \in M$, and let $\varepsilon \colon \pi^{-1}(U) \to L$ be a bundle chart such that $p \in U$, and let $\mathrm{pr}_1 \colon U \times L \to U$ and $\mathrm{pr}_2 \colon U \times L \to L$ denote the two projections. These are both submersions. Fix $u \in E_p$. Since $\hat{\varepsilon}$ is a diffeomorphism, its derivative at u is a bijection $T_u E \to T_{\hat{\varepsilon}(u)}(M \times L)$. Differentiating the equation $\pi = \mathrm{pr}_1 \circ \hat{\varepsilon}$, we see that $D\pi(u)$ is the composition

$$D\pi(u) = D\,\mathrm{pr}_1(\hat{\varepsilon}(u)) \circ D\hat{\varepsilon}(u),$$

and thus is surjective. Since u was an arbitrary point of E_p, this shows that p is a regular value of π, and since p was arbitrary, we see that π is submersion. The Implicit Function Theorem 6.10 then tells us that each fibre is naturally an embedded submanifold of E. Finally, $\hat{\varepsilon}$ maps E_p diffeomorphically onto the embedded submanifold $\{p\} \times L$ of $U \times L$, which is itself diffeomorphic to L via pr_2. ■

Remark 16.5. Suppose (W, ε) is a bundle chart on E. Let (U, x) and (V, y) be (manifold) charts on M and L respectively with $W \subset U$. Then $(x \circ \pi, y \circ \varepsilon)$ is a manifold chart on an open set in E which is compatible with the given smooth structure on E.

It is often useful to work backwards. Suppose we begin with a *set* E and a surjective map $\pi \colon E \to M$, where M is a smooth manifold.

Suppose in addition we are given another smooth manifold L and an open cover $\{U_a \mid a \in A\}$ of M, together with a collection of bijections

$$\hat{\varepsilon}_a \colon \pi^{-1}(U_a) \to U_a \times L$$

such that $\mathrm{pr}_1 \circ \hat{\varepsilon}_a = \pi$. We can then attempt to *define* a smooth structure by declaring that charts on E are of the form $(x \circ \pi, y \circ \varepsilon_a)$, where x is a chart on M defined on an open subset of U_a, and y is some chart on L. Of course, now there is something to check. By Proposition 1.17, if one can verify that the transition functions are diffeomorphisms, this will endow E with a smooth manifold structure in such a way that the (U_a, ε_a) become a fibre bundle atlas.

DEFINITION 16.6. Suppose we have two fibre bundles

$$L_1 \to E \xrightarrow{\pi_1} M, \qquad L_2 \to F \xrightarrow{\pi_2} N$$

such that $L_1 \subset L_2$, $E \subset F$ and $M \subset N$ are all embedded submanifolds. We say that E is a **subbundle** of F if $\pi_2|_E = \pi_1$, that is

$$E_p \subset F_p, \qquad \forall\, p \in M.$$

EXAMPLE 16.7. If $L \to E \xrightarrow{\pi} M$ is any fibre bundle and $\hat{\varepsilon} \colon \pi^{-1}(U) \to U \times L$ is a local trivialisation, then we can consider $L \to \pi^{-1}(U) \xrightarrow{\pi} U$ as a fibre bundle in its own right. This fibre bundle is trivial and is a subbundle of E. As a result we often say that E is **trivial over** U if there exists a local trivialisation with domain $\pi^{-1}(U)$.

Suppose $L \to E \xrightarrow{\pi} M$ is a fibre bundle and $\{(U_a, \varepsilon_a) \mid a \in A\}$ is a bundle atlas. If $U_a \cap U_b \neq \emptyset$ then for each $p \in U_a \cap U_b$, the fibre parts ε_a and ε_b restrict to define diffeomorphisms $\varepsilon_{a|p}, \varepsilon_{b|p} \colon E_p \to L$. Thus there is a well-defined map

$$\varepsilon_{ab} \colon U_a \cap U_b \to \mathrm{Diff}(L), \qquad p \mapsto \varepsilon_{a|p} \circ \varepsilon_{b|p}^{-1} \tag{16.1}$$

We usually call ε_{ab} the **transition function** from the bundle chart ε_a to the bundle chart ε_b, and refer to the collection $\{\varepsilon_{ab}\}$ of all transitions functions arising from the bundle atlas as the transition functions of the bundle atlas. By definition one has

Warning: This is a slightly different meaning of the word "transition function" than was used in Definition 1.10.

$$\varepsilon_{ab}(p) \circ \varepsilon_{b|p} = \varepsilon_{a|p} \tag{16.2}$$

as maps $E_p \to L$. If $U_a \cap U_b \cap U_c \neq \emptyset$ then the following **cocycle condition** is automatically satisfied:

The name "cocycle" comes from Čech cohomology. This is not important here.

$$\varepsilon_{ac}(p) = \varepsilon_{ab}(p) \circ \varepsilon_{bc}(p), \qquad \forall\, p \in U_a \cap U_b \cap U_c.$$

The composition on the right-hand side occurs in $\mathrm{Diff}(L)$. In particular,

$$\varepsilon_{ab}(p)^{-1} = \varepsilon_{ba}(p).$$

As remarked at the beginning of the lecture, in this level of generality fibre bundles are not particularly useful in differential geometry. One way to understand this is the following: the transition functions

(16.1) take values in the infinite-dimensional manifold $\mathrm{Diff}(L)$. This space is simply "too large" to work with. We therefore seek a way to cut down the possible options for the transition functions, and for this, we introduce a Lie group into the mix.

Suppose σ is an *effective* action of a Lie group G on L. Then the homomorphism $g \mapsto \sigma_g$ is an injective map $G \to \mathrm{Diff}(L)$, and hence we can regard G as a subgroup of $\mathrm{Diff}(L)$.

Definition 16.8. Suppose $L \to E \xrightarrow{\pi} M$ is a fibre bundle and σ is an effective action of G on L. A bundle atlas $\{(U_a, \varepsilon_a) \mid a \in A\}$ is said to be a **(G, σ)-bundle atlas** if the transition functions (16.1) take values in G, i.e., if $U_a \cap U_b$ is non-empty then there exists a smooth map $g_{ab} \colon U_a \cap U_b \to G$ such that

$$\varepsilon_{ab}(p) = \sigma_{g_{ab}(p)}. \tag{16.3}$$

If such an atlas exists, we say that E is a **(G, σ)-fibre bundle**, and we call G the **structure group** of the bundle.

We sometimes refer to a (G, σ)-fibre bundle simply as a **G-fibre bundle**, particularly when the action σ is either unimportant or clear from the context.

Remark 16.9. Definition 16.8 still makes perfect sense if we drop the assumption that σ is effective. However if σ is not effective then the maps g_{ab} are not uniquely determined – for example, if σ is the trivial action then *any* maps g_{ab} will work. This is not the end of the world, but it occasionally annoying, and for this reason we will only work with effective actions when discussing fibre bundles.

For example, it complicates the uniqueness part of the Fibre Bundle Construction Theorem 17.5.

Moreover Proposition 12.14 shows that if we start with any (not necessarily) effective action of G on L, we can convert it into an effective action without changing its image in $\mathrm{Diff}(L)$. Since the definition of (G, σ)-bundle atlas only uses σ through its image in $\mathrm{Diff}(L)$, this shows that working only with effective actions does not actually involve any loss of generality.

Remark 16.10. Just as with smooth atlases on manifolds, since (G, σ)-bundle atlases come with compatibility conditions, the union of two (G, σ)-bundle atlases may not be still be a (G, σ)-bundle atlas. However we can define an equivalence relation on the set of (G, σ)-bundle atlases by declaring two atlases to be equivalent if their union is another (G, σ)-bundle atlas. We then define a **(G, σ)-bundle structure** to be an equivalence class. Alternatively, a (G, σ)-bundle structure can be thought of as a maximal (G, σ)-bundle atlas. (Compare Remark 1.12). In practice however, just as with smooth atlases versus smooth structures on manifolds, the distinction is usually unimportant.

Remark 16.11. A given fibre bundle $L \to E \xrightarrow{\pi} M$ may have structure group G for many different Lie groups G (and thus we should really say "a structure group" rather than "the structure group"). It is often advantageous to make the structure group as small as possible: if E

has structure group G and $H \subset G$ is a Lie subgroup, then sometimes it is possible to find a (G, σ)-bundle atlas such that each transition function ε_{ab} takes image in $\{\sigma_h \mid h \in H\} \subset \mathrm{Diff}(L)$. Then this (G, σ)-bundle atlas is actually an $(H, \sigma|_H)$-bundle atlas, and we say that we have **reduced the structure group to H**. A concrete example of this awaits you on Problem Sheet G.

Passing from general fibre bundles to G-fibre bundles thus replaces the infinite-dimensional group $\mathrm{Diff}(L)$ with the finite-dimensional group G. This is already a major improvement over a general fibre bundle, but it is still not enough. There are two special types of G-fibre bundles that are of particular importance in differential geometry, and we introduce them now.

These two special types of fibre bundles come from the two "canonical" choices of Lie group actions we have met so far:

(i) If V is a vector space, then there is a canonical representation of $\mathrm{GL}(V)$ on V, cf. part (iv) of Examples 12.13.

(ii) If G is a Lie group, then G acts naturally on itself via left translation.

Option (i) gives rise to *vector bundles*, and option (ii) gives rise to *principal bundles*.

Definition 16.12. Let M be a smooth manifold. A **vector bundle** over M is a $\mathrm{GL}(V)$-fibre bundle $V \to E \xrightarrow{\pi} M$, where V is a vector space and $\mathrm{GL}(V)$ acts on V via the canonical representation. We say that E has **rank** l if $\dim V = l$.

Definition 16.13. Let M be a smooth manifold and G a Lie group. A **G-principal bundle** over M is a G-fibre bundle $G \to P \xrightarrow{\pi} M$, where G acts on itself via left translation.

In contrast to other fibre bundles, principal bundles are usually written with the letters P and Q.

Although it is not obvious from the definitions, the theories of vector bundles and principal bundles are essentially analogous, and it is largely a matter of taste whether one primarily works with vector bundles or principal bundles. Roughly speaking: principal bundles are slightly more general, whereas vector bundles are slightly easier to understand. We will return to this at the end of Lecture 18.

Our canonical example of a vector bundle is the tangent bundle.

Example 16.14. Let M be a smooth manifold. Then the tangent bundle $\pi \colon TM \to M$ is a vector bundle of rank m over M. It is clear that the fibres T_pM are vector spaces, so we need only check that the transition functions are linear. Let $\{(U_a, x_a) \mid a \in A\}$ denote a smooth atlas on M. Define

$$\hat{\varepsilon}_a \colon \pi^{-1}(U_a) \to U_a \times \mathbb{R}^m, \qquad \hat{\varepsilon}_a(p, \xi) = \big(p, (dx_a^i)_p(\xi) e_i\big)$$

With our new notation, the corresponding chart $\tilde{x}_a$ on TM constructed in the proof of Theorem 5.6 is given by

$$\tilde{x}_a = (x_a \circ \pi, \varepsilon_a)$$

which is compatible with the first paragraph of Remark 16.5. Moreover if $U_a \cap U_b \neq \emptyset$ then by (5.1) we have

$$\varepsilon_{ab}(p) = D(x_a \circ x_b^{-1})(x_b(p)),$$

which lies in $\mathrm{GL}(m) \subset \mathrm{Diff}(\mathbb{R}^m)$. A similar argument shows that the cotangent bundle T^*M is another vector bundle of rank m over M.

We have also already met many principal bundles in this course, via the Quotient Manifold Theorem 13.6, although at the moment this is not easy to deduce directly from the definition. We will come back to this at the end of the lecture.

Suppose $V \to E \xrightarrow{\pi} M$ is a fibre bundle with fibre a vector space. What does it mean to say that E is a vector bundle? The next result clarifies this.

Proposition 16.15. *Let $\pi\colon E \to M$ be a fibre bundle with fibre a vector space V. Then E is a vector bundle if and only if it is possible to endow each fibre E_p with a vector space structure and find a bundle atlas $\{(U_a, \varepsilon_a)\}$ with the property that for any $p \in U_a$ the map $\varepsilon_{a|p}\colon E_p \to V$ is a vector space isomorphism.*

Proof. Sufficiency is clear, for if (U_a, ε_a) and (U_b, ε_b) are two overlapping bundle charts as in the statement then

$$\varepsilon_{ab}(p) = \varepsilon_{a|p} \circ \varepsilon_{b|p}^{-1}$$

is the composition of linear maps, and hence is linear. Conversely if $V \to E \xrightarrow{\pi} M$ is a vector bundle of rank l then Problem B.1 implies that each fibre E_p admits the structure of a vector space, and moreover that this vector space structure has the property that each $\varepsilon_{a|p}\colon E_p \to V$ is a vector space isomorphism. ■

Proposition 16.15 allows us to make the following alternative definition of a vector bundle.

Definition 16.16. Let $\pi\colon E \to M$ be a surjective smooth map between two smooth manifolds, and set $E_p := \pi^{-1}(p)$. We say that E is a **vector bundle of rank** n if each E_p admits the structure of an n-dimensional vector space, and any $p \in M$ has a neighbourhood U together with a smooth map $\varepsilon\colon \pi^{-1}(U) \to \mathbb{R}^n$ such that:

(i) $(\pi, \varepsilon)\colon \pi^{-1}(U) \to U \times \mathbb{R}^n$ is a diffeomorphism,

(ii) if $\varepsilon_q := \varepsilon|_{E_q}$ then $\varepsilon_q\colon E_q \to \mathbb{R}^n$ is a vector space isomorphism for all $q \in U$.

We will call such a map ε a **vector bundle chart**, and the collection of such charts is called a **vector bundle atlas**.

Here is the analogous statement for principal bundles.

PROPOSITION 16.17. *Let $G \to P \xrightarrow{\pi} M$ be a fibre bundle with fibre G. Then P is a principal bundle if and only if there exists a smooth fibre preserving free right action τ on P, together with a bundle atlas $\{(U_a, \varepsilon_a)\}$ such that each map ε_a is (τ, r)-equivariant.*

Here by fibre preserving we mean that $\tau_g(P_p) \subset P_p$, i.e. $\pi \circ \tau_g = \pi$.

The fact that τ is a *right* action is because in Definition 16.12 we chose the convention that G acts on itself by *left* translations.

Proof. Suppose such an action τ exists. Since the action τ is fibre preserving, equivariance also holds in the fibres, that is,

$$\varepsilon_{a|p} \circ \tau_g = r_g \circ \varepsilon_{a|p} \tag{16.4}$$

for all $g \in G$ and $p \in U_a$. Our goal is to show that the map

$$\varepsilon_{ab} \colon U_a \cap U_b \to \operatorname{Diff}(G)$$

is actually of the form

$$\varepsilon_{ab}(p) = l_{g_{ab}(p)},$$

for $g_{ab} \colon U_a \cap U_b \to G$ a smooth function. Define

$$g_{ab}(p) := \varepsilon_{a|p} \circ \varepsilon_{b|p}^{-1}(e).$$

This is the composition of smooth maps and hence is smooth. Moreover it depends smoothly on p since ε_a and ε_b are smooth. Now take an arbitrary element $h \in G$. We compute

$$\begin{aligned}
\varepsilon_{ab}(p)(h) &= \varepsilon_{a|p} \circ \varepsilon_{b|p}^{-1}(h) \\
&= \varepsilon_{a|p} \circ \varepsilon_{b|p}^{-1} \circ r_h(e) \\
&= \varepsilon_{a|p} \circ \tau_h \circ \varepsilon_{b|p}^{-1}(e) \\
&= r_h \circ \varepsilon_{a|p} \circ \varepsilon_{b|p}^{-1}(e) \\
&= r_h(g_{ab}(p)) \\
&= l_{g_{ab}(p)}(h).
\end{aligned}$$

For the converse, suppose $\{(U_a, \varepsilon_a) \mid a \in A\}$ is a (G, l)-bundle atlas. We define a map

$$\tau \colon G \times P \to P$$

by declaring that

$$\tau_g(u) := \varepsilon_{a|p}^{-1} \circ r_g \circ \varepsilon_{a|p}(u)$$

where $a \in A$ is any element of A such that $\pi(u) \in U_a$. This is well defined, since if $\pi(u) \in U_a \cap U_b$ then

$$\begin{aligned}
\varepsilon_{b|p}^{-1} \circ r_g \circ \varepsilon_{b|p}(u) &= \varepsilon_{a|p}^{-1} \circ \varepsilon_{ab}(p) \circ r_g \varepsilon_{b|p}(u) \\
&= \varepsilon_{a|p}^{-1} \circ l_{g_{ab}(p)} \circ r_g \circ \varepsilon_{b|p}(u) \\
&= \varepsilon_{a|p}^{-1} \circ r_g \circ l_{g_{ab}(p)} \circ \varepsilon_{b|p}(u) \\
&= \varepsilon_{a|p}^{-1} \circ r_g \circ \varepsilon_{a|p}(u),
\end{aligned}$$

where the last line used (16.2). Now that we know τ is well defined, it is immediate that τ is a smooth fibre preserving free right action of G on P. Finally (16.4) holds by definition. ∎

In fact, we can further improve Proposition 16.17.

Proposition 16.18. *Let $\pi\colon P \to M$ be a surjective submersion and let G be a Lie group. Then P is a principal G bundle if and only if there exists a smooth free right action τ of G on P which is fibre preserving and transitive on the fibres.*

i.e. $P_{\pi(u)} = \mathsf{orb}_\tau(u)$ for all $u \in P$.

Proof. Suppose P is a principal G bundle. By Proposition 16.17 this means there exists a bundle atlas $\{(U_a, \varepsilon_a)\}$ such that each ε_a is (τ, r)-equivariant. Fix $p \in M$ and let $u_1, u_2 \in P_p$. Suppose $p \in U_a$. Let $g_i := (\varepsilon_a)_p(u_i)$ for $i = 1, 2$. Then by definition

$$\hat{\varepsilon}_a\big(\tau_{g_1^{-1}g_2}(u_1)\big) = (p, g_2) = \hat{\varepsilon}_a(u_2).$$

Since $\hat{\varepsilon}_a$ is a diffeomorphism, we have $u_2 = \tau_{g_1^{-1}g_2}(u_1)$. This shows that τ is transitive on the fibres.

Now suppose that τ is a smooth free fibre preserving action of G on P which is transitive on the fibres. Since π is a surjective submersion, by Proposition 6.13, for each $p \in M$ there is a neighbourhood U of p and a smooth local section $\psi\colon U \to P$ of π. Now consider the map

$$\varphi\colon U \times G \to \pi^{-1}(U), \qquad \varphi(q, g) := \tau_g(\psi(q)).$$

By hypothesis the map φ is a smooth injection. Under the splitting $T_{(q,g)}(U \times G) \cong T_qU \oplus T_gG$ from Problem C.1 the derivative of φ is given by

$$D\varphi(q, g) = D\tau_g(\psi(q)) \circ D\psi(q) + D\tau^{\psi(q)}(g).$$

By Corollary 13.2 this map has maximal rank $\dim M + \dim G$ at (q, g), and hence by the Inverse Function Theorem 5.10, the map φ is a diffeomorphism. Thus we can write $\varphi^{-1} = (\pi, \varepsilon)$ for a uniquely determined smooth function $\varepsilon\colon \pi^{-1}(U) \to G$. This will form our desired principal bundle chart once we check (τ, r)-equivariance. Let $u \in \pi^{-1}(U)$ and assume that $\pi(u) = q \in U$. Then

$$(\pi, \varepsilon)(\tau_g(u)) = \big(q, \varepsilon(\tau_g(u))\big),$$

and hence

$$\tau_g(u) = \varphi\big(q, \varepsilon(\tau_g(u)\big) \tag{16.5}$$

In particular for $g = e$ we get

$$u = \tau_e(u) = \varphi\big(q, \varepsilon(u)\big). \tag{16.6}$$

Then for $g \in G$ we compute:

$$\begin{aligned} \varphi(q, r_g(\varepsilon(u)) &= \tau_{\varepsilon(u)g}(\psi(q)) \\ &= \tau_g \circ \tau_{\varepsilon(u)}(\psi(q)) \\ &= \tau_g\big(\varphi(q, \varepsilon(u)\big) \\ &= \tau_g(u) \\ &= \varphi\big(q, \varepsilon(\tau_g(u)\big), \end{aligned}$$

where the last two lines used (16.6) and (16.5) respectively. Since φ is a diffeomorphism this shows that $r_g(\varepsilon(u)) = \varepsilon(\tau_g(u))$. ∎

Remark 16.19. An action τ satisfying (either) of the hypotheses of Proposition 16.18 is automatically proper. Indeed, the assertion that $\mathsf{orb}_\tau(u) = P_{\pi(u)}$ for all $u \in P$ implies that M is topologically the orbit space P/G. Problem G.2 then implies that τ is automatically proper.

Propositions 16.17 and 16.18 allow us to make the following alternative definition of a principal bundle, which is analogous to Definition 16.16.

Definition 16.20. Let $\pi\colon P \to M$ be a surjective smooth submersion, and set $P_p := \pi^{-1}(p)$. We say that P is a **principal G-bundle** if there exists a smooth free right action τ on P which is both fibre preserving and transitive on the fibres. A bundle chart (U, ε) which is (τ, r)-equivariant is called a **principal bundle chart** – Proposition 16.17 shows that we can find a bundle atlas of such charts, which we call a **principal bundle atlas**.

Definition 16.20 and Remark 16.19 allow us to use Quotient Manifold Theorem 13.6 to produce principal bundles.

Corollary 16.21. *Let τ be a proper free action of a Lie group G on a smooth manifold P. Then $\rho\colon P \to P/G$ is a principal G bundle.*

As a special case of Corollary 16.21 we have:

Corollary 16.22. *Let $M \cong G/H$ be a homogeneous space. Then M can be seen as the base space of a principal H-bundle $H \to G \xrightarrow{\pi} M$.*

LECTURE 17

The Fibre Bundle Construction Theorem

In this lecture we discuss morphisms between G-fibre bundles, give a recipe for constructing such bundles, and show they are determined up to isomorphism by its transition functions. We conclude by showing how a vector bundle canonically determines a principal bundle. Next lecture we will investigate the converse direction: producing vector bundles from principal bundles.

DEFINITIONS 17.1. Let

$$L_1 \to E \xrightarrow{\pi_1} M, \qquad \text{and} \qquad L_2 \to F \xrightarrow{\pi_2} N$$

be two fibre bundles. A **fibre bundle morphism** is a pair (φ, Φ) of smooth maps

$$\varphi \colon M \to N, \qquad \Phi \colon E \to F.$$

such that the following commutes:

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M & \xrightarrow[\varphi]{} & N \end{array}$$

We also say that Φ is a **fibre bundle morphism** along φ. If $\Phi_p \colon E_p \to F_{\varphi(p)}$ is a diffeomorphism for each $p \in M$ then we call Φ a **fibre bundle isomorphism along** φ.

If (φ, Φ) is a fibre bundle morphism then

$$\Phi_p := \Phi|_{E_p} \colon E_p \to F_{\varphi(p)}$$

is a smooth map for each $p \in M$. If $\Phi_p \colon E_p \to F_{\varphi(p)}$ is a diffeomorphism for each $p \in M$ then we call Φ a **fibre bundle isomorphism along** φ. Since $E_p \cong L_1$ and $F_{\varphi(p)} \cong L_2$, we see that a fibre bundle isomorphism along φ can only exist when $L_1 \cong L_2$.

Of particular interest is the case where $M = N$ and $\varphi = \mathrm{id}$.

DEFINITION 17.2. Let $L_1 \to E \xrightarrow{\pi_1} M$ and $L_2 \to F \xrightarrow{\pi_2} M$ be two fibre bundles over the same base space M. A smooth map $\Phi \colon E \to F$ is said to be **fibre bundle homomorphism** if (id, Φ) is a fibre bundle morphism. If in addition Φ is a fibre bundle isomorphism along id, then we call Φ a **fibre bundle isomorphism**, and we say that E and F are **isomorphic fibre bundles**.

So much for general fibre bundles. The notion of morphisms between G-fibre bundles is rather messier to define, and for simplicity we focus only on isomorphisms. Let

$$L \to E \xrightarrow{\pi_1} M, \qquad \text{and} \qquad L \to F \xrightarrow{\pi_2} N$$

be two (G,σ)-fibre bundles. Let $\{(U_a,\varepsilon_a) \mid a\in A\}$ be a (G,σ)-bundle atlas for E and let $\{(V_b,\gamma_b) \mid b\in B\}$ be a (G,σ)-bundle atlas for F. Let $\varphi\colon M\to N$ be a smooth map and let $\Phi\colon E\to F$ be a fibre bundle isomorphism along φ. If $a\in A$ and $b\in B$ are such that $U_a\cap\varphi^{-1}(V_b)\neq\emptyset$, then for $p\in U_a\cap\varphi^{-1}(V_b)$ we can consider the composition

$$L \xrightarrow{\varepsilon_{a|p}^{-1}} E_p \xrightarrow{\Phi_p} F_{\varphi(p)} \xrightarrow{\gamma_{b|\varphi(p)}} L$$

Denote this composition by $f_a^b(p)$. Thus $f_a^b(p)\in\operatorname{Diff}(L)$ and we can regard f_a^b as a map

$$f_a^b\colon U_a\cap\varphi^{-1}(V_b)\to\operatorname{Diff}(L),\qquad p\mapsto f_a^b(p). \tag{17.1}$$

Now recall that the reason we introduced (G,σ)-fibre bundles was to "cut down" the possible values the transition functions from the infinite-dimensional space $\operatorname{Diff}(L)$ to a finite-dimensional subgroup $\{\sigma_g \mid g\in G\}$. It therefore stands to reason that an isomorphism between such bundles along φ should also respect this restriction – in other words, the functions f_a^b from (17.1) should also take values in $\{\sigma_g \mid g\in G\}$.

Here is the formal definition.

Definition 17.3. Let σ be a smooth effective action of a Lie group G on a smooth manifold L. Assume we are given two fibre bundles

$$L\to E\xrightarrow{\pi_1} M,\qquad\text{and}\qquad L\to F\xrightarrow{\pi_2} N$$

together with a (G,σ)-bundle atlas $\{(U_a,\varepsilon_a) \mid a\in A\}$ on E and a (G,σ)-bundle atlas $\{(V_b,\gamma_b) \mid b\in B\}$ on F. Let $\varphi\colon M\to N$ be a smooth map and let $\Phi\colon E\to F$ be a fibre bundle isomorphism along φ. We say that Φ is a (G,σ)**-fibre bundle isomorphism along** φ if for each $a\in A$ and $b\in B$ such that $U_a\cap\varphi^{-1}(V_b)\neq\emptyset$, there exists a smooth map

$$h_a^b\colon U_a\cap\varphi^{-1}(V_b)\to G$$

such that if f_a^b is defined as in (17.1) then

$$f_b^a(p)=\sigma_{h_a^b(p)}.$$

As before, when $M=N$ and $\varphi=\mathrm{id}$ then we call Φ a (G,σ)**-fibre bundle isomorphism**, and we say that E and F are **isomorphic** (G,σ)**-fibre bundles**.

Remark 17.4. As a fun exercise, try and correctly write down the definition of a morphism in the most general setting where one has two fibre bundles with different fibres, different Lie groups, and different actions.

We now give a recipe for constructing fibre bundles starting from the transition functions.

Theorem 17.5 (The Fibre Bundle Construction Theorem). *Let* $\{U_a \mid a\in A\}$ *be an open covering of a manifold* M. *Let* G *be a Lie* *group. Suppose for each $a, b \in A$ such that $U_a \cap U_b \neq \emptyset$, we are given a smooth map $g_{ab} \colon U_a \cap U_b \to G$ such that the following cocycle conditions are satisfied:*

$$\begin{cases} g_{ac}(p) = g_{ab}(p) g_{bc}(p), & \forall\, p \in U_a \cap U_b \cap U_c, \\ g_{aa}(p) = e, & \forall\, p \in U_a, \ \forall\, a \in A. \end{cases} \tag{17.2}$$

Suppose in addition we are given a smooth effective action σ of G on a smooth manifold L. Then there exists a fibre bundle $L \to E \xrightarrow{\pi} M$, which admits a (G, σ)-bundle atlas $\{(U_a, \varepsilon_a) \mid a \in A\}$ such that the transition functions ε_{ab} are given by

$$\varepsilon_{ab}(p) = \sigma_{g_{ab}(p)}.$$

As you might expect from a theorem with such complicated hypotheses (compare the Proposition 1.17), the proof is basically trivial – most of the work is in formulating the hypotheses correctly!

Proof. Let

$$E := \left(\bigsqcup_{a \in A} (U_a \times L) \right) \Big/ \sim,$$

where we identify $(p, u) \in U_a \times L$ with $(q, v) \in U_b \times L$ if and only if $p = q$ and

$$u = \sigma_{g_{ab}(p)}(v).$$

Let

$$\rho \colon \bigsqcup_{a \in A} (U_a \times L) \to E$$

denote the quotient map, and endow E with the quotient topology. Let $\pi \colon E \to M$ denote the unique map so that

$$\begin{array}{ccc} \bigsqcup_{a \in A} (U_a \times L) & \xrightarrow{\mathrm{pr}_1} & \bigsqcup_{a \in A} U_a \\ \Big\downarrow \rho & & \Big\downarrow \text{“ id ”} \\ E & \xrightarrow[\pi]{} & M \end{array}$$

Then $\pi \colon E \to M$ is continuous by definition of the quotient topology. For each $a \in A$, the restriction of ρ to $U_a \times L$ onto its image in E is a homeomorphism. Its inverse is of the form (π, ε_a), where $\varepsilon_a \colon \pi^{-1}(U_a) \to L$. This is our desired bundle atlas on E: we first make E into a smooth manifold using the procedure outlined in the second half of Remark 16.5 – the fact that this gives a well-defined smooth structure follows from (17.2). It is then immediate from the definition that the transition functions of this bundle atlas are given by the maps ε_{ab}. This completes the proof. ∎

Example 17.6. Take $L = \mathbb{R}$, and identify $\mathrm{GL}(\mathbb{R}) = \mathbb{R} \setminus \{0\}$. We take $M = S^1 \subset \mathbb{C}$. Let $U_1 = S^1 \setminus \{i\}$ and $U_2 := S^1 \setminus \{-i\}$. By the Fibre Bundle Construction Theorem 17.5 a smooth map $g_{12} \colon U_1 \cap U_2 \to \mathbb{R} \setminus \{0\}$ determines a vector bundle of rank 1 over M. If we set

$$g_{21}(z) := \begin{cases} 1, & \Re(z) > 0, \\ -1, & \Re(z) < 0, \end{cases}$$

then the vector bundle so obtained is called the **Möbius bundle**. On Problem Sheet G you will show that there are exactly two rank 1 vector bundles over S^1 (up to isomorphism): the trivial bundle $S^1 \times \mathbb{R}$ and the Möbius bundle.

The Möbius bundle is a Möbius band of infinite width.

The next result clarifies the relation between the isomorphism class of a vector or principal bundle and its transition functions.

PROPOSITION 17.7. *Let σ be an effective action of a Lie group G on a smooth manifold L. Assume we are given two fibre bundles*

$$L \to E \xrightarrow{\pi_1} M, \qquad \text{and} \qquad L \to F \xrightarrow{\pi_2} M$$

Let $\{U_a \mid a \in A\}$ be an open cover of M such that both E and F admit (G, σ)-bundle atlases over the U_a. Let

This can always be achieved by taking intersections.

$$g^1_{ab} \colon U_a \cap U_b \to G, \qquad \text{and} \qquad g^2_{ab} \colon U_a \cap U_b \to G$$

denote the transition functions of E and F with respect to these bundle atlases. Then E and F are isomorphic as (G, σ)-fibre bundles if and only if there exists a family $f_a \colon U_a \to G$ of smooth functions such that

$$f_a(p) \circ g^1_{ab}(p) = g^2_{ab}(p) \circ f_b(p), \qquad \forall\, p \in U_a \cap U_b, \ \forall\, a, b \in A.$$

Similarly to the Fibre Bundle Construction Theorem, the most difficult part of Proposition 17.7 is formulating the correct hypotheses. The proof is left to you on Problem Sheet G.

COROLLARY 17.8. *The (G, σ)-fibre bundle constructed in the Fibre Bundle Construction Theorem 17.5 is unique up to (G, σ)-fibre bundle isomorphism.*

We now specialise the preceding definitions to vector and principal bundles. The definitions are not quite special cases of what we have already done (since in Definition 17.3 we only looked at (G, σ)-fibre bundle isomorphisms along maps).

DEFINITION 17.9. Let (φ, Φ) be a fibre bundle morphism between two vector bundles $\pi_1 \colon E \to M$ and $\pi_2 \colon F \to N$. We say that (φ, Φ) is a **vector bundle morphism** if $\Phi_p \colon E_p \to F_{\varphi(p)}$ is a *linear* map for each $p \in M$. When φ is fixed, we also say that Φ is a **vector bundle morphism along** φ if (φ, Φ) is a vector bundle morphism. If Φ_p maps each fibre E_p isomorphically onto $F_{\varphi(p)}$ then Φ is called a **vector bundle isomorphism along** φ.

EXAMPLE 17.10. Let $\varphi \colon M \to N$ be a smooth map. Then $D\varphi \colon TM \to TN$ is a vector bundle morphism along φ.

DEFINITION 17.11. Let $\pi_1 \colon E \to M$ and $\pi_2 \colon F \to M$ be two vector bundles over the same base space M. A smooth map $\Phi \colon E \to F$ is said to be **vector bundle homomorphism** if (id, Φ) is a vector bundle morphism. If Φ is in addition a diffeomorphism we say that Φ is a **vector bundle isomorphism**, and that E and F are **isomorphic vector bundles**.

Remark 17.12. A vector bundle homomorphism is a vector bundle isomorphism if and only if it is a diffeomorphism. This is not true for vector bundle morphisms along a map. For instance, if M is a manifold and $p \in M$ then (thinking of $\{p\}$ as a zero-dimensional manifold) we have a smooth map $\iota_p \colon \{p\} \hookrightarrow M$ given by inclusion. If E is any vector bundle over M then the inclusion map $E_p \hookrightarrow E$ is a vector bundle isomorphism along ι_p, but of course it is not a diffeomorphism.

The definition of morphisms for principal bundles is analogous.

Definition 17.13. Let (φ, Φ) be a fibre bundle morphism between two principal G-bundles $\pi_1 \colon P \to M$ and $\pi_2 \colon Q \to N$. Let τ_1 and τ_2 denote the associated right actions on P and Q respectively. We say that (φ, Φ) is a **principal bundle morphism** if Φ is (τ_1, τ_2)-equivariant. When φ is fixed, we also say that Φ is a **principal bundle morphism along** φ if (φ, Φ) is a principal bundle morphism. If Φ is in addition a diffeomorphism then we say Φ is a **principal bundle isomorphism along** φ.

Whilst the definition of principal bundle morphisms looks superficially similar to that of vector bundle morphisms, there is already a major difference between vector and principal bundles.

Lemma 17.14. *Let $\pi_1 \colon P \to M$ and $\pi_2 \colon Q \to N$ be two G-principal bundles. Suppose $\Phi \colon P \to Q$ is a principal bundle morphism along a diffeomorphism $\varphi \colon M \to N$. Then Φ is automatically a diffeomorphism, and hence a principal bundle isomorphism along φ.*

As Remark 17.12 shows, Lemma 17.14 is *not* true for vector bundle morphisms! The proof of Lemma 17.14 is on Problem Sheet G.

Definition 17.15. Let $\pi_1 \colon P \to M$ and $\pi_2 \colon Q \to M$ be two principal G-bundles over the same base space M. A diffeomorphism $\Phi \colon P \to Q$ is said to be **principal bundle isomorphism** if (id, Φ) is a principal bundle morphism. If such a Φ exists we say that P and Q are **isomorphic principal G-bundles**.

Note there is no point mimicking Definition 17.11 by first defining a “principal bundle homomorphism”, and then declaring that a principal bundle isomorphism is a principal bundle homomorphism which is in addition a diffeomorphism. Indeed, any principal bundle homomorphism is automatically an isomorphism by Lemma 17.14, since the identity is a diffeomorphism.

Remark 17.16. If $\pi \colon P \to M$ is a principal bundle and $\Phi \colon P \to P$ is a principal bundle isomorphism from P to itself then we call Φ a **gauge transformation**. The name comes from physics. We will come back to the study of gauge transformations extensively next semester.

Having defined morphisms, we can now define vector and principal subbundles.

Definition 17.17. Let $\pi_1 \colon E \to M$ and $\pi_2 \colon F \to M$ are two vector bundles over the same manifold such that $E \subset F$ is an embedded submanifold. We say that E is a **vector subbundle** of F if the inclusion $E \hookrightarrow F$ is a vector bundle homomorphism.

EXAMPLE 17.18. If Δ is a distribution on M then one can think of Δ as a vector subbundle of TM.

The notion of principal subbundles is slightly subtler, because we also want to allow for a Lie subgroup. Suppose $\psi\colon G \to H$ is a Lie group homomorphism and τ is a smooth action of H on a space M. Define a new action τ^ψ of G on M by

Part (i) of Examples 12.13 is a special case of this.

$$\tau^\psi_g := \tau_{\psi(g)}, \qquad g \in G.$$

We now extend the notion of a principal bundle morphism for different Lie groups.

DEFINITION 17.19. Suppose G and H are two Lie groups. Let $\pi_1\colon P \to M$ be a principal G-bundle and let $\pi_2\colon Q \to N$ be a principal H-bundle. Suppose $\psi\colon G \to H$ is a Lie group homomorphism. A **principal bundle morphism from P to Q with respect to ψ** consists of a pair of smooth maps $\varphi\colon M \to N$ and $\Phi\colon P \to Q$ such that $\pi_2 \circ \Phi = \varphi \circ \pi_1$ and such that Φ is (τ_1, τ_2^ψ)-equivariant. If $M = N$ and $\varphi = \mathrm{id}$ then we call Φ a **principal bundle homomorphism with respect to ψ**.

Warning: In this case the analogue of Lemma 17.14 is *not* true, and so Φ does not need to be a principal bundle isomorphism with respect to ψ.

REMARK 17.20. Definition 17.19 is a special case of the more general setup you were invited to guess in Remark 17.4.

Here is the definition of a principal subbundle .

Principal subbundles will play no role in Differential Geometry I (apart from in Problem G.10). However they will be very important in Differential Geometry II.

DEFINITION 17.21. Let G be a Lie group and suppose $H \subset G$ is a Lie subgroup. Suppose $\pi_1\colon P \to M$ is a principal H-bundle and $\pi_2\colon Q \to M$ is a principal G-bundle such that $P \subset Q$ is a weakly embedded submanifold. We say that P is a **principal H-subbundle** of Q if the inclusion $P \hookrightarrow Q$ is a principal bundle homomorphism with respect to the inclusion $H \hookrightarrow G$.

The Fibre Bundle Construction Theorem 17.5 allows us to produce a principal G-bundle from any G-fibre bundle. Indeed, suppose $L \to E \xrightarrow{\pi} M$ is a (G, σ)-fibre bundle. Let $\{(U_a, \varepsilon_a)\}$ be a (G, σ)-bundle atlas. This means we can write

$$\varepsilon_{ab}(p) = \sigma_{g_{ab}(p)}$$

for functions $g_{ab}\colon U_a \cap U_b \to G$. The functions $\{g_{ab}\}$ satisfy the cocycle condition, and hence by the Fibre Bundle Construction Theorem 17.5, there exists a principal G-bundle P whose transition functions are given by left translation by the g_{ab}. This principal bundle is unique up to principal bundle isomorphism by Corollary 17.8, and we give it a special name:

DEFINITION 17.22. We call P the **induced principal bundle of E**.

This is a slight abuse of terminology, as P is only unique up to isomorphism.

We conclude this lecture by giving an explicit construction of P in the case where E is a vector bundle.

Definition 17.23. Let $V \to E \xrightarrow{\pi} M$ be a vector bundle. Fix $p \in M$, and let $\mathrm{Fr}(E_p)$ denote the set of isomorphisms $\ell\colon V \to E_p$. Since any two isomorphisms $\ell_1, \ell_2\colon V \to E_p$ differ by element of $\mathrm{GL}(V)$, i.e. there exists $A \in \mathrm{GL}(V)$ such that $\ell_2 = \ell_1 \circ A$. In fact, if we fix our favourite isomorphism ℓ then the map $\mathrm{GL}(V) \to \mathrm{Fr}(E_p)$ given by $A \mapsto \ell \circ A$ is a bijection.

One can equivalently regard $\mathrm{Fr}(E_p)$ as the set of bases of the vector space E_p, since for any $\ell \in \mathrm{Fr}(E_p)$ the vectors (ℓe_i) form a basis of E_p, where e_i are the standard basis vectors in V, and conversely given a basis (v_i) there is a uniquely determined linear isomorphism $\ell\colon V \to E_p$ such that $\ell e_i = v_i$ for each i.

Definition 17.24. We now form the total space

$$\mathrm{Fr}(E) := \bigsqcup_{p \in M} \mathrm{Fr}(E_p),$$

and let $\Pi\colon \mathrm{Fr}(E) \to M$ denote the map that sends $\mathrm{Fr}(E_p)$ to p. We call $\mathrm{Fr}(E)$ the **frame bundle** of E.

Proposition 17.25. *Let $V \to E \xrightarrow{\pi} M$ be a vector bundle. Then $\Pi\colon \mathrm{Fr}(E) \to M$ is a principal $\mathrm{GL}(V)$-bundle over M. Moreover $\mathrm{Fr}(E)$ is the induced principal bundle of E.*

Proof. Let $\{(U_a, \varepsilon_a) \mid a \in A\}$ denote a vector bundle atlas. For each $p \in U_a$, $\varepsilon_{a|p}^{-1}\colon V \to E_p$ is a linear isomorphism, and thus $\varepsilon_{a|p}^{-1} \in \mathrm{Fr}(E_p)$. Define a map

$$\gamma_a\colon \Pi^{-1}(U_a) \to \mathrm{GL}(V),$$

by declaring that

$$\gamma_a(\varepsilon_{a|p}^{-1} \circ A) = A.$$

We will show that $\{(U_a, \gamma_a) \mid a \in A\}$ is a $\mathrm{GL}(V)$-principal bundle atlas on $\mathrm{Fr}(E)$. As in the proof of the Fibre Bundle Construction Theorem 17.5, we will simultaneously show that $\mathrm{Fr}(E)$ is a smooth manifold and a principal $\mathrm{GL}(V)$-bundle by computing the transition functions γ_{ab}. Fix $p \in U_a \cap U_b$. Then by definition

$$\begin{aligned}\gamma_{ab}(p)(A) &= \gamma_{a|p} \circ \gamma_{b|p}^{-1}(A) \\ &= \gamma_{a|p}\left(\varepsilon_{b|p}^{-1} \circ A\right) \\ &= \varepsilon_{ab}(p) \circ A.\end{aligned}$$

Thus the transition functions of $\mathrm{Fr}(E)$ are just left composition by the transition functions of E. That is,

$$\gamma_{ab}(p) = l_{\varepsilon_{ab}(p)},$$

where l is left translation in the Lie group $\mathrm{GL}(V)$. This means that the transition functions of E play the role of the functions g_{ab} in (16.3). Thus $\mathrm{Fr}(E)$ is indeed a principal $\mathrm{GL}(V)$ bundle, and by definition $\mathrm{Fr}(E)$ is the induced principal bundle of E. ∎

We have shown how to produce a principal bundle from a vector bundle. Next lecture we will show how to produce (many) vector bundles from a principal bundle.

LECTURE 18

Associated Bundles

We begin this lecture by explaining how to build fibre bundles from principal bundles.

DEFINITION 18.1. Let $\pi \colon P \to M$ be a principal G-bundle, and let σ be a smooth effective left action of G on another smooth manifold L. Define an equivalence relation $\sim$ on $P \times L$ by setting:

$$(\tau_g(u), q) \sim (u, \sigma_g(q)), \qquad u \in P,\ g \in G,\ q \in L, \tag{18.1}$$

Define $P \times_G L$ to be the quotient space $(P \times L)/\sim$. Writing $[u, q]$ for the equivalence class of (u, q), we define a map

$$\pi_L \colon P \times_G L \to M, \qquad [u, q] \mapsto \pi(u).$$

We call $P \times_G L$ an **associated bundle** of P.

REMARK 18.2. The notation $\pi_L \colon P \times_G L \to M$ is somewhat ambiguous, since we really should specify the action we are using. When confusion is possible, we will occasionally write $\pi_{L,\sigma} \colon P \times_{G,\sigma} L \to M$ or $\pi_\sigma \colon P \times_\sigma L \to M$ instead. Moreover the assumption that σ is effective is not used anywhere in the proof – it is simply there so as to fit in with the framework of Definition 16.8. As Remark 16.9 shows, restricting to effective actions does not actually involve any loss of generality, and thus assuming it here is harmless.

THEOREM 18.3 (The Associated Bundle Theorem). *Let $\pi \colon P \to M$ be a principal G-bundle, and let σ be a smooth effective action of G on L.*

(i) *The associated bundle $\pi_L \colon P \times_G L \to M$ is a (G, σ)-fibre bundle with fibre L, and moreover P is the induced principal bundle of $P \times_G L$.*

(ii) *The quotient map $\wp \colon P \times L \to P \times_G L$ given by $\wp(u, q) := [u, q]$ is another principal G-bundle, and the first projection $\mathrm{pr}_1 \colon P \times L \to P$ is a principal bundle morphism along π_L:*

$$\begin{array}{ccc} P \times L & \xrightarrow{\ \mathrm{pr}_1\ } & P \\ {\scriptstyle \wp}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ P \times_G L & \xrightarrow[\ \pi_L\ ]{} & M \end{array}$$

(iii) *For each $u \in P$, the map $\psi_u \colon L \to P \times_G L$ given by $q \mapsto [u, q]$ is a diffeomorphism from L to $\pi_L^{-1}(\pi(u))$.*

Proof. We will prove the result in four steps.

1. In this step we define a tentative bundle chart for $P \times_G L$ and prove that the associated local trivialisation is bijective. Suppose

$\varepsilon\colon \pi^{-1}(U) \to G$ is a principal bundle chart over an open set $U \subset M$. We define a map $\gamma\colon \pi_L^{-1}(U) \to L$ by

$$\gamma[u,q] := \sigma_{\varepsilon(u)}(q).$$

This is well defined because ε is (τ, r)-equivariant: if $[u,q] = [u_1, q_1]$ then there exists $g \in G$ such that $\tau_g(u_1) = u$ and $\sigma_g(q) = q_1$. Then

$$\begin{aligned}\gamma[u,q] &= \sigma_{\varepsilon(u)}(q)\\ &= \sigma_{\varepsilon(\tau_g(u_1))}(q)\\ &= \sigma_{\varepsilon(u_1)g}(q)\\ &= \sigma_{\varepsilon(u_1)} \circ \sigma_g(q)\\ &= \sigma_{\varepsilon(u_1)}(q_1)\\ &= \gamma[u_1, q_1].\end{aligned}$$

We claim that $\hat{\gamma} := (\pi_L, \gamma)\colon \pi_L^{-1}(U) \to U \times L$ is bijective. To see this, for each $p \in U$ let $u_p \in P_p$ denote the unique element such that $\varepsilon(u_p) = e$ (this is well defined as ε_p is a diffeomorphism). Now define $\varphi\colon U \times L \to \pi_L^{-1}(U)$ by $\varphi(p,q) := [u_p, q]$. We claim that φ is an inverse to $\hat{\gamma}$. Indeed,

$$\begin{aligned}\hat{\gamma} \circ \varphi(p,q) &= (\pi_L, \gamma)[u_p, q]\\ &= \big(p, \sigma_{\varepsilon(u_p)}(q)\big)\\ &= (p, \sigma_e(q))\\ &= (p,q).\end{aligned}$$

Going the other way round, if $p \in U$ and $u \in P_p$ then by equivariance

$$\begin{aligned}\hat{\varepsilon}\big(\tau_{\varepsilon(u)}(u_p)\big) &= \big(p, r_{\varepsilon(u)}\varepsilon(u_p)\big)\\ &= (p, \varepsilon(u)e)\\ &= (p, \varepsilon(u))\\ &= \hat{\varepsilon}(u)\end{aligned}$$

and thus as $\hat{\varepsilon}$ is a diffeomorphism we must have

$$\tau_{\varepsilon(u)}(u_p) = u. \tag{18.2}$$

We therefore have for $u \in P_p$ that

$$\begin{aligned}\varphi \circ \hat{\gamma}[u,q] &= \varphi(p, \sigma_{\varepsilon(u)}(q))\\ &= \big[u_p, \sigma_{\varepsilon(u)}(q)\big]\\ &= \big[\tau_{\varepsilon(u)}(u_p), q\big]\\ &= [u,q],\end{aligned}$$

where the last two equalities used (18.1) and (18.2) respectively. Thus $\hat{\gamma}$ is bijective.

2. In this step we prove (i). Let $\{(U_a, \varepsilon_a) \mid a \in A\}$ be a principal bundle atlas for P, and for each $a \in A$, let $\gamma_a\colon \pi_L^{-1}(U_a) \to L$ be defined as in Step 1. We claim that $\{(U_a, \gamma_a)\}$ can serve as a bundle atlas on $P \times_G L$. For this we must investigate the transition functions γ_{ab}. We want to show that

$$\gamma_{ab}(p)(q) = \sigma_{g_{ab}(p)}(q)$$

for some smooth functions $g_{ab} \colon U_a \cap U_b \to G$. This however is immediate from the previous step:

$$\gamma_{ab}(p)(q) = \sigma_{\varepsilon_{ab}(p)}(q), \tag{18.3}$$

that is,

$$g_{ab}(p) = \varepsilon_{ab}(p)$$

This is smooth, and hence as in Remark 16.5, we can endow $P \times_G L$ with a smooth structure by declaring all the maps $\hat{\gamma}_a$ to be diffeomorphisms. Then the collection $\{(U_a, \gamma_a)\}$ form a (G, σ)-bundle atlas, and $P \times_G L$ is a (G, σ)-fibre bundle. Moreover by definition P is the principal bundle induced by $P \times_G L$.

3. We now prove (ii). On the open set $\pi^{-1}(U_a) \times L$ of $P \times L$, the map $\wp$ is given by

$$\wp(u, q) = \hat{\gamma}_a^{-1}\big(\pi(u), \sigma_{\varepsilon_a(u)}(q)\big),$$

This shows that $\wp$ is smooth. If we differentiate this equation we obtain

$$D\wp(u, q) = D\hat{\gamma}_a^{-1}\big(\pi(u), \sigma_{\varepsilon_a(u)}(q)\big) \circ \Big(D\pi(u), D\sigma_{\varepsilon_a(u)}(q) + D\sigma^q(\varepsilon_a(u)) \circ D\varepsilon_a(u)\Big).$$

Since $\hat{\gamma}_a$ and $\sigma_{\varepsilon_a(u)}$ are diffeomorphisms, it follows from Proposition 16.4 that $\wp$ is a submersion. We now define a right action $\tilde{\tau}$ of G on $P \times L$ by

$$\tilde{\tau}_g(u, q) = \big(\tau_g(u), \sigma_{g^{-1}}(q)\big).$$

This action is free since τ is. Moreover $\tilde{\tau}$ preserves the fibres of $\wp$:

$$\begin{aligned}
\wp\big(\tilde{\tau}_g(u, q)\big) &= \wp\big(\tau_g(u), \sigma_{g^{-1}}(q)\big) \\
&= \big[\tau_g(u), \sigma_{g^{-1}}(q)\big] \\
&= [u, q] \\
&= \wp(u, q)
\end{aligned}$$

by the defining relationship (18.1). Finally $\tilde{\tau}$ is transitive on the fibres, since if

$$\wp(u_1, q_1) = \wp(u_2, q_2)$$

then by (18.1) again there exists $g \in G$ such that

$$\tau_g(u_1) = u_2, \qquad \sigma_g(q_2) = q_1,$$

and hence

$$\tilde{\tau}_g(u_1, q_1) = (u_2, q_2).$$

Since $P \times_G L$ is a manifold, Problem G.2 implies that $\tilde{\tau}$ is proper. It now follows from Proposition 16.18 that $\wp \colon P \times L \to P \times_G L$ is another principal G bundle. This proves (ii). The identity

$$\mathrm{pr}_1\big(\tau_g(u), \sigma_{g^{-1}}(q)\big) = \tau_g(u) = \tau_g(\mathrm{pr}_1(u, q))$$

shows that pr_1 is a principal G-bundle morphism along π_L.

4. It remains to prove (iii). Fix $p \in M$ and $u \in P_p$. The map $\psi_u\colon L \to \pi_L^{-1}(p)$ given by $q \mapsto [u, q]$ is smooth because $\wp$ is. Moreover if $p \in U_a$ then near $[u, q]$ a the map

$$\sigma_{\varepsilon_a(u)^{-1}} \circ \gamma_a\colon \pi_L^{-1}(p) \to L$$

is a smooth inverse to ψ_u. Thus ψ_u is a diffeomorphism. This completes the proof of part (iii), and hence also the theorem. ∎

Corollary 18.4. *Let $L \to E \xrightarrow{\pi} M$ be a (G, σ)-fibre bundle. Then there exists a principal bundle P such that E is isomorphic as a (G, σ)-fibre bundle to the associated bundle $P \times_G L$. Moreover P is unique up to principal bundle isomorphism.*

Proof. Let P denote the principal G-bundle induced by E. Thus the transition functions of E can be identified with the transition functions of P. Now consider the new fibre bundle $P \times_G L$. As the proof of Theorem 18.3 shows, the transition functions of $P \times_G L$ can also be identified with the transition functions of P. Thus E and $P \times_G L$ have the same transition functions, and by Proposition 17.7 it follows that E and $P \times_G L$ are isomorphic as (G, σ)-fibre bundles. ∎

Suppose $L = V$ is a vector space and σ is a linear action. Then (18.3) shows that the transition functions of $P \times_G V$ are linear, and hence $P \times_G V$ is a vector bundle over M. It will be useful later to have an explicit description of the vector space structure on the fibres. Fix $p \in M$, $v, w \in V$ and $c \in \mathbb{R}$. We define addition and scalar multiplication on $\pi_V^{-1}(p)$ as

$$[u, v] + c[u, w] := [u, v + cw],$$

where $u \in P_p$ is any element in the fibre over p and $v + cw$ is addition and scalar multiplication in the vector space V. This is well defined, i.e. independent of the choice of u, since if $g \in G$ then $[u, v] = [\tau_g(u), \sigma_{g^{-1}}(v)]$ and $[u, w] = [\tau_g(u), \sigma_{g^{-1}}(w)]$ and then since $\sigma_{g^{-1}}$ is linear

$$\begin{aligned}\big[\tau_g(u), \sigma_{g^{-1}}(v) + c\sigma_{g^{-1}}(w)\big] &= \big[\tau_g(u), \sigma_{g^{-1}}(c + vw)\big] \\ &= [u, v + cw].\end{aligned}$$

This also shows that the bundle charts γ constructed in the proof of Theorem 18.3 are vector bundle charts in the sense of Definition 16.16. Note that the map ψ_u from part (iii) of Theorem 18.3 satisfies

$$\begin{aligned}\psi_u(v + cw) &= [u, v + cw] \\ &= [u, v] + c[u, w] \\ &= \psi_u(v) + c\psi_u(w),\end{aligned}$$

and hence ψ_u is linear. We have thus proved:

Corollary 18.5. *Let $\pi\colon P \to M$ be a principal G-bundle. Suppose σ is a faithful representation of G on a vector space V. Then the associated bundle $P \times_G V$ is a vector bundle over M and the map ψ_u from part (iii) of Theorem 18.3 is a linear isomorphism.*

A faithful representation is simply another word for an effective linear action, cf. part (iii) of Examples 12.13.

Example 18.6. Let $V \to E \xrightarrow{\pi} M$ be a vector bundle, and let $\Pi\colon \mathrm{Fr}(E) \to M$ denote the frame bundle. The canonical representation of $\mathrm{GL}(V)$ on V produces a new vector bundle

$$\Pi_V : \mathrm{Fr}(E) \times_{\mathrm{GL}(V)} V \to M$$

It follows from Corollary 18.4 that this vector bundle is isomorphic to E. If $p \in M$ then a bundle chart (U, ε) about p provides an explicit isomorphism $E_p \cong \Pi_V^{-1}(p)$ via:

$$E_p \xrightarrow{\varepsilon_p} V \xrightarrow{\psi_{\varepsilon_p^{-1}}} \Pi_V^{-1}(p)$$

This may seem a bit silly: starting from a vector bundle E, we constructed its frame bundle, and then used the Associated Bundle Theorem to produce... E again. But this is missing the point: the real power of Corollary 18.5 is that we are free to choose *any* representation of $\mathrm{GL}(V)$.

Strictly speaking, all we have done is produced a new bundle which is isomorphic to E as a vector bundle.

Definition 18.7. Let $V \to E \xrightarrow{\pi} M$ be a vector bundle. Let $\Pi\colon \mathrm{Fr}(E) \to M$ denote the frame bundle of E. The group $\mathrm{GL}(V)$ acts on the dual space $V^* = \mathrm{Hom}(V, \mathbb{R})$ by

$$\lambda \mapsto \lambda \circ A^{-1} \tag{18.4}$$

for $A \in \mathrm{GL}(V)$ and $\lambda \in V^*$. This gives an associated vector bundle $\Pi_{V^*} : \mathrm{Fr}(E) \times_{\mathrm{GL}(V)} V^* \to M$. We call this vector bundle the **dual bundle** of E, and write it as E^*.

One can also give a very explicit construction of E^*. Namely, the total space E^* is given as the union

$$E^* := \bigsqcup_{p \in M} E_p^* \tag{18.5}$$

where $E_p^* = \mathrm{Hom}(E_p, \mathbb{R})$ is the dual vector space to E_p. A bundle chart (U, ε) on E gives rise to a bundle chart (U, ε^*) on E^* via

$$\varepsilon_p^*(\lambda)(v) := \lambda(\varepsilon_p^{-1}(v)), \qquad \lambda \in E_p^*,\ v \in V$$

together with an explicit isomorphism

$$E_p^* \xrightarrow{\varepsilon_p^*} V^* \xrightarrow{\psi_{(\varepsilon_p^*)^{-1}}} \Pi_{V^*}^{-1}(p)$$

Example 18.8. The cotangent bundle is the dual bundle to the tangent bundle.

The explicit construction of the dual bundle E^* in (18.5) is only isomorphic to the dual bundle from Definition 18.7. Nevertheless, we will suppress this in the discussion that follows, and regard them as being "the same". It is important to understand why this is harmless.

In general it is **not** alright to simply work with vector bundles up to isomorphism – at least, if one did then the whole dual bundle construction would be pointless, since any vector bundle is isomorphic to its dual bundle.

See Problem Sheet I.

However there is a stronger notion than isomorphism: **canonical isomorphism**. Roughly speaking, to say two mathematical objects are *canonically* isomorphic is to say that the isomorphism does not involve making any choices. This really is a stronger property. Indeed, for any finite-dimensional vector space V, you hopefully remember from linear algebra that:

Also called **natural isomorphism**.

$$V \cong V^* \qquad \text{(non-canonical isomorphism)}$$
$$V \cong V^{**} \qquad \text{(canonical isomorphism)} \tag{18.6}$$

The precise mathematical definition of canonical isomorphism will appear in the bonus section of the next lecture. For now it is only important for you to understand that:

> If two mathematical objects are **canonically** isomorphic, it is **harmless** to regard them as actually being the same object; whereas when the isomorphism is non-canonical then regarding them as the same object "loses" information.

Returning to the situation at hand: the construction of the dual bundle in (18.5) is canonically isomorphic to Definition 18.7. This is not too hard to prove directly (namely, one just checks that the obvious isomorphism between the two doesn't depend on choices). The general statement (which covers all possible cases of interest) is given in Theorem 19.64 next lecture. For all subsequent vector bundle constructions, we will suppress canonical isomorphisms from our discussion without further comment.

Definition 18.9. Let $G \to P \xrightarrow{\pi_1} M$ and $H \to Q \xrightarrow{\pi_2} M$ be principal bundles over the same base manifold M with corresponding right actions τ_1 and τ_2. Let

Note that the total space of $P \star Q$ is *not* the product space $P \times Q$.

$$P \star Q := \bigsqcup_{p \in M} P_p \times Q_p.$$

Define a right action τ of $G \times H$ on $P \star Q$ by

$$\tau_{(g,h)}(u_1, u_2) := \big(\tau_{1|g}(u_1), \tau_{2|h}(u_2)\big).$$

This is a free proper right action of $G \times H$ on $P \star Q$ whose orbits are exactly the fibres. Thus Remark 16.5 and Proposition 16.18 tells us that $P \star Q$ is a principal $G \times H$ bundle over M whose fibre over $p \in M$ is $P_p \times Q_p$. We call it the **product principal bundle**.

Let us apply this to vector bundles: if $V \to E \xrightarrow{\pi_1} M$ and $W \to F \xrightarrow{\pi_2} M$ are two vector bundles over the same manifold M, then $\mathrm{Fr}(E)$ is a principal $\mathrm{GL}(V)$-bundle over M and $\mathrm{Fr}(F)$ is a principal $\mathrm{GL}(W)$-bundle over M. Thus $\mathrm{Fr}(E) \star \mathrm{Fr}(F)$ is a principal $\mathrm{GL}(V) \times \mathrm{GL}(W)$ bundle over M.

Definition 18.10. The group $\mathrm{GL}(V) \times \mathrm{GL}(W)$ acts on $V \times W$ by the canonical representation, and thus we can form the vector bundle

$$\big(\mathrm{Fr}(E) \star \mathrm{Fr}(F)\big) \times_{\mathrm{GL}(V)\times\mathrm{GL}(W)} (V \oplus W)$$

Warning: As vector spaces, the direct sum $V \oplus W$ is the same thing as the product $V \times W$, and we often use the notation interchangeably. For vector bundles, we always use the notation $\oplus$ notation. This is because $E \times F$ is used to denote a different bundle; see Problem H.5.

over M. This is a vector bundle with fibre $V \oplus W$ and we denote it by $E \oplus F$ and call it the **direct sum** of E and F.

The direct sum bundle can also be explicitly constructed as

$$E \oplus F := \bigsqcup_{p \in M} E_p \oplus F_p.$$

This discussion can be summed up by the following:

METATHEOREM. *Anything you can do with vector spaces, you can also do with vector bundles.*

The "proof" of the Metatheorem is Corollary 18.5. Or more accurately: the statement of Corollary 18.5 is one way of turning the Metatheorem into an precise mathematical statement.

In the bonus section of the next lecture, we will present an entirely different way of formulating and proving the Metatheorem, using category-theoretic tools instead of principal bundles.

In the next lecture, we will see three further applications of the Metatheorem:

(i) If V and W are vector spaces, then the set $\mathrm{Hom}(V, W)$ of linear maps from V to W is a vector space; thus if E and F are vector bundles then there is a well-defined **homomorphism bundle** $\mathrm{Hom}(E, F)$.

(ii) If V and W are vector spaces, their tensor product $V \otimes W$ is another vector space; thus if E and F are vector bundles then there is a well-defined **tensor bundle** $E \otimes F$.

(iii) If V is a vector space, its exterior algebra $\bigwedge V$ is another vector space; thus if E is a vector vector there is a well-defined **exterior algebra bundle** $\bigwedge E$.

We conclude this lecture by returning to the difference between principal bundles and vector bundles. Corollaries 18.4 and 18.5 tell us that studying vector bundles over a given manifold M is essentially the same thing as studying principal G-bundles over M for G a *matrix* Lie group. However not all Lie groups are matrix Lie groups, and thus principal bundles are more general than vector bundles.

This can be formulated in a precise categorical way.

LECTURE 19

Tensor and Exterior Algebras

In this lecture we continue our theme of constructing new vector bundles from old, but this time we focus on two constructions you may be less familiar with on the linear algebra level.

DEFINITION 19.1. Let V and W be two vector spaces. Their **tensor product** is the vector space $V \otimes W$ which is defined as follows. First, let $F(V \times W)$ denote (infinite-dimensional) vector space which has as basis all pairs (v, w) where $v \in V$ and $w \in W$. Thus an element of $F(V \times W)$ consists of a finite linear combination of pairs (v, w) with $v \in V$ and $w \in W$. Now let $R(V, W)$ denote the linear subspace of $F(V \times W)$ generated by the set of all elements of the form

$$\begin{cases} (v_1 + v_2, w) - (v_1, w) - (v_2, w), & v_1, v_2 \in V,\ w \in W, \\ (v, w_1 + w_2) - (v, w_1) - (v, w_2), & v \in V,\ w_1, w_2 \in W, \\ c(v, w) - (cv, w), & v \in V,\ w \in W,\ c \in \mathbb{R}, \\ c(v, w) - (v, cw), & v \in V,\ w \in W,\ c \in \mathbb{R}. \end{cases}$$

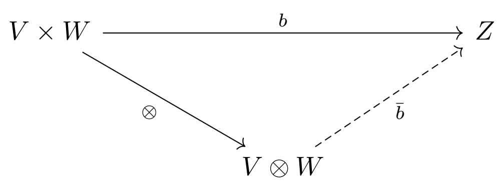
Let $V \otimes W$ denote the quotient vector space $F(V \times W)\big/R(V, W)$. The coset in $V \otimes W$ containing (v, w) is denoted by $v \otimes w$. By construction one has

$$\begin{cases} (v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, & v_1, v_2 \in V,\ w \in W, \\ v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2, & v \in V,\ w_1, w_2 \in W, \\ c(v \otimes w) = (cv) \otimes w, & v \in V,\ w \in W,\ c \in \mathbb{R}, \\ c(v \otimes w) = v \otimes (cw), & v \in V,\ w \in W,\ c \in \mathbb{R}. \end{cases}$$

A typical element in $V \otimes W$ is a finite sum $\sum_i c_i\, v_i \otimes w_i$ where the c_i are real numbers. An element of the form $v \otimes w$ is called **decomposable**.

There is a natural bilinear map $\otimes \colon V \times W \to V \otimes W$ that sends $(v, w) \mapsto v \otimes w$. Here is a useful property of the tensor product.

LEMMA 19.2. *Let V, W and Z be vector spaces and suppose $b \colon V \times W \to Z$ is a bilinear map. Then there exists a unique linear map $\bar{b} \colon V \otimes W \to Z$ such that the following diagram commutes:*



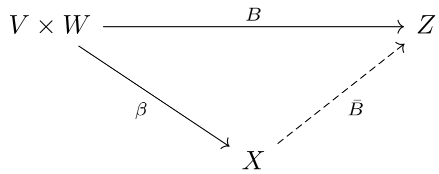
Moreover this property uniquely characterises $V \otimes W$.

Proof. Let $b \colon V \times W \to Z$ be a bilinear function. We extend b by linearity to a map $F(V \times W) \to Z$. Bilinearity then tells us that

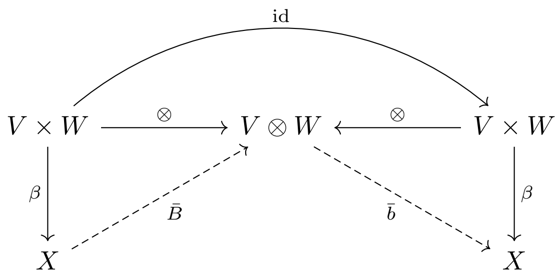
$R(V,W) \subset \ker b$, and hence b factors to define a homomorphism $\bar{b}\colon V \otimes W \to Z$ such that $\bar{b}(v \otimes w) = b(v,w)$ for all $(v,w) \in V \times W$. Moreover the map $\bar{b}$ is unique, since the decomposable elements generate $V \otimes W$.

Finally, to see why this property uniquely determines $\otimes$, suppose that X is another vector space equipped with a bilinear map $\beta\colon V \times W \to X$ with the property that if $B\colon V \times W \to Z$ is bilinear then there exists a *unique* linear map $\bar{B}\colon X \to Z$ such that the diagram commutes:

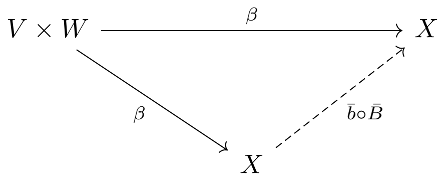
The philosophy behind this style of proof is explained in the bonus section below.



We apply this with $Z = V \otimes W$ and $B := \otimes$. This gives us a unique linear map $\bar{B}\colon X \to V \otimes W$ such that $\bar{B} \circ \beta = \otimes$. Now we go back to our original diagram and chose $Z = X$ and $b := \beta$. Thus we get a unique linear map $\bar{b}\colon V \otimes W \to X$ such that the following spliced diagram commutes.



Thus the composition $\bar{b} \circ \bar{B}$ makes this diagram commute:



But there is meant to only be one map that makes this diagram commute, and another choice is the identity map $X \to X$. Thus $\bar{b} \circ \bar{B} = \mathrm{id}_X$. Similarly $\bar{B} \circ \bar{b} = \mathrm{id}_{V \otimes W}$, and we conclude that X and $V \otimes W$ are isomorphic. This completes the proof. ∎

Given two vector spaces V and W, let $\mathrm{Hom}(V,W)$ denote the set of linear maps from V to W. Thus $\mathrm{Mat}(m) = \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^m)$ and $V^* = \mathrm{Hom}(V, \mathbb{R})$.

Corollary 19.3. *Let V and W denote vector spaces. There is a natural isomorphism* $\mathrm{Hom}(V,W) \cong V^* \otimes W$.

Proof. Define $b\colon V^* \times W \to \mathrm{Hom}(V,W)$ by $b(\lambda, w)(v) := \lambda(v)w$. This gives us a linear map $\bar{b}\colon V^* \otimes W \to \mathrm{Hom}(V,W)$ by Lemma 19.2. This map is an isomorphism, as an inverse $\mathrm{Hom}(V,W) \to V^* \otimes W$ is given by $\ell \mapsto e^i \otimes \ell e_i$, where (e_i) is any basis of V and (e^i) is the dual basis of V^*. ∎

Corollary 19.4. *If (e_i) is a basis for V and (e'_j) is a basis for W then $(e_i \otimes e'_j)$ is a basis for $V \otimes W$. Thus $\dim(V \otimes W) = \dim V \cdot \dim W$.*

The proof of the next result in on Problem Sheet H.

Lemma 19.5. *If V, W and Z are vector spaces then there are natural isomorphisms $V \otimes W \cong W \otimes V$ and $(V \otimes W) \otimes Z \cong V \otimes (W \otimes Z)$.*

Lemma 19.5 implies that we can unambiguously write $V \otimes W \otimes Z$.

Definition 19.6. Let V be a vector space and let h and k be non-negative integers. Define a new vector space

Warning: Do not confuse this notation with the tangent space of V!

$$T^{h,k}V := \underbrace{V \otimes \cdots \otimes V}_{h \text{ copies}} \otimes \overbrace{V^* \otimes \cdots \otimes V^*}^{k \text{ copies}}.$$

One calls an element of $T^{h,k}V$ a **tensor of type** (h, k). The vector space $T^{h,k}V$ has dimension $(\dim V)^{h+k}$.

Note we are using Lemma 19.5 to write the right-hand side without brackets. Lemma 19.5 also shows us that it is unimportant in which order we present the factors: for convenience we write the V factors first and the V^* factors afterwards.

Definition 19.7. Let $\mathrm{Mult}_{h,k}(V)$ denote the space of multilinear maps

$$\underbrace{V \times \cdots \times V}_{h \text{ copies}} \times \overbrace{V^* \times \cdots \times V^*}^{k \text{ copies}} \to \mathbb{R}.$$

Thus $\mathrm{Mult}_{1,0}(V) = V^*$ and $\mathrm{Mult}_{0,1}(V) = V^{**} \cong V$.

Proposition 19.8. *There is a canonical isomorphism between the vector space $T^{h,k}V$ and the vector space $\mathrm{Mult}_{k,h}(V)$.*

Note the h and the k swapped round—this is *not* a typo!

The proof of Proposition 19.8 uses the following piece of linear algebra.

Definition 19.9. Let V and W be vector spaces. A **perfect pairing** of V with W is a bilinear map $\beta \colon V \times W \to \mathbb{R}$ such that $\beta(v, \cdot)$ is identically zero if and only if $v = 0$, and $\beta(\cdot, w)$ is identically zero if and only if w is zero.

Proposition 19.10. *If there exists a perfect pairing β between V and W then the map $V \to W^*$ given by $v \mapsto \beta(v, \cdot)$ is a linear isomorphism.*

This result fails if V is infinite-dimensional.

Example 19.11. The natural isomorphism $V \cong V^{**}$ arises from the perfect pairing $V \times V^* \to \mathbb{R}$ given by $(v, \lambda) \mapsto \lambda(v)$.

Proof of Proposition 19.8. We prove the result in two steps.

1. In this step we define a perfect pairing of $T^{h,k}V$ with $T^{h,k}V^*$. Namely, if

$$v = v_1 \otimes \cdots \otimes v_h \otimes \lambda^1 \otimes \cdots \otimes \lambda^k \in T^{h,k}V$$

and

$$w = \eta^1 \otimes \cdots \otimes \eta^h \otimes w_1 \otimes \cdots \otimes w_k \in T^{h,k}V^*$$

then we can naturally feed them each other

$$\beta(v,w) \coloneqq \prod_{i=1}^{h} \eta^i(v_i) \cdot \prod_{j=1}^{k} \lambda^j(w_j) \tag{19.1}$$

We are *not* using the Einstein Summation Convention in this formula – this is a product not a sum!

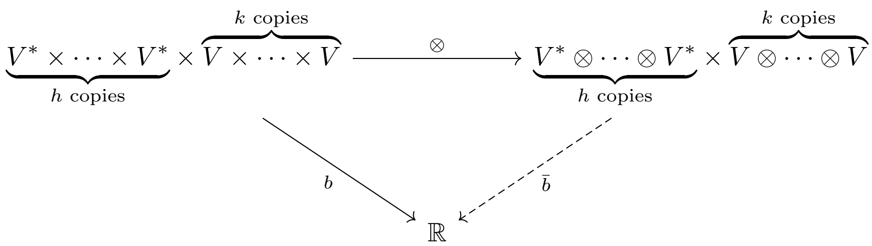
Now extend this bilinearly to all elements. It is immediate that this is a perfect pairing; thus by Proposition 19.10 we have

$$T^{h,k}V \cong \left(T^{h,k}V^*\right)^*$$

2. In this step we prove that

$$\mathrm{Mult}_{h,k}(V^*) \cong \left(T^{h,k}V^*\right)^* \tag{19.2}$$

The proof is by induction on $h + k$. The case $h + k = 1$ is exactly Lemma 19.2. The inductive step follows Lemma 19.5 and Lemma 19.2 again. Explicitly, given a map $b \in \mathrm{Mult}_{h,k}(V^*)$ there exists a unique $\bar{b}\colon T^{h,k}V^* \to \mathbb{R}$, i.e. $T \in \left(T^{h,k}V^*\right)^*$ such that the following diagram commutes.



This diagram is rotated compared to the diagram in the statement of Lemma 19.2 so it fits on the page better.

Since b is uniquely determined by $\bar{b}$, this sets up the desired isomorphism (19.2). Finally, by definition we have $\mathrm{Mult}_{h,k}(V^*) \cong \mathrm{Mult}_{k,h}(V)$, and so the proof is complete. ■

Remark 19.12. On decomposable elements the isomorphism $T^{h,k}V \cong \mathrm{Mult}_{k,h}(V)$ is easier to describe. Suppose for simplicity $A \in T^{2,3}V$ is the decomposable element

$$A = v_1 \otimes v_2 \otimes \lambda^1 \otimes \lambda^2 \otimes \lambda^3.$$

If we use Proposition 19.8 to regard A as an element of $\mathrm{Mult}_{3,2} V$ then A is given by

$$(w_1, w_2, w_3, \eta^1, \eta^2) \mapsto \eta^1(v_1)\, \eta^2(v_2)\, \lambda^1(w_1)\, \lambda^2(w_2)\, \lambda^3(w_3).$$

We can use the Metatheorem to transfer these linear algebra constructions to vector bundles:

Corollary 19.13. *Let $V \to E \xrightarrow{\pi_1} M$ and $W \to F \xrightarrow{\pi_2} M$ be two vector bundles. Then there is a vector bundle $V \otimes W \to E \otimes F \xrightarrow{\pi} M$ whose fibre over p is $E_p \otimes F_p$.*

Proof. We form the principal $\mathrm{GL}(V) \times \mathrm{GL}(W)$ bundle $\mathrm{Fr}(E) \star \mathrm{Fr}(F)$ as in Definition 18.9. The canonical action of $\mathrm{GL}(V) \times \mathrm{GL}(W)$ on $V \times W$ induces by Lemma 19.2 an action of $\mathrm{GL}(V) \times \mathrm{GL}(W)$ on $V \otimes W$. Corollary 18.5 then produces our desired vector bundle

$$E \otimes F \overset{\text{def}}{=} \big(\mathrm{Fr}(E) \star \mathrm{Fr}(F)\big) \times_{\mathrm{GL}(V)\times\mathrm{GL}(W)} (V \otimes W).$$

This completes the proof. ■

COROLLARY 19.14. *Let* $V \to E \xrightarrow{\pi_1} M$ *and* $W \to F \xrightarrow{\pi_2} M$ *be two vector bundles. Then there is a vector bundle* $\mathrm{Hom}(V, W) \to \mathrm{Hom}(E, F) \xrightarrow{\pi} M$ *whose fibre over* p *is* $\mathrm{Hom}(E_p, F_p)$.

We call $\mathrm{Hom}(E, F)$ the **homomorphism bundle**. In the special case $E = F$ we write $\mathrm{End}(E) := \mathrm{Hom}(E, E)$ and call it the **endomorphism bundle**.

Proof. The group $\mathrm{GL}(V) \times \mathrm{GL}(W)$ acts on $\mathrm{Hom}(V, W)$ via

$$\Phi \mapsto B \circ \Phi \circ A^{-1},$$

for $A \in \mathrm{GL}(V)$ and $B \in \mathrm{GL}(W)$. Now argue as above. ∎

COROLLARY 19.15. *Let* E *and* F *be two vector bundles over* M*. Then there is a natural vector bundle isomorphism*

$$\mathrm{Hom}(E, F) \cong E^* \otimes F.$$

Proof. Under the isomorphism from Corollary 19.3, the action of $\mathrm{GL}(V) \times \mathrm{GL}(W)$ on $\mathrm{Hom}(V, W)$ 19 is exactly the same as the action of $\mathrm{GL}(V) \times \mathrm{GL}(W)$ on $V^* \otimes W$ given by combining (18.4) with the action used in the proof of with Corollary 19.13. Corollary 18.4 then supplies the desired isomorphism. ∎

Alternatively, one can argue directly: define

$$\Phi \colon \mathrm{Hom}(E, F) \to E^* \otimes F$$

fibrewise by declaring that the map $\Phi_p \colon \mathrm{Hom}(E_p, F_p) \to E_p^* \otimes F_p$ is the isomorphism from Corollary 19.3. This assignment $p \mapsto \Phi_p$ is smooth and thus Φ is a vector bundle isomorphism.

If you are worried about why $p \mapsto \Phi_p$ is smooth, you could use Proposition 20.23 from next lecture.

Similarly Lemma 19.5 tells us that the tensor product of vector bundles is commutative and associative:

COROLLARY 19.16. *Let* D, E *and* F *be three vector bundles over* M*. Then the bundles* $E \otimes F$ *and* $F \otimes E$ *are isomorphic, and the bundles* $D \otimes (E \otimes F)$ *and* $(D \otimes E) \otimes F$ *are isomorphic.*

More generally:

COROLLARY 19.17. *Let* E *be a vector bundle over* M*. Then there is a vector bundle* $T^{h,k}E$ *over* M *whose fibre over* p *is the vector space* $T^{h,k}E_p$.

Again: do not confuse $T^{h,k}E$ with the tangent bundle TE. $T^{h,k}E$ is a bundle over M, whereas TE is a bundle over E.

Let us recall the formal definition of an algebra.

DEFINITION 19.18. A vector space V is said to be an **algebra** if there exists a bilinear map $V \times V \to V$, which we call **multiplication** and write as $(v, w) \mapsto vw$. If V and W are algebras, an **algebra morphism** $\ell \colon V \to W$ is a linear map such that

$$\ell(vw) = \ell(v)\ell(w), \qquad \forall\, v, w \in V.$$

By Lemma 19.2 this is equivalent to a linear map $V \otimes V \to V$.

Whilst each $T^{h,k}V$ is not an algebra, if we sum them all together we obtain one.

Definition 19.19. The **tensor algebra** of V is defined to be

$$\widetilde{T}V := \bigoplus_{h,k \geq 0} T^{h,k}V,$$

where $T^{0,0}V := \mathbb{R}$. This is a *graded algebra*, in the sense that $\otimes$ gives a natural map

$$\otimes \colon T^{h,k}V \times T^{h_1,k_1}V \to T^{h+h_1,k+k_1}V. \tag{19.3}$$

The natural map is defined as one would guess: on decomposable elements it simply tensors everything together and then rearranges the factors so the V elements come first, so as to fit with our convention. We illustrate this with $(h,k) = (1,2)$ and $(h_1,k_1) = (2,1)$:

$$\Big((v_1 \otimes \lambda^1 \otimes \lambda^2), (w_1 \otimes w_2 \otimes \eta^1)\Big) \mapsto v_1 \otimes w_1 \otimes w_2 \otimes \lambda^1 \otimes \lambda^2 \otimes \eta^1. \tag{19.4}$$

If $(h,k) = (0,0)$ then tensor multiplication with a scalar is defined to be normal scalar multiplication, i.e.

$$c \otimes v := cv, \qquad c \in \mathbb{R},\ v \in V.$$

The tilde is there to prevent confusion with the tangent bundle of V.

Remark 19.20. The space $\widetilde{T}V$ is an *infinite-dimensional* vector space. This means we use **cannot** use Corollary 18.5 to produce a "vector bundle" $\widetilde{T}V \to \widetilde{T}E \to M$ out of a vector bundle $V \to E \to M$.

Actually this is only half the story: $\widetilde{T}E$ is an example of an infinite-dimensional vector bundle over M. This goes beyond the scope of this course, however, and we won't need it.

We now introduce another linear algebra construction, called the **exterior algebra**. This will associate to a vector space V another (finite-dimensional) vector space $\bigwedge V$ which, like the tensor algebra $\widetilde{T}V$, admits an algebra structure.

Let V be a vector space. Let $\widetilde{T}^+V$ denote the subalgebra given by $\widetilde{T}^+V := \bigoplus_{h\geq 0} T^{h,0}V$. Let $\widetilde{I}V$ denote the two-sided ideal in $\widetilde{T}^+V$ generated by all elements of the form $v \otimes v$ for $v \in V$. Thus for instance $u \otimes v \otimes v \otimes w$ belongs to $\widetilde{I}V$.

Definition 19.21. The **exterior algebra** is defined to be the quotient algebra $\bigwedge V := \widetilde{T}^+V/\widetilde{I}V$. We denote the image of $v_1 \otimes \cdots \otimes v_h$ in $\bigwedge V$ by $v_1 \wedge \cdots \wedge v_h$ and call $\wedge$ the **wedge product**.

Such an element $v_1 \wedge \cdots \wedge v_h$ is called **decomposable**. The space $\bigwedge V$ is the quotient of one infinite-dimensional vector space by another. As we will shortly see, $\bigwedge V$ is always finite-dimensional.

Let $\bigwedge^h V$ to be the image of $T^{h,0}V$ in $\bigwedge V$ under the projection $\widetilde{T}^+V \to \bigwedge V$ there is a canonical vector space isomorphism

$$\bigwedge^h V \cong T^{h,0}V \Big/ I^hV,$$

where $I^hV := T^{h,0}V \cap \widetilde{I}V$. Note that $\bigwedge^1 V = V$ and $\bigwedge^0 V = \mathbb{R}$. This definition may seem a little abstract, so let us unpack things a bit.

Proposition 19.22 (Properties of the wedge product). *Let V be a vector space. Then:*

(i) *For all* $v, w \in V$, $v \wedge w = -w \wedge v$.

(ii) *Assume* $h, k > 0$. *If* $v \in \bigwedge^h V$ *and* $w \in \bigwedge^k V$ *then* $v \wedge w \in \bigwedge^{h+k} V$ *and*

$$v \wedge w = (-1)^{hk} w \wedge v,$$

This continues to hold if either $h = 0$ *or* $k = 0$ *provided we use the convention that for a real number* c *and a vector* v*, one has* $c \wedge v := cv$.

(iii) *If* $v_1 \wedge \cdots \wedge v_h \in \bigwedge^h V$ *is a decomposable element then transposing* v_i *with* v_j *acts as multiplication by -1:*

$$v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_h = -v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_h$$

(iv) *If* $\varrho \in \mathfrak{S}_h$ *is a permutation on* h *letters and* $v_i \in V$ *then*

$$v_{\varrho(1)} \wedge \cdots \wedge v_{\varrho(h)} = \operatorname{sgn}(\varrho) v_1 \wedge \cdots \wedge v_h.$$

Proof. To prove part (i), we note that for any $u \in V$, $u \otimes u$ belongs to $\widetilde{I}V$, and thus in $\bigwedge V$, $u \wedge u = 0$. Applying this with $u = v + w$ we have

$$\begin{aligned} 0 &= (v + w) \wedge (v + w) \\ &= v \wedge v + v \wedge w + w \wedge v + w \wedge w \\ &= v \wedge w + w \wedge v. \end{aligned}$$

To prove part (ii), as both sides are linear in v and w, it suffices to verify it for decomposable elements, and for such, the conclusion follows by repeated applications of part (i). Next, to prove part (iii), we may assume $i < j$. Set $u := v_{i+1} \wedge \cdots \wedge v_{j-1}$. Then by part (ii) one has

$$v_i \wedge u \wedge v_j = -v_j \wedge u \wedge v_i,$$

and thus part (iii) follows. Finally, part (iv) is immediate from the fact that any permutation may be written as a product of transpositions. ■

There is an analogous universal mapping property for the exterior algebra.

Definition 19.23. Let V and W be vector spaces. Let $\operatorname{Alt}_h(V, W)$ denote the space of **alternating h-linear maps**, i.e. multilinear maps $A \colon V \times \cdots \times V \to W$ (h times) that vanish whenever any two of the arguments are equal:

$$A(\cdots, v, \cdots, v, \cdots) = 0.$$

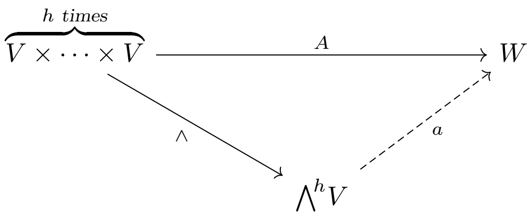
We abbreviate $\operatorname{Alt}_h(V) = \operatorname{Alt}_h(V, \mathbb{R})$.

The map $\wedge \colon V \times \cdots \times V \to \bigwedge^h V$ given by sending $(v_1, \ldots, v_h) \mapsto v_1 \wedge \cdots \wedge v_h$ is an example of such a map. We aim to prove the following alternating version of Proposition 19.8:

Proposition 19.24. *There is a canonical isomorphism between* $\bigwedge^h V^*$ *and* $\operatorname{Alt}_h(V)$.

The proof strategy is similar to that of Proposition 19.8, and we will be brief. First, we need an analogue of Lemma 19.2.

Lemma 19.25. *Let V and W be vector spaces. For any $A \in \mathrm{Alt}_h(V, W)$ there is a unique linear map $a \colon \bigwedge^h V \to W$ such that the following diagram commutes:*



Moreover $\bigwedge^h V$ is uniquely characterised by this property.

The proof of Lemma 19.25 is on Problem Sheet H.

Proof of Proposition 19.24. Just as in the proof of step 2 of Proposition 19.8, an inductive argument based on Lemma 19.25 tells us that we can identify

$$\mathrm{Alt}_h(V) \cong \left(\bigwedge^h V\right)^*.$$

The next step is to exhibit a perfect pairing of $\bigwedge^h V^*$ with $\bigwedge^h V$. This formula is a little harder to guess than in (19.1), but once you know the formula it is easy to check. Namely, we define

$$\alpha \colon \bigwedge^h V^* \times \bigwedge^h V \to \mathbb{R}$$

by declaring on decomposable elements that

$$\alpha\big(\lambda^1 \wedge \cdots \wedge \lambda^h, v_1 \wedge \cdots \wedge v_h\big) := \det A,$$

where A is the $h \times h$ matrix whose (i, j)th entry is $\lambda^i(v_j)$. Then extend α by bilinearity to all of $\bigwedge^h V^* \times \bigwedge^h V^*$. We invite you to verify this is indeed a perfect pairing. ■

For later use, let us state part of the proof of Proposition 19.24 as a separate corollary.

Corollary 19.26. *Let $\lambda^1, \dots, \lambda^h \in V^*$ and $v_1, \dots, v_h \in V$. Then viewing $\lambda^1 \wedge \cdots \wedge \lambda^h$ as an element of $\mathrm{Alt}_h(V)$, one has*

$$\lambda^1 \wedge \cdots \wedge \lambda^h(v_1, \dots, v_h) = \det A,$$

where A is the $h \times h$ matrix whose (i, j)th entry is $\lambda^i(v_j)$.

On Problem Sheet H you are also asked to show:

Lemma 19.27. *Let V be a vector space of dimension n with basis $\{e_1, \dots, e_n\}$. Then*

$$\{e_{i_1} \wedge \cdots \wedge e_{i_h} \mid 1 \le i_1 < \cdots < i_h \le n\}$$

is a basis of $\bigwedge^h V$ and $\bigwedge^h V = 0$ for $h > n$. Thus $\dim \bigwedge^h V = \binom{n}{h}$ and $\dim \bigwedge V = 2^n$.

We end today's lecture by applying the Metatheorem to the operations $\bigwedge^h$ and $\bigwedge$.

Corollary 19.28. *Let $V \to E \xrightarrow{\pi} M$ be a vector bundle of rank n. Then for any $0 \le h \le n$, there is a vector bundle $\bigwedge^h E \to M$ whose fibre over $p \in M$ is given by $\bigwedge^h E_p$. This vector bundle has rank $\binom{n}{h}$. Similarly there is a vector bundle $\bigwedge E \to M$ of rank 2^n whose fibre over $p \in M$ is given by $\bigwedge E_p$. It is the direct sum of the vector bundles $\bigwedge^h E$.*

The bundle $\bigwedge E$ inherits an algebra structure from the algebra structure on $\bigwedge V$.

Definition 19.29. Let V be vector space which is also an algebra in the sense of Definition 19.18, and suppose that $V \to E \xrightarrow{\pi} M$ is a vector bundle. We say that E is an **algebra bundle** if each fibre E_p admits the structure of an algebra, and there exists a vector bundle atlas $\{(U_a, \varepsilon_a)\}$ such that for each $p \in U_a$ the map $(\varepsilon_a)_p \colon E_p \to V$ is not only a linear isomorphism but also an algebra isomorphism.

Corollary 19.30. *Let $V \to E \xrightarrow{\pi} M$ be a vector bundle. Then $\bigwedge V \to \bigwedge E \to M$ is an algebra bundle.*

The proof of Corollary 19.30 is left for you on Problem Sheet H.

Remark 19.31. In Lecture 34 we will work with **Lie algebra bundles**, which are algebra bundles with the additional property that the algebra structure is actually a Lie algebra.

Bonus Material for Lecture 19

In the bonus material for this lecture, we introducing elements of a field of mathematics called **category theory**. This material will not be needed at any point during Differential Geometry I or II. Our aim is to give a category-theoretic proof of the Metatheorem from the previous lecture – see Theorem 19.60 below – which entirely bypasses principal and associated bundles.

In a nutshell, category theory is the an attempt to make "proof by analogy" a valid proof tactic. That is, category theory is an interdisciplinary language that allows one to describe certain general phenomena that crop up in mathematical arguments across the board. The advantage of possessing such a language is clear – it allows one to isolate the essence of a given statement or proof technique, thus allowing for concise and clean proofs. It is also efficient: a single category-theoretic blueprint can simultaneously prove diverse statements in number theory, geometry, algebra, analysis, and so on. Category theory is also useful in theoretical computer science; indeed, many functional programming languages (eg. Haskel, Scala) are almost literal interpretations of categorical methods. The generality comes at a price, though: category theory is often (lovingly) referred to as abstract nonsense.

You probably have already met several "category-theoretic" arguments in your mathematical career so far. Roughly speaking, a category-theoretic argument is one that focuses on transformations between objects of a given type, rather than on the objects themselves. An example of this is Lemma 19.2 above, which characterised the tensor product via a *universal property*. In general, a universal property typically expresses a certain role that a given mathematical object plays in relation to other objects of its type, and the abstract categorical theorem is: *if an object can be described by a universal property, then it is unique up to isomorphism*. If we were allowed to quote this result, the proof of Lemma 19.2 would have been over at the end of the first paragraph.

Before we get started on the definitions, let us list a few results which you have probably already met that can all be proved used categorical methods:

- If A and B are sets and $f\colon A \times B \to \mathbb{R}$ is a function, then

  $$\sup_{a\in A}\inf_{b\in B} f(a,b) \leq \inf_{b\in B}\sup_{a\in A} f(a,b)$$

  whenever the infima and suprema exist.
- *Cayley's Theorem:* Any finite group is isomorphic to a subgroup of a permutation group.
- Every row operation on matrices with m rows is given by left multiplication by some $m \times m$ matrix.
- A continuous bijection between compact Hausdorff spaces is a homeomorphism.

And now the definitions:

DEFINITIONS 19.32. A **category** C consists of three ingredients. The first is a *class* $\mathrm{obj}(\mathsf{C})$ of **objects**. Secondly, for each ordered pair of objects (A, B) there is a *set* $\mathrm{Hom}(A, B)$ of **morphisms** from A to B. Sometimes instead of $f \in \mathrm{Hom}(A, B)$ we write $f : A \to B$ or $A \xrightarrow{f} B$. Finally, there is a rule, called **composition**, which associates to every ordered triple (A, B, C) of objects a map

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C),$$

written

$$(f, g) \mapsto g \circ f,$$

which satisfies the following three axioms:

(i) The Hom sets are pairwise disjoint; that is, each $f \in \mathrm{Hom}(A, B)$ has a unique **domain** A and a unique **target** B.

(ii) Composition is associative whenever defined, i.e. given

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

one has

$$(h \circ g) \circ f = h \circ (g \circ f).$$

(iii) For each $A \in \mathrm{obj}(\mathsf{C})$ there is a unique morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ called the **identity** which has the property that $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$ for every $f : A \to B$.

REMARK 19.33. Note that we said that $\mathrm{obj}(\mathsf{C})$ was a *class* and $\mathrm{Hom}(A, B)$ was a *set*. There is (an important, but technical) difference between a class and a set. If you've ever taken a class on logic/set theory, you'll know that not every "collection" of objects is formally a set. For instance, the collection of all sets is itself not a set. A class is a more general concept (the collection of all sets is a class). Nevertheless, as far as we're concerned, the distinction is essentially irrelevant.

this is Russel's Paradox.

Here are six examples of categories. The first three are algebraic in nature.

EXAMPLE 19.34. The category Sets of **sets**. The objects of Sets are all the sets, and $\mathrm{Hom}(A, B)$ is just the set $\mathrm{Maps}(A, B)$ of all functions from A to B, and composition is just the usual composition of functions.

EXAMPLE 19.35. The category Groups of **groups**. The objects of Groups are just groups, and $\mathrm{Hom}(G, H)$ is the set of all *group homomorphisms* from G to H, and composition is just the usual composition of homomorphisms.

EXAMPLE 19.36. The category $\mathsf{Vect} = \mathsf{Vect}_{\mathbb{R}}$ of **finite-dimensional real vector spaces**. The objects of Vect are finite-dimensional real vector spaces, and $\mathrm{Hom}(V, W)$ is the set $\mathrm{Hom}(V, W)$ of all *linear maps* from V to W.

Here are three more examples more pertinent to this course.

EXAMPLE 19.37. The category Top of **topological spaces**. The objects of Top are all the topological spaces, and $\mathrm{Hom}(X, Y)$ is just the set $C(X, Y)$ of all *continuous* functions from X to Y, and composition is just the usual composition of functions.

EXAMPLE 19.38. The category Man of **smooth manifolds**. The objects of Man are smooth manifolds, and $\mathrm{Hom}(M, N)$ is the set $C^\infty(M, N)$ of all *smooth* maps $\varphi \colon M \to N$. Composition is given by normal composition of maps; this is well defined by Proposition 1.21.

EXAMPLE 19.39. The category $\mathsf{VectBundles}$ of **vector bundles**. The objects of $\mathsf{VectBundles}$ are vector bundles $\pi \colon E \to M$, and morphism from $\pi_1 \colon E_1 \to M_1$ to $\pi_2 \colon E_2 \to M_2$ is a pair (Φ, φ), where $\varphi \colon M_1 \to M_2$ is a smooth map and $\Phi \colon E_1 \to E_2$ is a vector bundle morphism from E_1 to E_2 along φ.

Remark 19.40. The category Vect is rather special: its morphism sets are themselves objects of the category. That is, if V and W are vector spaces then $\operatorname{Hom}(V, W)$ is itself naturally a vector space. This is not true in the category of Groups—the set of all group homomorphisms from one group to another typically does not have a group structure. Similarly the set $C^\infty(M, N)$ of smooth maps between two smooth manifolds is never itself a (finite-dimensional) manifold when $\dim M > 0$.

Remark 19.41. The fact that we require the morphism sets to be pairwise disjoint has several pedantic consequences. For example, suppose $A \subsetneq B$ are two sets. Then the inclusion $\imath\colon A \hookrightarrow B$ and the identity map $\mathrm{id}_A\colon A \to A$ are different morphisms, since they have different targets. One should be aware that we only allow the composition $g \circ f$ when the range of f is *exactly* the same as the domain of g. Suppose L, M, N and P are manifolds, and suppose M is an embedded submanifold of N. Let $\varphi\colon L \to M$ be smooth and let $\psi\colon N \to P$ be smooth. Then as we have seen, the composition $\psi \circ \varphi\colon L \to P$ is also smooth (since M is embedded). Nevertheless, from the point of view of category theory, the composition $\psi \circ \varphi$ *does not exist!* Rather, one must first take the inclusion $\imath\colon M \hookrightarrow N$ and then consider the composition $\psi \circ \imath \circ \varphi$, which is a well-defined element of the morphism space $C^\infty(L, P)$.

Definition 19.42. Suppose C and D are two categories. We say that C is a **subcategory** of D if:

1. $\operatorname{obj}(\mathsf{C}) \subseteq \operatorname{obj}(\mathsf{D})$;
2. $\operatorname{Hom}_{\mathsf{C}}(A, B) \subseteq \operatorname{Hom}_{\mathsf{D}}(A, B)$ for all $A, B \in \operatorname{obj}(\mathsf{C})$;
3. if $f \in \operatorname{Hom}_{\mathsf{C}}(A, B)$ and $g \in \operatorname{Hom}_{\mathsf{C}}(B, C)$ then the composite $g \circ f \in \operatorname{Hom}_{\mathsf{C}}(A, C)$ is equal to the composite $g \circ f \in \operatorname{Hom}_{\mathsf{D}}(A, C)$;
4. if $C \in \operatorname{obj}(\mathsf{C})$ then $\mathrm{id}_C \in \operatorname{Hom}_{\mathsf{C}}(C, C)$ is equal to $\mathrm{id}_C \in \operatorname{Hom}_{\mathsf{D}}(C, C)$.

If for every pair $A, B \in \operatorname{obj}(\mathsf{C})$ one always has $\operatorname{Hom}_{\mathsf{C}}(A, B) = \operatorname{Hom}_{\mathsf{D}}(A, B)$ then we say that C is a **full subcategory** of D.

Example 19.43. Here are two examples of subcategories:

(i) The category Ab of *abelian* groups is a full subcategory of the category Groups.

(ii) Let $\mathsf{Vect}^{\le\infty}$ denote the category of *all* real vector spaces (finite-dimensional or infinite-dimensional). Then Vect is a full subcategory of $\mathsf{Vect}^{\le\infty}$.

A **functor** is a map from one category to another. These come in two flavours: *covariant* and *contravariant*. We discuss the former first.

Definition 19.44. Suppose C and D are two categories. A **covariant functor** $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ associates to each $A \in \operatorname{obj}(\mathsf{C})$ an object $\mathsf{F}(A) \in \operatorname{obj}(\mathsf{D})$, and to each morphism $A \xrightarrow{f} B$ in C a morphism $\mathsf{F}(A) \xrightarrow{\mathsf{F}(f)} \mathsf{F}(B)$ in D which satisfies the following two axioms:

1. If $A \xrightarrow{f} B \xrightarrow{g} C$ in C then $\mathsf{F}(A) \xrightarrow{\mathsf{F}(f)} \mathsf{F}(B) \xrightarrow{\mathsf{F}(g)} \mathsf{F}(C)$ in D and

$$\mathsf{F}(g \circ f) = \mathsf{F}(g) \circ \mathsf{F}(f).$$

2. $\mathsf{F}(\mathrm{id}_A) = \mathrm{id}_{\mathsf{F}(A)}$ for every $A \in \mathrm{obj}(\mathsf{C})$.

The easiest example of a functor is a **forgetful functor**:

Example 19.45. The forgetful functor $\mathsf{Top} \to \mathsf{Sets}$ simply "forgets" the topological structure. Thus it assigns to each topological space its underlying set, and to each continuous function it assigns the same function, considered now simply as a map between two *sets* (i.e. it "forgets" the function is continuous). The same thing works as a functor $\mathsf{Man} \to \mathsf{Top}$, where one "forgets" the smooth manifold structure.

Example 19.46. There is slightly more interesting forgetful functor $\mathsf{VectBundles} \to \mathsf{Man}$ that sends a vector bundle $\pi\colon E \to M$ to its base space M (i.e. it "forgets" the vector bundle sitting over the base). On morphisms, this functor just "forgets" the vector bundle morphism: $(\varphi, \Phi) \mapsto \varphi$.

Here is a pertinent example of functor from the category Vect to itself:

Examples 19.47. Let V be a fixed vector space. There is a covariant functor

$$\mathrm{Hom}(V, \cdot)\colon \mathsf{Vect} \to \mathsf{Vect}$$

that assigns to a vector space W the vector space $\mathrm{Hom}(V, W)$. If $\ell\colon W_1 \to W_2$ is a linear map then

$$\mathrm{Hom}(V, \cdot)(\ell)\colon \mathrm{Hom}(V, W_1) \to \mathrm{Hom}(V, W_2)$$

is given by $\ell_1 \mapsto \ell_1 \circ \ell$.

In today's lecture we constructed several functors:

Example 19.48. There is a covariant functor $\otimes\colon (\mathsf{Vect}, \mathsf{Vect}) \to \mathsf{Vect}$ given by $(V, W) \mapsto V \otimes W$.

Example 19.49. There is a covariant functor $\bigwedge\colon \mathsf{Vect} \to \mathsf{Vect}$

Algebraic topology is an excellent source of functors. For instance, the **fundamental group** π_1 is a covariant functor from the **pointed homotopy category** hTop_* to Groups, and the **higher homotopy groups** are covariant functors $\pi_n\colon \mathsf{hTop}_* \to \mathsf{Ab}$. **Singular homology** (or indeed, any homology theory) is a covariant functor $\mathsf{hTop}^2 \to \mathsf{Ab}$, where hTop^2 is the homotopy category of pairs.

One can also formulate the definition of a functor of more than one variable. This requires us to define the notion of a product category.

Definition 19.50. Let C and D be two categories. The **product category** (C, D) is the category whose objects are ordered pairs (C, D) where $C \in \mathrm{obj}(\mathsf{C})$ and $D \in \mathrm{obj}(\mathsf{D})$, and

$$\mathrm{Hom}((C, D), (C', D')) = \{(f, g) \mid f \in \mathrm{Hom}_{\mathsf{C}}(C, C'),\ g \in \mathrm{Hom}_{\mathsf{D}}(D, D')\}.$$

The composition $(f, g) \circ_{(\mathsf{C},\mathsf{D})} (f', g')$ is defined as you expect:

$$(f, g) \circ_{(\mathsf{C},\mathsf{D})} (f', g') \coloneqq ((f \circ_{\mathsf{C}} f'), (g \circ_{\mathsf{D}} g')) \,.$$

The identity element $\mathrm{id}_{(C,D)}$ is simply the pair $(\mathrm{id}_C, \mathrm{id}_D)$.

Example 19.51. The category $(\mathsf{Vect}, \mathsf{Vect})$ has objects ordered pairs (V, W) of vector spaces, and morphisms pairs of linear maps.

Definition 19.52. A **covariant functor of two variables** is a covariant functor defined on a product category: $\mathsf{F}\colon (\mathsf{C}, \mathsf{D}) \to \mathsf{E}$.

Example 19.53. Let V and W be vector spaces. Then the **direct sum** $V \oplus W$ of V and W is another vector space. Thus we get a functor $\oplus\colon (\mathsf{Vect}, \mathsf{Vect}) \to \mathsf{Vect}$ that assigns to (V, W) the vector space $V \oplus W$, and assigns to a pair (ℓ_1, ℓ_2) of linear maps $\ell_1 \colon V_1 \to V_2$ and $\ell_2 \colon W_1 \to W_2$ the linear map $\ell_1 \oplus \ell_2 \colon V_1 \oplus W_1 \to V_2 \oplus W_2$.

We use the direct sum notation here as well (instead of the product) so as not to confuse you with the product category.

In the same way, one can form a **k-fold product category** $(\mathsf{C}_1, \dots, \mathsf{C}_k)$ of categories C_i, and a **covariant functor of k variables** is a covariant functor of the form $\mathsf{F}\colon (\mathsf{C}_1, \dots, \mathsf{C}_k) \to \mathsf{D}$. For example, there is a functor

$$(\mathsf{Vect}, \dots, \mathsf{Vect}) \to \mathsf{Vect}, \qquad (V_1, \dots, V_k) \to V_1 \oplus \dots \oplus V_k.$$

A contravariant functor is defined in almost the same way, but it reverses the arrows.

Definition 19.54. Suppose C and D are two categories. A **contravariant functor** $\mathsf{G}\colon \mathsf{C} \to \mathsf{D}$ associates to each $A \in \mathrm{obj}(\mathsf{C})$ an object $\mathsf{G}(A) \in \mathrm{obj}(\mathsf{D})$, and to each morphism $A \xrightarrow{f} B$ in C a morphism $\mathsf{G}(B) \xrightarrow{\mathsf{G}(f)} \mathsf{G}(A)$ in D which satisfies the following two axioms:

1. If $A \xrightarrow{f} B \xrightarrow{g} C$ in C then $\mathsf{G}(C) \xrightarrow{\mathsf{G}(g)} \mathsf{G}(B) \xrightarrow{\mathsf{G}(f)} \mathsf{G}(A)$ in D and
$$\mathsf{G}(g \circ f) = \mathsf{G}(f) \circ \mathsf{G}(g).$$
2. $\mathsf{G}(\mathrm{id}_A) = \mathrm{id}_{\mathsf{G}(A)}$ for every $A \in \mathrm{obj}(\mathsf{C})$.

Here is a simple example of a contravariant functor on the category of vector spaces.

Example 19.55. Let W be a fixed vector space. Then there is a contravariant functor

$$\mathrm{Hom}(\cdot, W)\colon \mathsf{Vect} \to \mathsf{Vect}$$

that assigns to a vector space V the vector space $\mathrm{Hom}(V, W)$. If $\ell\colon V_1 \to V_2$ is a linear map then

Note the order of V_1 and V_2!

$$\mathrm{Hom}(\cdot, W)(\ell)\colon \mathrm{Hom}(V_2, W) \to \mathrm{Hom}(V_1, W)$$

is given by $\ell_2 \mapsto \ell \circ \ell$. It is important for you to understand why $\mathrm{Hom}(V, \cdot)$ is covariant but $\mathrm{Hom}(\cdot, W)$ is contravariant.

Taking $W = \mathbb{R}$ shows that $V \mapsto V^*$ is a contravariant functor.

Remark 19.56. Going back to algebraic topology, **singular cohomology** is a contravariant functor $\mathsf{hTop}^2 \to \mathsf{Ab}$. Later in this course we will look at **de Rham cohomology**.

Similarly one can consider contravariant functors of more than one variable. In fact, one can even consider functors that are covariant in some variables and contravariant in others. This is easiest to see with an example.

Example 19.57. Let $\mathrm{Hom}(\cdot,\cdot)\colon (\mathsf{Vect},\mathsf{Vect}) \to \mathsf{Vect}$ denote the functor that sends a pair (V,W) to the vector space $\mathrm{Hom}(V,W)$. As Example 19.47 and 19.55 showed, this is contravariant in the first variable and covariant in the second variable. If $\ell_1\colon V_1 \to V_2$ and $\ell_2\colon W_1 \to W_2$ then

$$\mathrm{Hom}(\cdot,\cdot)(\ell_1,\ell_2)\colon \mathrm{Hom}(V_2,W_1) \to \mathrm{Hom}(V_1,W_2)$$

sends a linear map $a\colon V_2 \to W_1$ to the linear map $\ell_2 \circ a \circ \ell_1\colon V_1 \to W_2$.

We have now almost arrived at the correct setting for which to prove the Metatheorem. The only thing that is left is to take into the account that we require our functors to be smooth.

Definition 19.58. Let $\mathsf{F}\colon \mathsf{Vect} \to \mathsf{Vect}$ be a covariant functor. We say that F is **smooth** if for any two vector spaces V, W, the map

$$\mathrm{Hom}(V,W) \to \mathrm{Hom}(\mathsf{F}(V),\mathsf{F}(W)), \qquad \ell \mapsto \mathsf{F}(\ell)$$

is itself smooth in the normal sense.

A similar definition makes sense for functors of k variables which are covariant in some variables and contravariant in others, provided one remembers to flip the domain and target in each contravariant variable:

Definition 19.59. Let $\mathsf{F}\colon (\mathsf{Vect},\dots,\mathsf{Vect}) \to \mathsf{Vect}$ be a functor of k variables of either (or mixed) variance. We say that F is a **smooth functor** if for any vector spaces $V_1,\dots V_k$ and $W_1,\dots,W_k$, the induced map

$$\bigoplus_{i=1}^{r} \tilde{\mathrm{Hom}}(V_i,W_i) \to \mathrm{Hom}(\mathsf{F}(V_1,\dots,V_k),\mathsf{F}(W_1,\dots W_k)),$$
$$(\ell_1,\dots,\ell_k) \mapsto \mathsf{F}(\ell_1,\dots,\ell_k) \tag{19.5}$$

where

$$\tilde{\mathrm{Hom}}(V_i,W_i) := \begin{cases} \mathrm{Hom}(V_i,W_i), & \text{if } \mathsf{F} \text{ is covariant in the } i\text{th variable,} \\ \mathrm{Hom}(W_i,V_i), & \text{if } \mathsf{F} \text{ is contravariant in the } i\text{th variable,} \end{cases}$$

is a smooth map in the usual sense (note again each side is simply a vector space).

In fact, in all the examples we have seen, the map (19.5) is actually a linear map (and so is certainly smooth). We emphasise though that for a general functor this may not be the case. Here now is a precise statement of the Metatheorem.

THEOREM 19.60. *Let* $\mathsf{F}\colon (\mathsf{Vect}, \dots, \mathsf{Vect}) \to \mathsf{Vect}$ *be a smooth functor of* k *variables of either variance in each variable. Let* $\pi_i \colon E_i \to M$ *be* k *vector bundles. Define*

$$\mathsf{F}(E_1, \dots, E_k) := \bigsqcup_{p \in M} \mathsf{F}(E_{1|p}, \dots, E_{k|p}),$$

with associated projection $\pi\colon \mathsf{F}(E_1, \dots, E_k) \to M$*. Then* $\mathsf{F}(E_1, \dots, E_k)$ *is a vector bundle.*

The proof is very easy, but it is notationally quite challenging. We recommend you write out for yourself the case $k = 2$ where F is say, contravariant in the first variable and covariant in the second (the $\mathrm{Hom}(\cdot, \cdot)$ functor from Example 19.57 is such an example). Once you understand this, the general case is just messier.

Proof. Choose an open set $U \subset M$ over which all the E_i are trivial, i.e. so that there exist vector bundle charts $\varepsilon_i \colon \pi_i^{-1}(U) \to \mathbb{R}^{n_i}$, where E_i has rank n_i. Then for each $p \in U$ and each i, we have a linear isomorphism $\varepsilon_{i|p}\colon E_{i|p} \to \mathbb{R}^{n_i}$. Set

$$\tilde{\varepsilon}_{i|p} := \begin{cases} \varepsilon_{i|p}\colon E_{i|p} \to \mathbb{R}^{n_i}, & \text{if } \mathsf{F} \text{ is covariant in the } i\text{th variable,} \\ \varepsilon_{i|p}^{-1}\colon \mathbb{R}^{n_i} \to E_{i|p}, & \text{if } \mathsf{F} \text{ is contravariant in the } i\text{th variable.} \end{cases}$$

Since F is a functor, we can feed it the morphisms $\tilde{\varepsilon}_{i,p}$ to get a map

$$\tilde{\varepsilon}_p = \mathsf{F}(\tilde{\varepsilon}_{1|p}, \dots, \tilde{\varepsilon}_{k|p}) \in \mathrm{Hom}\Big(\mathsf{F}\big(E_{1|p}, \dots, E_{k|p}\big), \mathsf{F}\big(\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_k}\big)\Big)$$

By functoriality, $\tilde{\varepsilon}_p$ is linear isomorphism. Define $\tilde{\varepsilon}\colon \pi^{-1}(U) \to \mathsf{F}(\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_k})$ by letting $\tilde{\varepsilon}$ be equal to $\tilde{\varepsilon}_p$ on $\mathsf{F}(E_{1|p}, \dots, E_{k|p})$. We now declare $\tilde{\varepsilon}$ to be a bundle chart for $\mathsf{F}(E_1, \dots, E_k)$ over U.

To complete the proof, we need to show that the transition functions are smooth linear isomorphisms. For this, suppose $\gamma_i \colon \pi_i^{-1}(U) \to \mathbb{R}^{k_i}$ were different choices of vector bundle chart on each E_i, with corresponding chart $\tilde{\gamma}$ on $\mathsf{F}(E_1, \dots, E_k)$. Let $g_i(p) = \varepsilon_{i|p} \circ \gamma_{i|p}^{-1}$ denote the transition functions on E_i from the ε_i to the γ_i. We must show that the transition function $g(p) := \tilde{\varepsilon}_p \circ \tilde{\gamma}_p^{-1}$ is smooth and linear. But this again follows almost immediately from functoriality. If $p \in U$ then

$$\begin{aligned} g(p) &= \tilde{\varepsilon}|_{\mathsf{F}(E_{1|p}, \dots, E_{k|p})} \circ \tilde{\gamma}|^{-1}_{\mathsf{F}(E_{1|p}, \dots, E_{k|p})} \\ &= \mathsf{F}(\tilde{\varepsilon}_{1|p}, \dots, \tilde{\varepsilon}_{k|p}) \circ \mathsf{F}(\tilde{\gamma}_{1|p}, \dots, \tilde{\gamma}_{k|p})^{-1} \\ &= \mathsf{F}(\tilde{\varepsilon}_{1|p}, \dots, \tilde{\varepsilon}_{k|p}) \circ \mathsf{F}(\tilde{\gamma}_{1|p}^{-1}, \dots, \tilde{\gamma}_{k|p}^{-1}) \\ &= \mathsf{F}\big(\tilde{\varepsilon}_{1|p} \circ \gamma_{1|p}^{-1}, \dots \tilde{\varepsilon}_{k|p} \circ \gamma_{k|p}^{-1}\big) \\ &= \mathsf{F}\big(\tilde{g}_1(p), \dots \tilde{g}_k(p)\big), \end{aligned}$$

where

$$\tilde{g}_i(p) := \begin{cases} g_i(p) & \text{if } \mathsf{F} \text{ is covariant in the } i\text{th variable,} \\ g_i(p)^{-1} & \text{if } \mathsf{F} \text{ is contravariant in the } i\text{th variable.} \end{cases}$$

Thus since F is a functor, $\mathsf{F}(\tilde{g}_1(p), \dots \tilde{g}_k(p))$ is a linear isomorphism. Moreover since F is a smooth functor, $p \mapsto \mathsf{F}(\tilde{g}_1(p), \dots \tilde{g}_k(p))$ depends smoothly on p. This completes the proof. ■

Theorem 19.60 bypasses the Corollary 18.5. But it is not yet enough by itself, as we have not yet proved the analogue of Corollary 18.4 – namely, that the vector bundle given by Theorem 19.60 is unique up to isomorphism.

This however is easily rectified, but it requires introducing **natural transformations**, which, roughly speaking, are functors between functors.

Definition 19.61. Let C and D be two categories, and let $\mathsf{F}, \mathsf{G} \colon \mathsf{C} \to \mathsf{D}$ be two functors. A **natural transformation** $\tau \colon \mathsf{F} \to \mathsf{G}$ is a family of morphisms $\tau_C \colon \mathsf{F}(C) \to \mathsf{G}(C)$ for each $C \in \operatorname{obj}(\mathsf{C})$ such that for any morphism $f \colon A \to B$ in C the following diagram commutes:

$$\begin{array}{ccc} \mathsf{F}(A) & \xrightarrow{\tau_A} & \mathsf{G}(A) \\ {\scriptstyle \mathsf{F}(f)}\downarrow & & \downarrow{\scriptstyle \mathsf{G}(f)} \\ \mathsf{F}(B) & \xrightarrow[\tau_B]{} & \mathsf{G}(B) \end{array}$$

If each morphism τ_C is an isomorphism then we say that τ is a **natural isomorphism**.

In the previous lecture we briefly discussed the difference between canonical isomorphisms and non-canonical isomorphisms. To illustrate this, let us give a proper (18.6) from the previous lecture.

Theorem 19.62. *A finite-dimensional vector space V is canonically isomorphic to its double dual V^{**}.*

Proof. Let $\mathsf{F} \colon \mathsf{Vect} \to \mathsf{Vect}$ denote the functor

$$\mathsf{F}(V) := V^{**} = \operatorname{Hom}(\mathsf{Hom}(V, \mathbb{R}), \mathbb{R}).$$

If $A\ell \colon V \to W$ is a linear map then $\mathsf{F}(\ell) \colon \mathsf{F}(V) \to \mathsf{F}(W)$ is the linear map usually written as $\ell^{**} \colon V^{**} \to W^{**}$ and defined by

$$\ell^{**}(\varphi)(\eta) = \varphi(\eta \circ \ell) \qquad \varphi \in V^{**}, \eta \in W^*.$$

Let $\mathsf{ev}_V \colon V \to V^{**}$ denote the map

$$\mathsf{ev}_V(v)(\eta) := \eta(v), \qquad \eta \in V^*.$$

We claim that ev is a natural isomorphism from the identity functor to F. This comes down to showing that the following diagram commutes for any pair of vector spaces V, W and any linear map $\ell \colon V \to W$:

$$\begin{array}{ccc} V & \xrightarrow{\mathsf{ev}_V} & V^{**} \\ {\scriptstyle \ell}\downarrow & & \downarrow{\scriptstyle \ell^{**}} \\ W & \xrightarrow[\mathrm{ev_W}]{} & W^{**} \end{array}$$

This is trivial: if $\eta \in W^*$ and observe:

$$\begin{aligned} \ell^{**}\mathsf{ev}_V(v)(\eta) &= \mathsf{ev}_V(v)(\eta \circ \ell) \\ &= \eta(\ell v) \\ &= \mathsf{ev}_{\ell v}(\eta). \end{aligned}$$

The proof is complete. ∎

REMARK 19.63. If we work on the larger category $\mathsf{Vect}^{\leq\infty}$ of all (not necessarily finite-dimensional) vector spaces, ev is still a natural transformation, but no longer a natural isomorphism.

Theorem 19.60 admits the following enhancement.

THEOREM 19.64. *Let F_1 and F_2 be two functors as in the statement of Theorem 19.60. Assume there exists a smooth natural isomorphism $\tau\colon \mathsf{F}_1 \to \mathsf{F}_2$. Then the vector bundles obtained by applying Theorem 19.60 to F_1 and F_2 are naturally isomorphic.*

Corollary 19.3 and Corollary 19.16 are special cases of Theorem 19.64.

LECTURE 20

Sections of Vector Bundles

A fibre bundle $\pi\colon E \to M$ is a surjective submersion between manifolds with the property that the domain E has extra structure. Smooth maps that go in the opposite direction are—from the point of view of fibre bundles—uninteresting unless they respect this extra structure.

Definition 20.1. Let $L \to E \xrightarrow{\pi} M$ be a fibre bundle. A **section** of E is a smooth map $s\colon M \to E$ such that $\pi \circ s = \mathrm{id}$, that is, a smooth map $s\colon M \to E$ such that

We should really say "smooth section", but since we will never consider non-smooth sections, we omit the adjective.

$$s(p) \in E_p, \qquad \forall\, p \in M. \tag{20.1}$$

The set of all sections is denoted by $\Gamma(E)$. A **local section** of E is a section of the bundle $\pi^{-1}(U) \to U$ of E over an open set $U \subset M$. We denote by $\Gamma(U, E)$ the set of all local sections with domain U.

Examples 20.2. Here are some examples of sections:

(i) Let M be a manifold. A vector field X on M is a section of the tangent bundle. Thus

$$\mathfrak{X}(M) = \Gamma(TM).$$

Similarly a vector field X defined on an open subset of M is a local section:

$$\mathfrak{X}(U) = \Gamma(U, TM).$$

In particular, if (U, x) is a chart on M with local coordinates (x^i) then $\frac{\partial}{\partial x^i}$ is an element of $\Gamma(U, TM)$.

(ii) In a similar vein, if $f \in C^\infty(M)$ then in Example 5.2 we defined a section df of T^*M. If $f \in C^\infty(U)$ then $df \in \Gamma(U, T^*M)$.

(iii) A section of the trivial fibre bundle $M \times L \to M$ is the same thing as a smooth map $M \to L$. Thus for instance, a section of $M \times \mathbb{R} \to M$ is just a smooth function on M.

(iv) Let $\pi_1\colon E \to M$ and $\pi_2\colon F \to M$ denote two vector bundles, and consider the bundle vector bundle $\mathrm{Hom}(E, F)$. A section $\Phi \in \Gamma(\mathrm{Hom}(E, F))$ is a smooth map $p \mapsto \Phi_p$ where $\Phi_p\colon E_p \to F_p$ is a linear map. Thus:

$$\Gamma(\mathrm{Hom}(E, F)) = \{\text{vector bundle homomorphisms } \Phi\colon E \to F\}.$$

Local sections always exist:

Lemma 20.3. *Let $L \to E \xrightarrow{\pi} M$ be a fibre bundle and let $p \in M$. Then there exists a neighbourhood U of p and a local section $s \in \Gamma(U, E)$.*

Proof. The map π is a surjective submersion by Lemma 16.4. Now apply Proposition 6.13. ∎

The existence of a global section is sometimes not automatic:

PROPOSITION 20.4. *Let M be a smooth manifold.*

(i) *Let $\pi\colon E \to M$ be a vector bundle. Then $\Gamma(E) \neq \emptyset$.*

(ii) *Let $\pi\colon P \to M$ be a principal G-bundle. Then $\Gamma(P) \neq \emptyset$ if and only if $P = M \times G$ is trivial.*

Proof. We prove the two cases separately.

- (i): The map
$$o\colon M \to E, \qquad p \mapsto 0 \in E_p$$
is a global smooth section. We call o the **zero section**.

- (ii): If $P = M \times G$ is the trivial bundle, then for any $g \in G$ the map $s(p) := (p, g)$ is a section. Conversely, let τ denote the free right action and suppose $s\colon M \to P$ is a section. Then since p and $s(\pi(p))$ belong to the same fibre for each $p \in P$, there is a well-defined equivariant map $\varepsilon\colon P \to G$ such that
$$p = \tau_{\varepsilon(p)}(s(\pi(p))), \qquad \forall\, p \in P.$$
We claim that ε is a principal bundle chart, whence P is a trivial bundle. For this we need to prove that $(\pi, \varepsilon)\colon P \to M \times G$ is a diffeomorphism. But this follows from Lemma 17.14, since (π, ε) is a principal bundle morphism along the identity map on M. ∎

For vector bundles we can do more than just ask for a single local section.

DEFINITION 20.5. Let $\pi\colon E \to M$ be a vector bundle of rank n and let $U \subset M$ be open. A **local frame** for E over U is a collection $(e_1, \dots, e_n)$ of sections $e_i \in \Gamma(U, E)$ such that $\{e_1(p), \dots, e_n(p)\}$ form a basis of the vector space E_p for each $p \in U$.

Here are three equivalent ways to think of local frames:

LEMMA 20.6. *Let $\pi\colon E \to M$ be a vector bundle and suppose $U \subset M$ is an open set. The following are equivalent.*

(i) *There exists a local frame for E over U.*

(ii) *There exists a vector bundle chart $\varepsilon\colon \pi^{-1}(U) \to \mathbb{R}^n$.*

(iii) *There exists a section of the frame bundle over U: $\Gamma(U, \mathrm{Fr}(E)) \neq \emptyset$.*

Proof. Suppose (e_i) is a local frame over U. Then every point $v \in \pi^{-1}(U)$ can be written as uniquely as linear combination $v = a^i\, e_i(p)$. We define
$$\varepsilon\colon \pi^{-1}(U) \to \mathbb{R}^n, \qquad v \mapsto (a^1, \dots, a^n). \tag{20.2}$$
This is a vector bundle chart. This shows that (i) ⇒ (ii).

Conversely if $\varepsilon\colon \pi^{-1}(U) \to \mathbb{R}^n$ is a vector bundle chart then if we define

$$e_i(p) := \varepsilon_p^{-1}(e_i),$$

where e_i is the standard basis vector in $\mathbb{R}^n$, then e_i is smooth (use the argument from the proof of Lemma 20.9) and the collection $\{e_i(p)\}$ is a basis of E_p since ε_p is a linear isomorphism. This shows (ii) $\Rightarrow$ (i).

A section s of the frame bundle $\mathrm{Fr}(E)$ over U is by definition a smooth map $s\colon U \to \mathrm{Fr}(E)$ such that $s(p) \in \mathrm{Fr}(E_p)$, that is, $s(p)$ is a linear isomorphism $\mathbb{R}^n \to E_p$. Define $e_i \in \Gamma(U, E)$ by $e_i(p) := s(p)e_i$. Then (e_i) is a local frame for E over U. Conversely starting with (e_i) and defining s by the same equation produces a local section of $\mathrm{Fr}(E)$ over U. This shows (i) $\Leftrightarrow$ (iii). ■

A **global frame** of a vector bundle is a frame defined on $U = M$. The next statement is the generalisation to arbitrary vector bundles of of Problem F.1.

Corollary 20.7. *A vector bundle $\pi\colon E \to M$ admits a global frame if and only if it is trivial.*

Lemma 20.8. *Let $\pi\colon E \to M$ be a fibre bundle and let $s \in \Gamma(U, E)$. Then $s(U)$ is an embedded submanifold of E of dimension equal to the dimension of M.*

Proof. If (V, x) is a chart on M with $V \subset U$ then $x \circ \pi$ is a chart on $s(V)$. ■

Applying this to the zero section allows us of a vector bundle allows us to see $M \cong o(M)$ as an embedded submanifold of E.

The space of sections of a vector bundle has extra structure not present in normal fibre bundles. We already saw this for vector fields in Lecture 8, but let us go over it again here.

Lemma 20.9. *Let $\pi\colon E \to M$ be a vector bundle. Then for any non-empty open set $U \subset M$, the set $\Gamma(U, E)$ is an infinite-dimensional real vector space and a module over the ring $C^\infty(U)$.*

Here and elsewhere one should implicitly assume that all vector bundles have strictly positive rank.

Proof. Suppose $s \in \Gamma(U, E)$. Let $x\colon V \to \mathcal{O}$ be a chart on $V \subset U$ and let ε be a vector bundle chart on E defined on $\pi^{-1}(V)$. Then as in Remark 16.5, we may take $(x \circ \pi, \varepsilon)$ as a chart on E. The assumption that s is smooth means that the composition

$$(x \circ \pi, \varepsilon) \circ s \circ x^{-1}\colon \mathcal{O} \to \mathcal{O} \times \mathbb{R}^n$$

is smooth. Moreover the section property tells us that this local map is of the form

$$(x \circ \pi, \varepsilon) \circ s \circ x^{-1} = (\mathrm{id}, \tilde{s}) \tag{20.3}$$

where $\tilde{s}\colon \mathcal{O} \to \mathbb{R}^n$ is some smooth map. Just as in the proof of Proposition 8.2, this argument reverses, and we see that a map s satisfying the section property is smooth if and only if each local map $\tilde{s}$ is smooth.

With this in hand the lemma is trivial: if r and s are two sections and $c \in \mathbb{R}$ then $p \mapsto cr(p) + s(p)$ certainly satisfies the section property (20.1), and its local expression is given by $c\tilde{r} + \tilde{s}$ which is smooth if $\tilde{r}$ and $\tilde{s}$ are. Moreover if $f \in C^\infty(U)$ then we define

$$(fs)(p) := f(p)s(p), \qquad p \in U.$$

The local expression of fs is $\tilde{f}\tilde{s}$ where $\tilde{f} = f \circ x^{-1}$. This is smooth.

Finally, the vector space $C^\infty(U)$ is infinite-dimensional by Lemma 2.13, and hence any non-zero module over it is *a fortiori* infinite-dimensional over $\mathbb{R}$. ■

The analogue of part *(iii)* of Proposition 8.2 also holds for general vector bundles.

Remark 20.10. If (e_i) is a local frame for E over U then any (not necessarily smooth) map $s\colon U \to E$ satisfying the section property (20.1) can be written as

$$s = f^i e_i, \qquad \text{for some functions } f^i\colon U \to \mathbb{R}.$$

If we take the vector bundle chart ε on E from (20.2) associated to the local frame (e_i) then for any chart x on M with appropriate domain, the function $\tilde{s}$ associated to s from (20.3) is given by

$$\tilde{s}(q) = \big(f^1(x^{-1}(q)), \dots, f^n(x^{-1}(q))\big).$$

This tells us that s is smooth (and hence belongs to $\Gamma(U, E)$) if and only if the functions a^i are smooth functions on U.

The next lemma is analogous to Problem D.2 and is thus left as an exercise.

Lemma 20.11. *Let $\pi\colon E \to M$ be a vector bundle. Let $p \in M$ and $v \in E_p$. Then there exists a section $s \in \Gamma(E)$ with $s(p) = v$.*

An application of the Bump Functions Lemma 3.2 gives us the following result.

Lemma 20.12. *Let $\pi\colon E \to M$ be a vector bundle and let $s \in \Gamma(U, E)$. Fix $p \in U$. Then there exists a global section $r \in \Gamma(E)$ such that r agrees with s on a neighbourhood of p.*

Proof. Choose a neighbourhood V of p with $\overline{V} \subset U$. Choose a bump function $\chi\colon M \to \mathbb{R}$ such that $\chi(q) = 1$ for all $q \in V$ and such that $\operatorname{supp}(\chi) \subset U$. Define $r\colon M \to E$ by

$$r(q) := \begin{cases} \chi(q)s(q), & q \in U, \\ 0, & q \in M \setminus U. \end{cases}$$

Then r is smooth and agrees with s on the neighbourhood V of p. ■

Definition 20.13. A local frame (e_i) of E over U determines a local frame (e^i) of the dual bundle E^* over U by requiring that

Exercise: Why is this smooth?

$$e^i(p)(e_j(p)) = \delta^i_j, \qquad \text{for all } p \in U.$$

We call (e^i) the **dual frame** to (e_i).

Convention. If $\lambda \in \Gamma(U, E^*)$ is a section of a *dual* bundle, we normally write λ_p instead of $\lambda(p)$ for the value of λ at p. Thus $\lambda_p \colon E_p \to \mathbb{R}$ is a linear map.

This is consistent with our notation df_p for the differential of f at p, i.e. the value of the section $df \in \Gamma(T^*M)$ at p.

Remark 20.14. If $s \in \Gamma(U, E)$ then if we write $s = f^i e_i$ for smooth functions f^i as per Remark 20.10 then observe that

$$e^i_p(s(p)) = f^i(p).$$

Similarly if $\lambda \in \Gamma(U, E^*)$ is any section of the dual bundle then we can write $\lambda = g_i e^i$ where the function $g_i \in C^\infty(U)$ are given by

$$g_i(p) = \lambda_p(e_i(p)).$$

Example 20.15. Let M be a smooth manifold, and let (U, x) be a chart on M with local coordinates (x^i). Then

$$\left\{ \frac{\partial}{\partial x^i} \mid i = 1, \dots, m \right\}$$

is a local frame for TM over U. Similarly

$$\{dx^i \mid i = 1, \dots, m\}$$

is a local frame for T^*M over U. This is the dual frame. Taking this one step further,

$$\left\{ \frac{\partial}{\partial x^i} \otimes dx^j \mid 1 \le i, j \le m \right\}$$

is a local frame for $TM \otimes T^*M$ over U.

We now introduce two key properties that a linear operator between spaces of sections may or may not have.

Definition 20.16. Let $\pi_1 \colon E \to M$ and $\pi_2 \colon F \to M$ be two vector bundles over the same manifold M. Suppose $\zeta \colon \Gamma(E) \to \Gamma(F)$ is an $\mathbb{R}$-linear operator.

- We say that ζ is a **local operator** if whenever a section $s \in \Gamma(E)$ vanishes on an open set $U \subset M$, $\zeta(s) \in \Gamma(F)$ also vanishes on U.
- We call ζ a **point operator** if whenever a section $s \in \Gamma(E)$ vanishes at a point p, $\zeta(s)$ also vanishes at p.

Any point operator is clearly a local operator, but the converse is not true.

Example 20.17. By part (iii) of Example 20.2, the space $C^\infty(\mathbb{R})$ can be identified with the space of all sections of the trivial bundle $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Differentiation

$$C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \qquad \gamma \mapsto \gamma'$$

is a local operator (since if γ is constant on an open set its derivative is also constant on that open set) but it is not a point operator.

More generally:

Example 20.18. Let M be a smooth manifold, and let $X \in \mathfrak{X}(M)$ denote a vector field. Regard X as a derivation of $C^\infty(M)$ as in Proposition 8.7, or equivalently, as a linear operator $\Gamma(M \times \mathbb{R}) \to \Gamma(M \times \mathbb{R})$ (as in part (iii) of Examples 20.1). Then $f \mapsto X(f)$ is a local operator by Corollary 3.4, but not a point operator.

Local operators behave well under restriction.

Proposition 20.19. *Let $\pi_1 \colon E \to M$ and $\pi_2 \colon F \to M$ be two vector bundles and suppose $\zeta \colon \Gamma(E) \to \Gamma(F)$ is a local operator. Then for each open set $U \subset M$, there is a unique $\mathbb{R}$-linear map $\zeta^U \colon \Gamma(U, E) \to \Gamma(U, F)$, called the* **restriction** *of ζ to U, such that for any global section s, one has*

$$\zeta^U(s|_U) = \zeta(s)|_U. \tag{20.4}$$

Proof. Let $s \in \Gamma(U, E)$ and fix $p \in U$. By Lemma 20.12 there exists a global section r of E that agrees with s in some neighbourhood V of p. We set

$$\zeta^U(s)(p) := \zeta(r)(p).$$

This is well-defined, i.e. independent of the choice of r, since ζ is a local operator. Since $\zeta(r)$ is smooth by assumption, it follows $\zeta^U(s)$ is smooth at p, and since p was arbitrary, $\zeta^U(s)$ is smooth. Finally, if s is a global section then s is an extension of $s|_U$ for any open U, and thus (20.4) follows. ∎

As the proof of Proposition 20.19 shows, the operator ζ^U is itself a local operator.

So far we have looked at operators which are $\mathbb{R}$-linear. Since $\Gamma(E)$ and $\Gamma(F)$ are modules over $C^\infty(M)$, we could instead restrict our attention to operators that are $C^\infty(M)$-linear, i.e.

Note $C^\infty(M)$-linear implies $\mathbb{R}$-linear, since we may regard $\mathbb{R} \subset C^\infty(M)$ via the constant functions.

$$\zeta(fs) = f\zeta(s), \qquad \forall\, f \in C^\infty(M),\ s \in \Gamma(E).$$

Actually this is the same thing as working with point operators, as the next result shows.

Theorem 20.20. *Let $\pi_1 \colon E \to M$ and $\pi_2 \colon F \to M$ be two vector bundles over M. An $\mathbb{R}$-linear operator $\zeta \colon \Gamma(E) \to \Gamma(F)$ is a point operator if and only if ζ is $C^\infty(M)$-linear.*

Proof. We prove the result in four steps.

1. In this first step we show that a $C^\infty(M)$-linear operator $\zeta \colon \Gamma(E) \to \Gamma(F)$ is a local operator. Suppose $s \in \Gamma(E)$ vanishes on an open set U. Let $p \in U$, and choose a bump function $\chi \colon M \to \mathbb{R}$ such that $\chi(p) = 1$ and $\operatorname{supp}(\chi) \subset U$. Then χs is identically zero on M, and so $\zeta(\chi s)$ is identically zero. However evaluating at p and using $C^\infty(M)$-linearity,

$$0 = \zeta(\chi s)(p) = \chi(p)\zeta(s)(p) = \zeta(s)(p).$$

Since p was an arbitrary point of U, we have $\zeta(s)|_U \equiv 0$ as required.

2. Since ζ is a local operator, by Proposition 20.19 the local operators ζ^U are defined. In this step we show that such a local operator ζ^U is itself $C^\infty(U)$-linear.

Fix $s \in \Gamma(U,E)$, $f \in C^\infty(U)$. We want to show that $\zeta^U(fs) = f\zeta^U(s)$. Fix $p \in U$ and let $r \in \Gamma(E)$ denote a global section that agrees with s on a neighbourhood of p, and let g be a global smooth function that agrees with f on a neighbourhood of p. Then

The existence of g is a special case of Lemma 20.12, cf. part (iii) of Examples 20.2, but it was also proved directly in Step 2 or Proposition 3.3.

$$\begin{aligned}\zeta^U(fs)(p) &= \zeta(gr)(p)\\ &= g(p)\zeta(r)(p)\\ &= f(p)\zeta^U(s)(p).\end{aligned}$$

Since p was arbitrary, we see that $\zeta^U(fs) = f\zeta^U(s)$, as required.

3. We now show that ζ is actually a point operator. Let $s \in \Gamma(E)$. Suppose $s(p) = 0$. Choose an neighbourhood U of p admitting a local frame (e_i). Then we can write

$$s|_U = f^i e_i, \qquad f^i \in C^\infty(U).$$

Since $s(p) = 0$ we have $f^i(p) = 0$ for each i. We now compute:

$$\begin{aligned}\zeta(s)(p) &= \zeta^U(s|_U)(p)\\ &= \zeta^U(f^i e_i)(p)\\ &= f^i(p)\zeta^U(e_i)(p)\\ &= 0,\end{aligned}$$

where the first equality used Proposition 20.19 and the penultimate equality used the previous step.

4. Finally we prove the converse: suppose $\zeta\colon \Gamma(E) \to \Gamma(F)$ is a point operator. Fix $f \in C^\infty(M)$, $s \in \Gamma(E)$ and $p \in M$. Let $c := f(p)$. Then $fs - cs$ vanishes at p, and thus $\zeta(fs - cs)(p) = 0$ as ζ is a point operator. Since ζ is $\mathbb{R}$-linear,

$$\begin{aligned}\zeta(fs)(p) &= \zeta(cs)(p)\\ &= c\zeta(s)(p)\\ &= f(p)\zeta(s)(p).\end{aligned}$$

Since p was arbitrary, $\zeta(fs) = f\zeta(s)$. This completes the proof. ■

Let us now return to part (iv) of Examples 20.2: a vector bundle homomorphism $\Phi\colon E \to F$ is the same thing as a section of the homomorphism bundle $\mathrm{Hom}(E,F)$. The aim of the rest of this lecture is to give yet another alternative description of a vector bundle homomorphism.

Definition 20.21. Let $\pi_1\colon E \to M$ and $\pi_2\colon F \to M$ denote two vector bundles over the same manifold M. Let $\Phi\colon E \to F$ denote a vector bundle homomorphism. We define an operator

$$\Phi_*\colon \Gamma(E) \to \Gamma(F), \qquad s \mapsto \Phi \circ s.$$

Proposition 20.22. *Let $\pi_1\colon E \to M$ and $\pi_2\colon F \to M$ denote two vector bundles over the same manifold M. Let $\Phi\colon E \to F$ denote a vector bundle homomorphism. Then $\Phi_*\colon \Gamma(E) \to \Gamma(F)$ is $C^\infty(M)$-linear, and hence a point operator.*

Proof. The map Φ_* is clearly a linear map between the two vector spaces $\Gamma(E)$ and $\Gamma(F)$. More is true: Φ_* is actually a *module homomorphism*, i.e. it is linear over $C^\infty(M)$. Indeed, if $f \in C^\infty(M)$, $s \in \Gamma(E)$, and $p \in M$ then

$$\begin{aligned} \Phi_*(fs)(p) &= \Phi \circ (fs)(p) \\ &= \Phi|_p(f(p)s(p)) \\ &= f(p)\Phi_p(s(p)) \\ &= (f\Phi_*(s))(p), \end{aligned}$$

where the penultimate equality used that Φ_p is a linear map. ■

The main result of today's lecture, the Hom-Gamma Theorem 20.25, states that every point operator $\Gamma(E) \to \Gamma(F)$ is a of the form Φ_*. This requires some more preparation.

As we have seen in Example 20.18, a vector field on a manifold can be thought of an operator on the space of sections of the trivial bundle $M \times \mathbb{R}$ via $f \mapsto X(f)$. The next result generalises part (ii) of Proposition 8.2 to arbitrary vector bundles.

PROPOSITION 20.23. *Let $\pi_1 \colon E \to M$ and $\pi_2 \colon F \to M$ be two vector bundles. Suppose $\Phi \colon E \to F$ is a fibre-preserving map such that $\Phi_p \colon E_p \to F_p$ is linear for every $p \in M$. Then Φ is smooth (and hence a vector bundle homomorphism) if and only if $\Phi_*(s) := \Phi \circ s$ belongs to $\Gamma(U, F)$ for every $s \in \Gamma(U, E)$.*

Proof. If Φ is smooth then certainly $\Phi \circ s$ is smooth. For the converse, let $p \in M$ and suppose (U, x) is a chart on M with local coordinates (x^i). We may assume that both E and F admit local frames over U; call them (e_j) and (e'_i) respectively. Since Φ_* maps smooth sections to smooth sections, there are functions $f^i_j \in C^\infty(U)$ such that

$$\Phi_*(e_j) = f^i_j \, e'_i.$$

Let ε_1 and ε_2 denote the vector bundle charts on E and F respectively associated to (e_j) and (e'_i) from part (ii) of Lemma 20.6. Then $(x \circ \pi_1, \varepsilon)$ is a manifold chart on E on $\pi_1^{-1}(U)$, and $(x \circ \pi_2, \varepsilon_2)$ is a manifold chart on F on $\pi_2^{-1}(U)$. Let $a^j \in C^\infty(\pi_1^{-1}(U))$ denote the smooth functions defined implicitly by the requirement

$$v = a^j(v) e_j(\pi_1(v)), \qquad \forall\, v \in \pi_1^{-1}(U).$$

Then the local expression of Φ is of the form:

$$(x \circ \pi_2, \varepsilon_2) \circ \Phi \circ (x \circ \pi_1, \varepsilon_1)^{-1} = \big(\mathrm{id}, (a^j f^1_j, \dots, a^j f^n_j) \circ x^{-1}\big),$$

which is smooth. ■

PROPOSITION 20.24. *Let $\pi_1 \colon E \to M$ and $\pi_2 \colon F \to M$ be two vector bundles. Suppose $\zeta \colon \Gamma(E) \to \Gamma(F)$ is a $C^\infty(M)$-linear map. Then for each $p \in M$ there is a unique linear map $\Phi_p \colon E_p \to F_p$ such that for all $s \in \Gamma(E)$, one has*

$$\Phi_p(s(p)) = \zeta(s)(p).$$

Proof. Fix $p \in M$ and $v \in E_p$. By Lemma 20.11 there exists a section s such that $s(p) = v$. Define $\Phi_p(v) := \zeta(s)(p)$. This definition is independent of the choice of s, since if s_1 was another such section then $(s - s_1)(p) = 0$, and thus $\zeta(s)(p) - \zeta(s_1)(p) = \zeta(s - s_1)(p) = 0$ since ζ is a point operator by Theorem 20.20.

We claim that Φ_p is a linear map. Indeed, if $v_1, v_2 \in E_p$ and $c \in \mathbb{R}$, then if s_1 and s_2 are sections such that $s_i(p) = v_i$ then $cs_1 + s_2$ is a section whose value at p is $cv_1 + v_2$ and

$$\begin{aligned}\Phi_p(cv_1 + v_2) &= \zeta(cs_1 + s_2)(p) \\ &= c\zeta(s_1)(p) + \zeta(s_2)(p) \\ &= c\Phi_p(v_1) + \Phi_p(v_2).\end{aligned}$$

This completes the proof. ■

We now move onto our main result.

Theorem 20.25 (The Hom-Gamma Theorem). *Let $\pi_1 \colon E \to M$ and $\pi_2 \colon F \to M$ be two vector bundles. Then there is a one-to-one correspondence between*

$$\{\text{vector bundle homomorphisms } \Phi \colon E \to F\}$$

and

$$\{\text{point operators } \zeta \colon \Gamma(E) \to \Gamma(F)\}$$

given by

$$\Phi \mapsto \Phi_*.$$

The reason for the name will be explained after the proof.

Proof. We first prove surjectivity. If $\zeta \colon \Gamma(E) \to \Gamma(F)$ is a $C^\infty(M)$-linear map then by Proposition 20.24 there exists a linear map $\Phi_p \colon E_p \to F_p$ such that for any $s \in \Gamma(E)$, $\Phi_p(s(p)) = \zeta(s)(p)$. Define $\Phi \colon E \to F$ by declaring that $\Phi|_{E_p} = \Phi_p$. Then by Proposition 20.23, the map Φ is a vector bundle homomorphism, and clearly $\Phi_* = \zeta$.

To prove injectivity, suppose $\Phi_* = \Psi_*$. Let $p \in M$ and $v \in E_p$ and let $s \in \Gamma(E)$ be a section such that $s(p) = v$ (using Lemma 20.11). Then

$$\begin{aligned}\Phi(v) &= \Phi(s(p)) \\ &= \Phi_*(s)(p) \\ &= \Psi_*(s)(p) \\ &= \Psi(s(p)) \\ &= \Psi(v).\end{aligned}$$

Since p and v were arbitrary we conclude $\Phi = \Psi$ as required. ■

Remark 20.26. Why the name "Hom-Gamma Theorem"? Suppose R is a commutative ring and X, Y are two R-modules. Let us write

$$\mathrm{Hom}(X, Y) = \{f \colon X \to Y \text{ is an } R\text{-module homomorphism}\}.$$

In the case $R = \mathbb{R}$, modules are just vector spaces, and this coincides with our notation for $\mathrm{Hom}(V, W)$ as the set of linear maps from V to W. We now apply this with $R = C^\infty(M)$. By Theorem 20.20, an element

$$\zeta \in \mathrm{Hom}\big(\Gamma(E), \Gamma(F)\big)$$

is the same thing as a point operator. Thus the Hom-Gamma Theorem 20.25 tells us that

$$\Gamma\big(\mathrm{Hom}(E, F)\big) \cong \mathrm{Hom}\big(\Gamma(E), \Gamma(F)\big), \tag{20.5}$$

in other words,

The operations Hom *and* Γ *commute.*

Bonus Material for Lecture 20

In today's bonus section, we make another algebraic interlude and introduce the notion of a **sheaf**. As with the category theory material from the previous lecture, this is for interest only, and is not needed to understand any of the subsequent lectures.

Roughly speaking, a presheaf is a way to assign data locally to open subsets of a topological space in such a way that it is compatible with restrictions. A sheaf is a presheaf for which it is possible to go backwards and reassemble global data from local data.

Definition 20.27. Let X denote a topological space. A **presheaf** $\mathcal{F}$ **of sets on** X consists of:

(i) A set $\mathcal{F}(U)$ for every open set $U \subset X$.

(ii) For every pair $U \subset V$ of open sets a map $\mathrm{res}^V_U \colon \mathcal{F}(V) \to \mathcal{F}(U)$ called the **restriction map** such that $\mathrm{res}^U_U = \mathrm{id}_{\mathcal{F}(U)}$ for every U and such that

$$\mathrm{res}^W_U = \mathrm{res}^V_U \circ \mathrm{res}^W_V, \qquad \text{whenever } U \subset V \subset W.$$

Definition 20.28. Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves on X. A **morphism of presheaves** $\zeta \colon \mathcal{F} \to \mathcal{G}$ is a family of maps $\zeta^U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ such that for every pair of open sets $U \subset V$ the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{F}(V) & \xrightarrow{\ \zeta^V\ } & \mathcal{G}(V) \\
{\scriptstyle \mathrm{res}^V_U}\downarrow & & \downarrow{\scriptstyle \mathrm{res}^V_U} \\
\mathcal{F}(U) & \xrightarrow[\ \zeta^U\ ]{} & \mathcal{G}(U)
\end{array}$$

If $\zeta \colon \mathcal{F} \to \mathcal{G}$ and $\xi \colon \mathcal{G} \to \mathcal{H}$ are two morphisms of presheaves over X then their composition $\xi \circ \zeta \colon \mathcal{F} \to \mathcal{H}$ is defined as one would guess:

$$(\xi \circ \zeta)_U := \xi^U \circ \zeta^U.$$

An **isomorphism** is a presheaf morphism such that each ζ^U is a bijection.

This gives us the category $\mathsf{PSh}(X;\mathsf{Sets})$ of presheaves on X whose objects are the presheaves on X and whose morphisms are presheaf morphisms.

Definition 20.29. Let C be an arbitrary category. A **presheaf $\mathcal{F}$ on X with values in C** is defined in almost the same way, only now each $\mathcal{F}(U)$ must be an object of C, each restriction map res^V_U must be a morphism in C, and morphisms between two presheaves must also be morphisms in C.

To give a concrete example, let's take $\mathsf{C} = \mathsf{Vect}$. A presheaf of vector spaces is thus an assignment of a vector space $\mathcal{F}(U)$ for every open set $U \subset X$, and the restriction maps res^V_U must be linear transformations $\mathcal{F}(V) \to \mathcal{F}(U)$. Finally if $\zeta \colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of vector spaces then each ζ^U must be a linear transformation $\mathcal{F}(U) \to \mathcal{G}(U)$. In particular, an isomorphism of presheaves of vector spaces requires each ζ^U to be a linear isomorphism.

Remark 20.30. Here is an alternative more categorical definition of a presheaf. Let $\mathsf{Open}(X)$ denote the category whose objects are the open sets of X and, for two open sets U, V, the morphism space $\mathrm{Hom}(U,V)$ consists of the inclusion map $U \hookrightarrow V$ if $U \subset V$ and is empty otherwise. Then a presheaf on X with values in C is simply a contravariant functor $\mathsf{Open}(X) \to \mathsf{C}$. A morphism $\zeta \colon \mathcal{F} \to \mathcal{G}$ is a *natural transformation* ζ between the two functors.

If $\mathcal{F}$ is a presheaf on X and $s \in \mathcal{F}(V)$ then for $U \subset V$ we normally abbreviate

$$s|_U := \mathrm{res}^V_U(s)$$

This fits in with the idea that we are "restricting" s to U. In fact, every single presheaf we will care about in the course will be a presheaf of *functions*, which we now define, and in this case restriction really *is* restriction.

Definition 20.31. Let X be a topological space and let S be a fixed set. A **presheaf of S-valued functions** is a presheaf with the property that $\mathcal{F}(U) \subset \mathrm{Maps}(U,S)$ for all open sets $U \subset X$, where $\mathrm{Maps}(U,S)$ denotes the set of all functions from U to S (i.e. the morphism set in category Sets).

Definition 20.32. Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves on X. We say that $\mathcal{F}$ is a **subpresheaf** of $\mathcal{G}$ if for every open set $U \subset X$, $\mathcal{F}(U) \subset \mathcal{G}(U)$, and for all $U \subset V$ open sets the restriction maps $\mathcal{F}(V) \to \mathcal{F}(U)$ are induced by the restriction maps $\mathcal{G}(V) \to \mathcal{G}(U)$.

Thus if $\mathcal{F}$ is any presheaf of S-valued functions on X then $\mathcal{F}$ is a subpresheaf of the presheaf of *all* S-valued functions on X.

Examples 20.33. Let us see some standard examples of presheaves that will be relevant to this course.

(i) Let X be a topological space and take $S = \mathbb{R}$. Let $\mathcal{C}_X$ denote the presheaf that assigns to an open set $U \subset X$ the set of continuous real-valued functions on X:

$$\mathcal{C}_X(U) \coloneqq C(U, \mathbb{R}) = \{f : U \to \mathbb{R} \text{ continuous}\}.$$

$\mathcal{C}_X$ is not just a presheaf of sets, but a presheaf of $\mathbb{R}$-algebras (and thus also a presheaf of rings and (infinite-dimensional) vector spaces).

(ii) We can also consider differentiable functions. Take $X = \mathbb{R}$ and let $\mathcal{F}(U) = C^\infty(U)$ denote the set of all smooth functions $U \to \mathbb{R}$. This is a subpresheaf of $\mathcal{C}_\mathbb{R}$. We can think of differentiation as a morphism $D : \mathcal{F} \to \mathcal{F}$. This is a morphism of presheaves of vector spaces, since

$$D(af + bg) = af' + bg' = aD(f) + bD(g)$$

but it is *not* a morphism of presheaves of algebras, since

$$D(fg) = fg' + f'g \neq D(f)D(g).$$

(iii) More generally, let M be a smooth manifold. Then the assignment $U \mapsto C^\infty(U)$ is a presheaf of $\mathbb{R}$-algebras on M. As before, differentiation is a morphism of presheaves of vector spaces, but not of algebras. We normally denote this presheaf by $\mathcal{C}^\infty_M$.

(iv) Let $\pi : E \to M$ be a vector bundle. Then $U \mapsto \Gamma(U, E)$ is a presheaf of (infinite-dimensional) vector spaces on M. It is not a presheaf of algebras, since in general there is no way to multiply two sections together. We usually denote this presheaf by $\mathcal{E}_E$.

Unless E is an algebra bundle, cf. Definition 19.29.

(v) Let X be any topological space and let S be any set. Let $\mathcal{F}(U)$ denote the set of all *constant* functions $U \to S$. Since a constant function $f : U \to S$ can be identified with its image $s \coloneqq f(U)$, one can simply think of $\mathcal{F}(U)$ as being equal to S itself. In this case, all restriction maps are the identity map id_S. We call this the **constant presheaf on X with values in S**.

Let us now introduce a sheaf, which is a presheaf with an additional property.

DEFINITION 20.34. Let $\mathcal{F}$ be a presheaf on X (of sets, rings, groups, etc.). We say that $\mathcal{F}$ is a **sheaf** if the following condition is satisfied: for any open set $U \subset X$ and any open cover $\{U_a \mid a \in A\}$ of U, if we are given a collection $s_a \in \mathcal{F}(U_\mathsf{a})$ such that

$$s_a|_{U_a \cap U_b} = s_b|_{U_a \cap U_b}, \qquad \forall\, a, b \in A \text{ such that } U_a \cap U_b \neq \emptyset, \tag{20.6}$$

then there exists a *unique* $s \in \mathcal{F}(U)$ such that $s|_{U_a} = s_a$ for all $a \in A$.

REMARK 20.35. Taking $U = \emptyset$ and choosing the covering with empty index set $A = \emptyset$ shows that if $\mathcal{F}$ is a sheaf then $\mathcal{F}(\emptyset)$ is a set consisting of one element.

A morphism $\zeta\colon \mathcal{F} \to \mathcal{G}$ of sheaves is simply a morphism of the underlying presheaves, and we denote by $\mathsf{Sh}(X; \mathsf{C})$ the category of sheaves on X with values in C.

Remark 20.36. If we start with a presheaf of functions, as in Definition 20.31, the condition (20.6) can be phrased in a slightly simpler fashion: if $\mathcal{F}$ is a presheaf of S-valued functions on X then $\mathcal{F}$ is a sheaf if and only if for any open set $U \subset X$ and any open cover $\{U_a \mid a \in A\}$ of U, if $f\colon U \to S$ is any function such that $f|_{U_a} \in \mathcal{F}(U_a)$ for each $a \in A$, then $f \in \mathcal{F}(U)$.

This reformulation makes it clear that the presheaf $\mathcal{C}_X$ of continuous functions on a topological space is actually a sheaf.

Proposition 20.37. *Let M be a smooth manifold. Then $\mathcal{C}^\infty_M$ is a sheaf. More generally, if $\pi\colon E \to M$ is any vector bundle over M then $\mathcal{E}_E$ is a sheaf.*

Not everything is a sheaf however: the presheaf of constant functions from part (v) of Example 20.33 is not a sheaf if X contains two disjoint non-empty open subsets and S has more than one element.

Exercise: Why?

There is a natural way to turn a presheaf into a sheaf. This procedure is called the **sheafification** of a presheaf. The definition is rather complicated, and for our purposes unimportant (since the relevant presheaves in this course are already sheaves thanks to Proposition 20.37). Thus we will content ourselves with giving the definition only in the special case of a presheaf of functions.

Proposition 20.38. *Let X be a topological space and let S be a set. Suppose $\mathcal{F}$ is a presheaf of S-valued functions on X. Let*

$$\tilde{\mathcal{F}}(U) := \Big\{ f\colon U \to S \mid \text{there exists an open covering } \{U_a \mid a \in A\} \text{ of } U \text{ such that } f|_{U_a} \in \mathcal{F}(U_a) \text{ for all } a \in A. \Big\}$$

Then $\tilde{\mathcal{F}}$ is a sheaf and the inclusion $\mathcal{F}(U) \hookrightarrow \tilde{\mathcal{F}}(U)$ induces a morphism of presheaves $\imath\colon \mathcal{F} \to \tilde{\mathcal{F}}$.

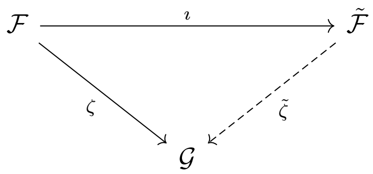
Proof. This is clear from the reformulation of the sheaf condition given in Remark 20.36 – we simply added in all the functions that were needed in order for $\mathcal{F}$ to be a sheaf. ■

Remark 20.39. If $\mathcal{F}$ already was a sheaf, then clearly $\mathcal{F} = \tilde{\mathcal{F}}$. Indeed, in this case all the functions we added were already in $\mathcal{F}$.

Example 20.40. Let $\mathcal{F}$ be the presheaf of constant S-valued functions on X. As we have remarked before, this is typically not a sheaf. However it is very easy to describe the sheaf obtained from $\mathcal{F}$ via Proposition 20.38. Indeed, a little thought shows that the sheaf $\tilde{\mathcal{F}}$ is exactly the *locally constant* functions on S:

$$\tilde{\mathcal{F}}(U) = \{ f\colon U \to S \mid f \text{ is locally constant} \}.$$

Remark 20.41. The sheafification can be defined via a universal property (compare Lemma 19.2): Let $\mathcal{F}$ be a presheaf on X. The sheafification $\tilde{\mathcal{F}}$ and the morphism $\iota\colon \mathcal{F} \to \tilde{\mathcal{F}}$ of presheaves has the property that if $\mathcal{G}$ is any sheaf on X and $\zeta\colon \mathcal{F} \to \mathcal{G}$ is any morphism of presheaves, then there exist a *unique* morphism of sheaves $\zeta\colon \tilde{\mathcal{F}} \to \mathcal{G}$ such that the following diagram commutes:



As such, via abstract nonsense, the sheafification is unique up to isomorphism.

We now move onto discussing the **stalk** of a presheaf. This generalises the notation of a germ of a function that we discussed in Lecture 2.

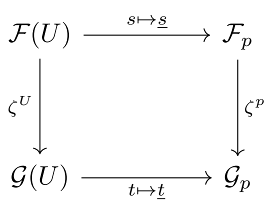
Definition 20.42. Let $\mathcal{F}$ be a presheaf on X, and let $x \in X$. We define the **stalk** of $\mathcal{F}$ at x to be:

$$\mathcal{F}_p \coloneqq \{(U, s) \mid U \text{ is a neighbourhood of } p,\ s \in \mathcal{F}(U)\} \Big/ \sim$$

where $(U, s) \sim (V, t)$ if there exists a neighbourhood $W \subset U \cap V$ such that $s|_W \equiv t|_W$.

Thus for any neighbourhood U of p there exists a canonical map $\mathcal{F}(U) \to \mathcal{F}_p$ that sends s to the equivalence class of (U, s) in $\mathcal{F}_p$, which we denote by $\underline{s}$.

Lemma 20.43. *Let $\zeta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. Then for each $x \in X$ there is a well-defined map $\zeta^p\colon \mathcal{F}_p \to \mathcal{G}_p$ defined as follows: if $\underline{s} \in \mathcal{F}_p$ is represented by (U, s), then we declare that $(U, \zeta^U(s))$ is a representative of $\zeta^p(\underline{s})$. Thus the following diagram commutes:*



Proof. We need only check this is well-defined. Suppose $(U, s) \sim (V, t)$. Then there exists $W \subset U \cap V$ such that $s|_W \equiv t|_W$. Since ζ is a presheaf morphism, one has that

$$\zeta^U(s)|_W = \zeta^W(s|_W) = \zeta^W(t|_W) = \zeta^V(t)|_W.$$

Thus $(U, \zeta^U(s)) \sim (V, \zeta^V(t))$. ∎

Remark 20.44. A more categorical way to define stalks is the following: given $p \in X$, let $\mathsf{Open}_p(X)$ denote the full subcategory of

$\mathsf{Open}(X)$ (cf. Remark 20.30) consisting of neighbourhoods of p. Then if $\mathcal{F}$ is a presheaf on X, one has

$$\mathcal{F}_p = \varinjlim \mathcal{F}(U)$$

where the filtered colimit runs over $\mathsf{Open}_p(X)$. Similarly if $\zeta\colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves then

$$\zeta^p = \varinjlim \zeta^U.$$

If $\mathcal{F}$ is a presheaf of groups, or rings, or modules, etc, then the stalks also inherit that structure. We saw this in the concrete example where $\mathcal{F} = \mathcal{C}^\infty_M$ just after Definition 2.7. As another example, suppose $\mathcal{F}$ is a sheaf of groups. Then $\mathcal{F}_p$ is also a group, where we define the group law as follows: if $\underline{s}$ is represented by (U, s) and $\underline{t}$ is represented by (V, t), then we declare $\underline{s} \cdot \underline{t}$ to be the element represented by $(U \cap V, s|_{U\cap V} \cdot t|_{U\cap V})$.

Remark 20.45. More generally, if C is a category in which filtered colimits exist then for any $p \in X$ there is a functor $\mathsf{PSh}(X; \mathsf{C}) \to \mathsf{C}$ given by $\mathcal{F} \mapsto \mathcal{F}_p$.

Let us now look at some operations on sheaves.

Definition 20.46. Let U be an open set of X. Then if $\mathcal{F}$ is any presheaf on X then we can define a presheaf $\mathcal{F}|_U$ on U by setting $\mathcal{F}|_U(V) \coloneqq \mathcal{F}(V)$ for $V \subset U$ open. If $\mathcal{F}$ is a sheaf then so is $\mathcal{F}|_U$.

Definition 20.47. Let $\varphi\colon X \to Y$ be a continuous map from one topological space to another. Suppose $\mathcal{F}$ is a presheaf on X. We define a presheaf $\varphi_*(\mathcal{F})$ on Y by declaring that

$$\varphi_*(\mathcal{F})(U) \coloneqq \mathcal{F}(\varphi^{-1}(U)), \qquad U \subset Y \text{ open}.$$

We call $\varphi_*(\mathcal{F})$ the **direct image** of $\mathcal{F}$ under φ. If $\zeta\colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves on X then $\varphi_*(\zeta)\colon \varphi_*(\mathcal{F}) \to \varphi_*(\mathcal{G})$ is a morphism on presheaves on Y, where

$$\varphi_*(\zeta)_U \coloneqq \zeta^{\varphi^{-1}(U)}\colon \varphi_*(\mathcal{F})(U) = \mathcal{F}(\varphi^{-1}(U)) \to \mathcal{G}(\varphi^{-1}(U)) = \varphi_*(\mathcal{G})(U).$$

In this way we get a functor from presheaves on X to presheaves on Y. If $\mathcal{F}$ is a sheaf on X then it is clear that $\varphi_*(\mathcal{F})$ is a sheaf on Y.

Definition 20.48. A **continuous ringed space** consists of a pair $(X, \mathcal{F})$ where X is a topological space and $\mathcal{F}$ is a subsheaf of the sheaf $\mathcal{C}_X$ of $\mathbb{R}$-algebras from part (i) of Example 20.33. Explicitly, this means:

- $\mathcal{F}$ is a sheaf and $\mathcal{F}(U) \subset C(U, \mathbb{R})$ for each open set $U \subset X$.
- If $f, g \in \mathcal{F}(U)$ and $a, b \in \mathbb{R}$ then $af + bg$ and fg both belong to $\mathcal{F}(U)$.

Remark 20.49. The name "continuous ringed space" is not quite standard. In algebraic geometry, given a commutative ring R, one studies the more general notion of a **ringed space**, which is defined to be a pair $(X, \mathcal{F})$, where X is a topological space and $\mathcal{F}$ is a sheaf of commutative, associative and unital R-algebras on X. Thus what I call a "continuous ringed space" is the special case where $R = \mathbb{R}$ and $\mathcal{F}$ is a subalgebra of the sheaf of continuous functions on X.

Algebraic geometers often restrict to a special class of ringed spaces, called **locally ringed spaces**, which are ringed spaces $(X, \mathcal{F})$ with the additional property that the stalk $\mathcal{F}_p$ is a *local ring* for every point $p \in X$ (i.e. it has a unique maximal ideal). All continuous ringed spaces in the sense of Definition 20.48 are locally ringed spaces; see Lemma 2.15.

Definition 20.50. Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be two continuous ringed spaces. A **morphism of continuous ringed spaces** is a continuous map $\varphi \colon X \to Y$ with the following property:

$$f \in \mathcal{G}(U) \quad \Rightarrow \quad f \circ \varphi \in \mathcal{F}(\varphi^{-1}(U)), \qquad \text{for all open } U \subset Y. \quad (20.7)$$

Property (20.7) implies there is a well-defined sheaf morphism $\mathcal{G} \to \varphi_*(\mathcal{F})$ given by

$$f \in \mathcal{G}(U) \mapsto f \circ \varphi \in \varphi_*(\mathcal{F})(U).$$

An **isomorphism of continuous ringed spaces** is a homeomorphism φ such that both φ and φ^{-1} are morphisms of continuous ringed spaces.

We will now use the notion of a continuous ringed space to give an equivalent definition of a manifold. This definition is more in the spirit of algebraic geometry, and it has several advantages over the standard one, as we will shortly explain.

Definition 20.51. Let $(M, \mathcal{F})$ be a continuous ringed space. We say $(M, \mathcal{F})$ is a **smooth ringed space of dimension** n if for every point $p \in M$ there exists a neighbourhood U of p and a homeomorphism $x \colon U \to \mathcal{O}$, where $\mathcal{O}$ is some open subset of $\mathbb{R}^n$, such that σ defines an isomorphism of continuous ringed spaces

$$(U, \mathcal{F}|_U) \cong (\mathcal{O}, \mathcal{C}^\infty_{\mathcal{O}}).$$

The next theorem tells us that this really is an alternative way to define a manifold.

Theorem 20.52. *Let M be a smooth manifold of dimension n. Then $(M, \mathcal{C}^\infty_M)$ is a smooth ringed space of dimension n. Conversely, assume that $(M, \mathcal{F})$ is a smooth ringed space, and assume in addition that M is Hausdorff and second countable. Then there exists a unique smooth structure on M such that $\mathcal{F}$ becomes the sheaf $\mathcal{C}^\infty_M$.*

The proof is easy: one direction is clear from the definition of a smooth function on a manifold (Definition 2.1), and for the other direction we (work a bit and then) apply Proposition 1.17.

Remark 20.53. In many ways, starting Lecture 1 by defining a manifold via Definition 20.51 would have been more efficient. Here are some reasons why:

(i) There is no need to worry about equivalence classes of smooth atlases (cf. Remark 1.12).

(ii) The definition of what it means for a continuous map $\varphi\colon (M, \mathcal{C}^\infty_M) \to (N, \mathcal{C}^\infty_N)$ between two smooth manifolds to be smooth is much cleaner: it simply has to be a morphism of continuous ringed spaces.

(iii) The definition of a tangent vector as a derivation on the space of germs (i.e. the stalks of the sheaf $\mathcal{C}^\infty_M$) is far more natural.

(iv) This algebraic approach dramatically reduces the need to use local coordinates, which are messy and irritating.

Nevertheless, the best part of differential geometry is the "geometry", and this algebraic approach deletes most of said geometry. So we will not pursue it beyond this lecture.

We conclude this lecture by giving a sheaf-theoretic definition of a vector bundle. This will also allow us to reinterpret the Hom-Gamma Theorem 20.25 in a more algebraic way. First, some preliminary definitions.

DEFINITION 20.54. Let $(X, \mathcal{F})$ be a continuous ringed space. Let $\mathcal{M}$ be a sheaf of abelian groups on X, and assume in addition that for every open set $U \subset X$, the abelian group $\mathcal{M}(U)$ has the structure of an $\mathcal{F}(U)$-module, and moreover the restriction morphisms respect this structure, ie.

$$\mathrm{res}^V_U(fs) = \mathrm{res}^V_U(f)\,\mathrm{res}^V_U(s), \qquad \forall\, f \in \mathcal{F}(V),\ s \in \mathcal{M}(V).$$

Then we say that $\mathcal{M}$ is a **sheaf of $\mathcal{F}$-modules**. A morphism ζ from one sheaf $\mathcal{M}$ of $\mathcal{F}$-modules to another sheaf $\mathcal{N}$ of $\mathcal{F}$-modules is one such that each map $\zeta^U\colon \mathcal{M}(U) \to \mathcal{N}(U)$ is $\mathcal{F}(U)$-linear. We call such a ζ an **$\mathcal{F}$-morphism** of sheaves.

Here is an example.

EXAMPLE 20.55. Let $\pi\colon E \to M$ be a vector bundle. Then the sheaf $\mathcal{E}_E$ of sections of E is a sheaf of $\mathcal{C}^\infty_M$-modules. Indeed, this is just a fancy way of rephrasing Lemma 20.9.

We can also rephrase some of the results from the previous lecture.

COROLLARY 20.56. *Let $\pi_1\colon E_1 \to M$ and $\pi_2\colon E_2 \to M$, and suppose $\zeta\colon \Gamma(E_1) \to \Gamma(E_2)$ is an $\mathbb{R}$-linear operator. Then ζ is a local operator in the sense of Definition 20.16 if and only if $\zeta = \zeta^M$ for a morphism of sheaves $\zeta\colon \mathcal{E}_{E_1} \to \mathcal{E}_{E_2}$.*

Proof. This is Proposition 20.19. ∎

COROLLARY 20.57. *Let $\pi_1\colon E_1 \to M$ and $\pi_2\colon E_2 \to M$ be two vector bundles over M. Suppose $\zeta\colon \mathcal{E}_{E_1} \to \mathcal{E}_{E_2}$ is a C^∞_M-morphism of sheaves. Then $\zeta^M\colon \Gamma(E_1) \to \Gamma(E_2)$ is a point operator in the sense of Definition 20.16.*

Proof. This is Theorem 20.20. ■

Here is another more abstract example of an $\mathcal{F}$-module.

EXAMPLE 20.58. Let $(X, \mathcal{F})$ be a continuous ringed space. Let $n \in \mathbb{N}$. Then the sum

$$\mathcal{F}^k(U) := \underbrace{\mathcal{F}(U) \oplus \cdots \oplus \mathcal{F}(U)}_{n \text{ copies}}$$

is a *free* $\mathcal{F}$-module of rank k.

More generally, if $\mathcal{M}$ is any $\mathcal{F}$-module over X then we say that $\mathcal{M}$ is **locally free of rank** n if for any $p \in X$ there exists a neighbourhood U of p and an $\mathcal{F}|_U$-isomorphism of sheaves $\mathcal{M}|_U \cong \mathcal{F}^k$. If $\mathcal{M}$ is locally free of rank n then with a little work one can show that the stalk $\mathcal{M}_p$ is a free $\mathcal{F}_p$-module of rank n.

EXAMPLE 20.59. Let $\pi \colon E \to M$ be a vector bundle of rank n. Then the sheaf $\mathcal{E}_E$ is locally free of rank n. Indeed, this follows from the fact that for any $p \in M$, there exists a neighbourhood U of p such that E admits a local frame (e_i). Then any $s \in \Gamma(U, E)$ can be written as a

$$s = f^i e_i, \qquad a^i \in C^\infty(U),$$

and the correspondence $s \mapsto (f^1, \dots, f^n)$ sets up an isomorphism $\mathcal{E}_E|_U$ with $(\mathcal{C}^\infty_M|_U)^n$.

Just as in Theorem 20.52, it is actually possible to work backwards and *define* a vector bundle this way.

THEOREM 20.60. *Let M be a smooth manifold and let $\mathcal{M}$ be a sheaf of locally free $\mathcal{C}^\infty_M$-modules of rank n. Then there exists a vector bundle $\pi \colon E \to M$ and a $\mathcal{C}^\infty_M$-isomorphism of sheaves from $\mathcal{M}$ to $\mathcal{E}_E$. Moreover E is unique up to vector-bundle isomorphism.*

Proof (sketch). The stalk $\mathcal{F}_pM$ of $\mathcal{C}^\infty_M$ is a local ring with maximal ideal $\mathfrak{m}_p$ equal to the kernel of the evaluation map. The stalk $\mathcal{M}_p$ is a free $\mathcal{F}_p$-module of rank n. Thus if we set

See Lemma 2.15.

$$E_p := \mathcal{M}_p / \mathfrak{m}_p \mathcal{M}_p$$

then E_p is a vector space of dimension n. Now set $E = \bigsqcup_{p \in M} E_p$. If $p \in M$ and $U \subset M$ is a neighbourhood such that $\mathcal{M}|_U \cong (\mathcal{C}^\infty_M|_U)^n$ then this gives us a basis $\{e_1(x), \dots, e_k(x)\}$ of E_p for every $p \in U$, and thus a local frame for E. This gives us a bundle chart via (20.2). We use this to define a fibre bundle structure on E via Remark 16.5. The transition functions arising from a different choice of local frame near p are linear by assumption, and thus we have built a vector bundle. ■

REMARK 20.61. Theorem 20.60 tells us that there is a one-to-one correspondence (up to isomorphism) between vector bundles and locally free sheaves. From the point of view of categories, this gives us a way to go from an object of the category of vector bundles to an object of the category of finite rank locally free sheaves. A souped-up version of Hom Γ Theorem from the previous lecture allows us to extend this to morphisms too: i.e. a vector bundle homomorphism $E \to F$ is equivalent to an $\mathcal{C}^\infty_M$-morphism of sheaves. This allows us to conclude the following result: *there is an equivalence of categories between the category of vector bundles over M and the category of finite rank locally free $\mathcal{C}^\infty_M$-modules.*

LECTURE 21

Tensor Fields

Today we investigate sections of the tensor bundles $T^{h,k}(TM)$. Such sections play a special role in differential geometry, and they are classically referred to as *tensor fields*. Tensor fields can be thought of as a generalisation of vector fields – indeed, a tensor field of type $(h,k) = (1,0)$ is exactly a vector field. We already briefly met tensor fields in Remark 8.17 – one of the goals of today's lecture is to fill in the details from this remark.

DEFINITION 21.1. A **tensor field** of type (h,k) on M is a section of $T^{h,k}(TM)$. We normally use the special notation $\mathcal{T}^{h,k}(M)$ for tensor fields. The space of sections $\mathcal{T}^{h,k}(U) := \Gamma(U, T^{h,k}(TM))$ is defined similarly; these are the tensor fields of type (h,k) over U. Let us unpack this a bit. The bundle $T^{h,k}(TM)$ is the bundle whose fibre over $p \in M$ is

$$T^{h,k}(T_pM) := \underbrace{T_pM \otimes \cdots \otimes T_pM}_{h \text{ copies}} \otimes \overbrace{T_p^*M \otimes \cdots \otimes T_p^*M}^{k \text{ copies}}.$$

If $A \in \mathcal{T}^{h,k}(M)$ then we can think of the value of A at p, which we write either as $A(p)$ or A_p (the latter is preferred if there are many variables) as a multilinear map

$$A_p \colon \underbrace{T_p^*M \times \cdots \times T_p^*M}_{h \text{ copies}} \times \overbrace{T_pM \times \cdots \times T_pM}^{k \text{ copies}} \to \mathbb{R}$$

thanks to Proposition 19.8.

A tensor field of type $(1,0)$ is just a vector field: in this case we think of $X(p) \colon T_p^*M \to \mathbb{R}$ as the linear map given by $X(p)(\lambda) := \lambda(X(p))$. If $A \in \mathcal{T}^{h,k}(M)$ and (U,x) is a chart on M then locally we can write

Expressions of this form are the main reason we introduced the Einstein Summation Convention!

$$A = A^{i_1 \cdots i_h}_{j_1 \cdots j_k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_h}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_k}$$

where the function $A^{i_1 \cdots i_h}_{j_1 \cdots j_k} \in C^\infty(U)$ is defined by

$$A^{i_1 \dots i_h}_{j_1 \dots j_k}(p) = A_p \left(dx_p^{i_1}, \dots, dx_p^{i_h}, \frac{\partial}{\partial x^{j_1}}\Big|_p, \dots, \frac{\partial}{\partial x^{j_k}}\Big|_p \right).$$

If $A \in \mathcal{T}^{h,k}(U)$ and $B \in \mathcal{T}^{h_1,k_1}(U)$ then we can form their tensor product

$$A \otimes B \in \mathcal{T}^{h+h_1,k+k_1}(U)$$

which is defined pointwise via Definition 19.19, see (19.3) and (19.4). This means that if we set

$$\mathcal{T}(U) := \bigoplus_{h,k \geq 0} \mathcal{T}^{h,k}(U),$$

Warning: $\mathcal{T}(M)$ is not the space of sections of a vector bundle, as the tensor algebra $\widetilde{T}E$ of a vector bundle is not a vector bundle, cf. Remark 19.20.

then $\mathcal{T}(U)$ is not just a module over the ring $C^\infty(U)$ but actually an algebra. By a slight abuse of language, we call $\mathcal{T}(M)$ the **tensor algebra** of M.

Definition 21.2. Let $\zeta\colon \mathcal{T}(M) \to \mathcal{T}(N)$ be an $\mathbb{R}$-linear map. We say that ζ has **degree** $(i, j) \in \mathbb{Z}^2$ if

$$\zeta\big(\mathcal{T}^{h,k}(M)\big) \subset \mathcal{T}^{h+i,k+j}(N)$$

Here i and j can be negative; we use the convention that $T^{h,k}(TM) = \{0\}$ if either $h < 0$ or $k < 0$.

In the case $M = N$, we say that such an ζ is a **local operator**, respectively a **point operator**, if the restriction of ζ to each $\mathcal{T}^{h,k}(M)$ is a local, respectively a point, operator.

Remark 21.3. As in (19.4), strictly speaking the tensor $A \otimes B$ defined above needs its factors rearranging. If for instance $A = X_1 \otimes \omega_1$ and $B = X_2 \otimes \omega_2$ for vector fields X_i and 1-forms ω_i, then $A \otimes B$ should really be written as $X_1 \otimes X_2 \otimes \omega_1 \otimes \omega_2$ (so that the vector field factors come first). In practice, this is inconvenient, and so we will often not bother and just keep the factors unchanged, thus writing $A \otimes B = X_1 \otimes \omega_1 \otimes X_2 \otimes \omega_2$. This is harmless, since it was merely a convention to put the vector fields first (cf. Corollary 19.16).

i.e. ω_i is a tensor of type $(0, 1)$.

A differential form is a section of the exterior algebra bundle of the cotangent bundle.

Definition 21.4. A **differential k-form** (often simply called "a k-form") on M is a section of $\bigwedge^k T^*M$. We use the special notation $\Omega^k(M)$ for the space of differential k-forms. If $\omega \in \Omega^k(M)$ and $p \in M$ then we can think of ω_p as an alternating map

$$\omega_p\colon \underbrace{T_pM \times \cdots \times T_pM}_{k \text{ copies}} \to \mathbb{R},$$

thanks to Proposition 19.24.

If $\omega \in \Omega^k(M)$ and (U, x) is a chart on M then locally we can write

$$\omega = \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \tag{21.1}$$

where $\omega_{i_1 \dots i_k} \in C^\infty(U)$. We define

$$\Omega(M) = \bigoplus_{0 \le k \le m} \Omega^k(M),$$

with $\Omega(U)$ defined similarly. Thus an element of $\Omega(M)$ is a sum $\sum_{i=0}^m \omega_i$ where $\omega_i \in \Omega^i(M)$. Since an alternating multilinear map is (in particular) a multilinear map, we see that any differential k-form may be regarded as a tensor of type $(0, k)$. For $k = 1$ this is an equality (since in dimension one every linear map is alternating):

$$\Omega^1(M) = \mathcal{T}^{0,1}(M)$$

But for $k \geq 2$, there are (many) multilinear maps that are not alternating, and thus not every tensor of type $(0, k)$ is a differential form.

$$\Omega^k(M) \subsetneq \mathcal{T}^{0,k}(M).$$

As above, an $\mathbb{R}$-linear map $\zeta\colon \Omega(M) \to \Omega(N)$ is said to have **degree** h if $\zeta(\Omega^k(M)) \subset \Omega^{h+k}(M)$.

Identifying tensors: For low values of (h,k) we have various different notations for $\mathscr{T}^{h,k}(M)$.

- $\mathscr{T}^{0,0}(M) = C^\infty(M)$.
- $\mathscr{T}^{1,0}(M) = \mathfrak{X}(M)$.
- $\mathscr{T}^{0,1}(M) = \Omega^1(M)$.
- $\mathscr{T}^{1,1}(M) = \Gamma(\mathrm{End}(TM))$.

The following theorem tells us how to recognise tensor fields and differential forms "in the wild".

THEOREM 21.5 (The Tensor and Differential Form Criterion). *Let M be a smooth manifold and let $U \subset M$ be a non-empty open set.*

(i) *There is a canonical identification between $\mathscr{T}^{h,k}(U)$ and $C^\infty(U)$-multilinear functions*

$$\underbrace{\Omega^1(U) \times \cdots \times \Omega^1(U)}_{h \text{ copies}} \times \overbrace{\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U)}^{k \text{ copies}} \to C^\infty(U).$$

(ii) *There is a canonical identification between $\Omega^k(U)$ and alternating $C^\infty(U)$-multilinear functions*

$$\underbrace{\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U)}_{k \text{ copies}} \to C^\infty(U).$$

Proof. The case $(h,k) = (1,0)$ or $(h,k) = (0,1)$ of part (i) follows from the Hom-Gamma Theorem 21.5. Indeed, suppose for definiteness $(h,k) = (0,1)$. We apply the Hom-Gamma Theorem with $E = TM$ and $F = M \times \mathbb{R}$ the trivial bundle. Then in the notation of (20.5),

$$\mathrm{Hom}\big(\Gamma(E), \Gamma(F)\big) = \big\{C^\infty(M)\text{-linear maps } \mathfrak{X}(M) \to C^\infty(M)\big\}.$$

and

$$\begin{aligned}\Gamma\big(\mathrm{Hom}(E,F)\big) &= \Gamma(E^* \otimes \mathbb{R}) \\ &= \Gamma(E^*) \\ &= \mathscr{T}^{0,1}(M),\end{aligned}$$

where the first equality used Corollary 19.14 and the second equality uses the fact that for any real vector space V there is a canonical isomorphism $V \cong V \otimes \mathbb{R}$.

The general case for part (i) can be proved by induction on $h+k$; the details of this argument are left to you on Problem Sheet I. The proof of part (ii) proceeds along similar lines, and is also omitted. ∎

We will typically suppress this isomorphism from our notation, and thus interchangeably regard a tensor field A over U either as an element of $\mathscr{T}^{h,k}(U)$, or as an appropriate multilinear map, and similarly with differential forms. In the bonus section below we explain why Theorem 21.5 could also be called a "Tensor-Gamma Theorem".

REMARK 21.6. Every differential geometer should at one point in their life compute tensor fields in local coordinates. This is left as a wholesome exercise for you to enjoy on Problem Sheet I.

Once is enough.

We will return to differential forms next lecture. For now we concentrate on tensors. Let us now think all the way back to Lecture 8. If $\varphi\colon M \to N$ is a diffeomorphism, we defined two isomorphisms

$$\varphi_*\colon C^\infty(M) \to C^\infty(N), \qquad \varphi_*\colon \mathfrak{X}(M) \to \mathfrak{X}(N).$$

The vector field $\varphi_* X$ is sometimes called the **push forward** of X by φ, and the map φ_* is called a **push forward** map.

Since $\mathscr{T}^{0,0}(M) = C^\infty(M)$ and $\mathscr{T}^{1,0}(M) = \mathfrak{X}(M)$, we can think of both of these maps φ_* as being defined on tensors of type $(0,0)$ or $(1,0)$. As hinted at in Remark 8.17, these two maps φ_* are both special cases of an $\mathbb{R}$-linear map between the tensor algebras:

$$\varphi_*\colon \mathscr{T}(M) \to \mathscr{T}(N)$$

which **preserves type**.

i.e. is of type $(0,0)$.

REMARK 21.7. The same is true of the Lie derivative. We will come back to this next lecture.

We shall take a slightly circuitous route to defining the map φ_* and first define a type-preserving map going the other way

$$\varphi^*\colon \mathscr{T}(N) \to \mathscr{T}(M).$$

The upper star indicates that $\varphi \mapsto \varphi^*$ is contravariant, i.e. it reverses the direction of the arrows.

In fact, some of this construction works for arbitrary smooth maps (not necessarily diffeomorphisms), and we cover this first as a special case. This will be useful next lecture when we talk about pullbacks of differential forms.

DEFINITION 21.8. Let $\varphi\colon M \to N$ be smooth (not necessarily a diffeomorphism). Let $k \geq 1$ and suppose $A \in \mathscr{T}^{0,k}(N)$ is a $(0,k)$ tensor field on N. We define the **pullback** of A by φ, written $\varphi^* A$, to be the tensor field on M defined pointwise as follows: for $p \in M$ and $\xi_1, \dots \xi_k \in T_p M$, set

$$(\varphi^* A)_p(\xi_1, \dots, \xi_k) := A_{\varphi(p)}\big(D\varphi(p)\xi_1, \dots, D\varphi(p)\xi_k\big),$$

We extend this to cover the case $k = 0$ by setting

$$\varphi^* f := f \circ \varphi, \qquad f \in C^\infty(N).$$

It is immediate that φ^* is $\mathbb{R}$-linear. Moreover if $f \in C^\infty(N)$ then

$$\varphi^*(fA) = (\varphi^* f)(\varphi^* A),$$

Note also that if $\psi\colon L \to M$ then $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$. If we consider all these maps φ_* at once we get a type-preserving operator

$$\varphi^*\colon \bigoplus_{k\geq 0} \mathscr{T}^{0,k}(N) \to \bigoplus_{k\geq 0} \mathscr{T}^{0,k}(M)$$

which moreover is an algebra morphism, i.e.

$$\varphi^*(A \otimes B) = \varphi^* A \otimes \varphi^* B.$$

To extend this definition to tensors of arbitrary type, we need to assume that φ is a diffeomorphism. First let us introduce the notion of a **cotangent lift**.

Definition 21.9. Let $\varphi\colon M \to N$ be a diffeomorphism. Then $D\varphi(p)\colon T_pM \to T_{\varphi(p)}N$ is a linear isomorphism for each p, and thus we can speak of its inverse $D\varphi(p)^{-1}\colon T_{\varphi(p)}N \to T_pM$. Thus there is a well-defined map

This coincides with the differential of φ^{-1} at the point $\varphi(p)$.

$$D\varphi^\dagger\colon T^*M \to T^*N,$$

called the **cotangent lift** defined for $p \in M$, $\lambda \in T_p^*M$ and $\xi \in T_{\varphi(p)}N$ by

$$\big(D\varphi^\dagger(p)\lambda\big)(\xi) := \lambda\big(D\varphi(p)^{-1}\xi\big)$$

The cotangent lift makes the following diagram commute:

$$\begin{array}{ccc} T^*M & \xrightarrow{D\varphi^\dagger} & T^*N \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow[\varphi]{} & N \end{array}$$

Thus $D\varphi^\dagger$ is a vector bundle morphism along φ.

We emphasise $D\varphi^\dagger$ is only defined for diffeomorphisms.

Definition 21.10. Now suppose $\varphi\colon M \to N$ is a diffeomorphism. Then we can define the pullback tensor $\varphi^* A \in \mathscr{T}^{h,k}(M)$ for a tensor $A \in \mathscr{T}^{h,k}(N)$ of arbitrary type (h,k) by setting

$$\begin{aligned} &\varphi^* A_p(\lambda_1, \ldots, \lambda_h, \xi_1, \ldots, \xi_k) \\ &\qquad := A_{\varphi(p)}\Big(D\varphi^\dagger(p)\lambda_1, \ldots, D\varphi^\dagger(p)\lambda_h, D\varphi(p)\xi_1, \ldots, D\varphi(p)\xi_k\Big) \end{aligned}$$

for $p \in M$, $\lambda_1, \ldots, \lambda_h \in T_p^*M$ and $\xi_1, \ldots, \xi_k \in T_pM$.

Remark 21.11. More generally, if $\varphi\colon U \subset M \to N$ is a locally defined map which is a diffeomorphism onto its image $V = \varphi(U)$ then we can still use φ to pull back tensor fields of arbitrary type from V to U.

Thus when φ is a diffeomorphism there is a well-defined operator $\varphi^*\colon \mathscr{T}(N) \to \mathscr{T}(M)$ which is an algebra morphism, that is,

$$\varphi^*(A \otimes B) = \varphi^* A \otimes \varphi^* B. \tag{21.2}$$

Summary:

- A tensor of type $(0, k)$ can be pulled back by an arbitrary smooth map.
- A tensor of type (h, k) for $h > 0$ can only be pulled back by a diffeomorphism.

So much for pullbacks. What about push forwards? The answer is simple:

DEFINITION 21.12. Let $\varphi\colon M \to N$ be a diffeomorphism. We define the **push forward** map

$$\varphi_*\colon \mathcal{T}(M) \to \mathcal{T}(N)$$

by

$$\varphi_* := (\varphi^{-1})^*.$$

Thus the push forward is the pullback of the inverse.

In the special case $h = k = 0$, the map φ_* sends a function f to $f \circ \varphi^{-1}$. In the special case $h = 1$ and $k = 0$, one has

$$(\varphi_* X)(p) = D\varphi(\varphi^{-1}(p))X(\varphi^{-1}(p)),$$

and thus in both cases these extend the definitions from Lecture 8.

Bonus Material for Lecture 21

In Lecture 19 we defined tensor products for finite-dimensional real vector spaces. However everything would have worked (without any changes at all) if we worked with finite rank modules over a fixed commutative ring R. A more interesting question is to what extent the finite rank hypothesis was needed. Indeed, suppose V is a module over a commutative ring R. The dual module is defined $V^* = \mathrm{Hom}_R(V, R)$, and the space $\mathrm{Mult}_{h,k}(V)$ is then defined be the set of multilinear maps

$$\underbrace{V \times \cdots \times V}_{h \text{ copies}} \times \overbrace{V^* \times \cdots \times V^*}^{k \text{ copies}} \to R.$$

One can then ask the question: *is it true that $T^{h,k}V$ and $\mathrm{Mult}_{s,r}(V)$ are isomorphic modules*?

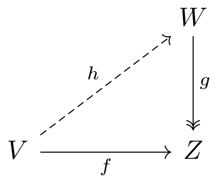
$$T^{h,k}V \overset{?}{\cong} \mathrm{Mult}_{k,h}(V). \tag{21.3}$$

The answer in general is **no**. Nevertheless (21.3) it is true for *some* infinite-rank modules.

DEFINITION 21.13. Let R be a commutative ring and V an R-module. We say that V is **projective** if for any R-module Z, if we are given a module homomorphism $f\colon V \to Z$ and a surjective module homomorphism $g\colon W \to Z$ from some other R-module W, there exists a module homomorphism $h\colon V \to W$ such that the following commutes:

All rings are assumed to be unital.

Warning: We do **not** require h to be unique. This is **not** a universal property!



On Problem Sheet I you will prove the following generalisation of Proposition 19.8.

PROPOSITION 21.14. *Let R be a commutative ring and let V be a finitely generated projective R-module. Then for all $h, k \geq 0$,*

$$T^{h,k}V \cong \mathrm{Mult}_{k,h}(V).$$

Moreover, as you will also prove on Problem Sheet I, the space of sections of a vector bundle satisfy the hypotheses of Proposition 21.14:

PROPOSITION 21.15. *Let $\pi\colon E \to M$ be a vector bundle. For any open set $U \subset M$, the space $\Gamma(U, E)$ is a finitely generated projective $C^\infty(U)$-module.*

This gives us the following cute corollary.

COROLLARY 21.16. *The operators Γ and $T^{h,k}$ also "commute". That is, there is an isomorphism of $C^\infty(M)$-modules*

$$\Gamma\big(T^{h,k}(TM)\big) \cong T^{h,k}\big(\Gamma(TM)\big),$$

Proof. The Tensor Criterion Theorem 21.5 tells us that

$$\mathcal{T}^{h,k}\big(\mathfrak{X}(U)\big) \cong \mathrm{Mult}_{k,h}\left(\mathfrak{X}(U)\right).$$

Combining this with Propositions 21.14 and 21.15 we obtain

$$\mathcal{T}^{h,k}(U) \cong T^{h,k}\big(\mathfrak{X}(U)\big),$$

which is just compact notation for

$$\Gamma\big(U, T^{h,k}(TM)\big) \cong T^{h,k}\big(\Gamma(U, TM)\big).$$

■

LECTURE 22

The Lie Derivative Revisited

In this lecture we introduce the notion of a **tensor derivation** on a manifold. We show that the Lie derivative is a tensor derivation, and thus unify the two definitions of the Lie derivative from Definition 10.1.

DEFINITION 22.1. Let V be a vector space and fix $h, k \geq 0$. Choose $i \leq h$ and $j \leq k$. The (i, j)**th contraction**, written $C^{i,j}$ is the linear operator

$$C^{i,j} \colon T^{h,k}V \to T^{h-1,k-1}V$$

defined on decomposable elements by feeding the ith V-factor to the jth V^* factor:

$$C^{i,j}(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_h \otimes \lambda^1 \otimes \cdots \otimes \lambda^j \otimes \cdots \otimes \lambda^k) := \lambda^j(v_i) \cdot v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{i+1} \otimes \cdots \otimes \lambda^{j-1} \otimes \lambda^{j+1} \otimes \cdots \otimes \lambda^k$$

and then extending by linearity.

LEMMA 22.2. *Assume* $\dim V = n$*. Let* $A \in T^{h,k}V$*, and regard* A *as defining an element of* $\mathrm{Mult}_{k,h}(V)$ *as in Proposition 19.8. Let* $\{e_1, \dots, e_n\}$ *be a basis of* V *with dual basis* $\{e^1, \dots, e^n\}$ *of* V^**. Then if we regard* $C^{i,j}A$ *also as an element of* $\mathrm{Mult}_{k-1,h-1}(V)$*, one has*

$$(C^{i,j}A)\big(w_1, \dots, w_{k-1}, \eta^1, \dots, \eta^{h-1}\big) = \sum_{l=1}^n A\big(w_1, \dots, \underbrace{e_l}_{j\text{th position}}, \dots, w_{k-1}, \eta^1, \dots, \underbrace{e^l}_{i\text{th position}}, \dots \eta^{h-1}\big).$$

Proof. It suffices to prove the equality for decomposable elements. Let us temporarily add a tilde to denote the multilinear map corresponding to a given tensor. For simplicity assume $(h, k) = (2, 3)$, and assume $A = v_1 \otimes v_2 \otimes \lambda^1 \otimes \lambda^2 \otimes \lambda^3$. Then as in Remark 19.12, the corresponding map $\widetilde{A} \in \mathrm{Mult}_{3,2}(V)$ is given by

The general case is only notationally more complicated.

$$\widetilde{A}(w_1, w_2, w_3, \eta^1, \eta^2) = \eta^1(v_1)\, \eta^2(v_2)\, \lambda^1(w_1)\, \lambda^2(w_2)\, \lambda^3(w_3).$$

Take $(i, j) = (1, 2)$. Then $C^{1,2}A = \lambda^2(v_1) \cdot v_2 \otimes \lambda^1 \otimes \lambda^3$. We compute

We include the summation signs to minimise the risk of confusion.

$$\begin{aligned}(C^{1,2}\widetilde{A})(w_1, w_2, \eta^1) &\overset{\text{def}}{=} \sum_{l=1}^n \widetilde{A}(w_1, e_l, w_2, e^l, \eta^1) \\ &= \sum_{l=1}^n e^l(v_1)\, \eta^1(v_2)\, \lambda^1(w_1)\, \lambda^2(e_l)\, \lambda^3(w_2) \\ &= \left(\sum_{l=1}^n e^l(v_1)\, \lambda^2(e_i)\right) \eta^1(v_2)\, \lambda^1(w_1)\, \lambda^3(w_2).\end{aligned}$$

But

$$\sum_{l=1}^n e^l(v_1)\, \lambda^2(e_l) = \lambda^2(v_1),$$

and thus

$$C^{1,2}\widetilde{A} = \widetilde{C^{1,2}A}$$

which is what we wanted to prove. ■

A contraction $C^{i,j}$ extends to define an operator on tensor fields in an obvious fashion. For instance, if $A \in \mathcal{T}^{2,1}(M)$ is the decomposable tensor $X \otimes Y \otimes \omega$ then $C^{1,1}A = \omega(X)Y$.

Lemma 22.3. *For any* $1 \leq i \leq h$ *and* $1 \leq j \leq k$*, the contraction* $C^{i,j}$ *defines a point operator* $\mathcal{T}^{h,k}(M) \to \mathcal{T}^{h-1,k-1}(M)$.

Proof. We need only check the point property, but this is immediate from the definition: if $A_p = 0$ then $(C^{i,j}A)_p = 0$. ■

Definition 22.4. Let $\zeta \colon \mathcal{T}(M) \to \mathcal{T}(N)$ be an operator which preserves type. We say that ζ **commutes with all contractions** if for any any $1 \leq i \leq h$ and $1 \leq j \leq k$ one has

$$\zeta \circ C^{i,j} = C^{i,j} \circ \zeta$$

as maps $\mathcal{T}^{h,k}(M) \to \mathcal{T}^{h-1,k-1}(N)$.

Example 22.5. Let $\varphi \colon M \to N$ be a diffeomorphism. Then φ^* and φ_* both commute with all contractions.

Here is the main definition for today.

Definition 22.6. A **tensor derivation** is an $\mathbb{R}$-linear operator $\zeta \colon \mathcal{T}(M) \to \mathcal{T}(M)$ that preserves type, commutes with all contractions, and satisfies

$$\zeta(A \otimes B) = \zeta A \otimes B + A \otimes \zeta B \tag{22.1}$$

for all $A, B \in \mathcal{T}(M)$.

We refer to (22.1) as the **derivation property**. Operators that satisfy this property are automatically local operators.

Lemma 22.7. *Suppose* $\zeta \colon \mathcal{T}(M) \to \mathcal{T}(M)$ *is an* $\mathbb{R}$*-linear map that satisfies the derivation property. Then* ζ *is a local operator. Thus in particular tensor derivations are local operators.*

Proof. This is a standard argument which we have seen many times by now. Suppose $A \in \mathcal{T}^{h,k}(M)$ vanishes on $U \subset M$. Let $p \in U$ and let $\chi \colon M \to \mathbb{R}$ denote a bump function with $\chi(p) = 1$ and $\operatorname{supp}(\chi) \subset U$. Then $\eta A = \chi \otimes A$ vanishes identically on M, and hence by $\mathbb{R}$-linearity $\zeta(\chi \otimes A) = 0$. Then by the derivation property

$$\begin{aligned} 0 &= \zeta(\chi \otimes A)(p) \\ &= \chi(p)(\zeta A)(p) + (\zeta\chi)(p)A(p) \\ &= (\zeta A)(p). \end{aligned}$$

As p was an arbitrary point of U, it follows that ζA vanishes on U. ■

Remark 22.8. Combining Lemma 22.7 and Proposition 20.19 shows that tensor derivations induce operators $\zeta^U : \mathscr{T}(U) \to \mathscr{T}(U)$ for each open set $U \subset M$. These operators are actually tensor derivations in their own right: the fact that ζ^U commutes with contractions follows from the fact that contractions are local operators, and the fact that ζ^U satisfies the derivation property is immediate.

Lemma 22.9. *Suppose ζ is a tensor derivation on M, and let $U \subset M$ be open. Fix $A \in \mathscr{T}^{h,k}(U)$, and suppose $X_1, \dots, X_k \in \mathfrak{X}(U)$, and $\omega_1, \dots, \omega_h \in \Omega^1(U)$. Then:*

$$\begin{aligned}\zeta^U(A)(\omega_1, \dots, \omega_h, X_1, \dots, X_k) = \zeta^U\big(A(\omega_1, \dots, \omega_h, X_1, \dots, X_k)\big) \\ - \sum_{i=1}^{h} A\big(\omega_1, \dots, \zeta^U(\omega_i), \dots, \omega_h, X_1, \dots X_k\big) \\ - \sum_{i=1}^{k} A\big(\omega_1, \dots, \omega_h, X_1, \dots, \zeta^U(X_i), \dots X_k\big).\end{aligned}$$

Proof. The $(0,0)$-tensor $A(\omega_1, \dots, \omega_h, X_1, \dots, X_k)$ can be thought of as being obtained from the $(h+k, h+k)$ tensor $A \otimes \omega_1 \otimes \cdots \otimes \omega_h \otimes X_1 \otimes \cdots \otimes X_k$ by repeated contractions. We write this symbolically as

As in Remark 21.3, we don't bother to reorder the factors in this expression.

$$A(\omega_1, \dots, \omega_h, X_1, \dots, X_k) = C(A \otimes \omega_1 \otimes \cdots \otimes \omega_h \otimes X_1 \otimes \cdots \otimes X_k),$$

where C stands for repeated contractions. The claim now follows by repeatedly using the fact that ζ commutes with contractions and satisfies the derivation property. ■

Corollary 22.10. *Suppose ζ and ξ are two tensor derivations that agree on functions and vector fields. Then they are identical.*

Proof. Let ω be a 1-form. Then by Lemma 22.9 with $A = \omega$ we see for an arbitrary vector field X that

$$\begin{aligned}\zeta(\omega)(X) &= \zeta(\omega(X)) - \omega(\zeta(X)) \\ &= \xi(\omega(X)) - \omega(\xi(X)) \\ &= \xi(\omega)(X).\end{aligned}$$

Since X was arbitrary, this shows that $\zeta(\omega) = \xi(\omega)$, and since ω was arbitrary this shows that ζ and ξ coincide on tensors of type $(0,1)$. Now for an arbitrary A, observe that Lemma 22.9 expands $\zeta(A)$ in such a way that all the other terms are of the form ζ eating a function, a vector field, or a 1-form. Thus $\zeta(A) = \xi(A)$ for arbitrary A. ■

The next result shows how one can work backwards and build a tensor derivation if we have something defined on functions and vector fields with the appropriate property.

Proposition 22.11. *Suppose we have a type-preserving local operator ζ defined on smooth functions and vector fields which satisfies*

i.e. we have two local operators $\zeta\colon \mathscr{T}^{0,0}(M) \to \mathscr{T}^{0,0}(M)$ and $\zeta\colon \mathscr{T}^{1,0}(M) \to \mathscr{T}^{1,0}(M)$.

$$\begin{aligned}\zeta(fg) &= \zeta(f)g + f\zeta(g), \\ \zeta(fX) &= \zeta(f)X + f\zeta(X)\end{aligned} \tag{22.2}$$

for all $f, g \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$. Then ζ extends uniquely to a tensor derivation.

Note the two conditions in (22.2) are forced if we want ζ to be a tensor derivation, since $f \otimes g = fg$ and $f \otimes X = fX$.

Proof. Uniqueness is immediate from the previous corollary, since we have prescribed what ζ must do to vector fields and functions. The proof proceeds very similarly to that of Corollary 22.10: the derivation property coupled with our requirement that ζ commutes with contractions means that at every stage we have no choice how to proceed.

Namely, we define ζ on 1-forms by setting

$$\zeta(\omega)(X) = \zeta(\omega(X)) - \omega(\zeta(X)). \tag{22.3}$$

The hypotheses imply that $\zeta \colon \mathcal{T}^{0,1}(M) \to \mathcal{T}^{0,1}(M)$ is a local operator. Next to define ζ on $\mathcal{T}^{1,1}(M)$ we start with a tensor of the form $X \otimes \omega$. The derivation property requires us define

$$\zeta(X \otimes \omega) := X \otimes \zeta(\omega) + \zeta(X) \otimes \omega.$$

If ζ commutes with the contraction $C^{1,1} \colon \mathcal{T}^{1,1}(M) \to \mathcal{C}^\infty(M)$ then we need

$$\begin{aligned} \zeta(\omega(X)) &= C^{1,1}\big(\zeta(X \otimes \omega)\big) \\ &= C^{1,1}\big(X \otimes \zeta(\omega) + \zeta(X) \otimes \omega\big) \\ &= \zeta(\omega)(X) + \omega(\zeta(X)), \end{aligned}$$

and this is true by (22.3). This also shows that (22.3) was forced – no other choice would have worked. Now we use the formula from Lemma 22.9 to define ζ on all higher tensors. A check similar to the one we just did shows that the resulting object is a derivation that commutes with all contractions. ∎

We now obtain our promised extension of the Lie derivative.

THEOREM 22.12. *Let $X \in \mathfrak{X}(M)$. There exists a unique tensor derivation $\mathcal{L}_X \colon \mathcal{T}(M) \to \mathcal{T}(M)$ that extends the Lie derivative defined on functions and vector fields from Lecture 8.*

Lemma 22.9 tells us how to compute $\mathcal{L}_X A$. For instance, if $A \in \mathcal{T}^{0,k}(M)$ then

$$\begin{aligned} (\mathcal{L}_X A)(Y_1, \dots, Y_k) = X\big(A(Y_1, \dots, Y_k)\big) \\ - \sum_{i=1}^{k} A\big(Y_1, \dots, Y_{i-1}, \mathcal{L}_X Y_i, Y_{i+1}, \dots, Y_k\big). \end{aligned} \tag{22.4}$$

The next result, whose proof is deferred to Problem Sheet I, classifies all tensor derivations.

PROPOSITION 22.13. *Let M be a smooth manifold.*

(i) *Suppose $A \in \mathcal{T}^{1,1}(M) \cong \Gamma(\mathrm{End}(TM))$. Then there exists a unique tensor derivation ζ_A on M with the property that $\zeta_A(Y)(p) = A_p(Y(p))$ for any vector field Y and satisfies $\zeta_A(f) = 0$ for any function f.*

(ii) *Let ξ be an arbitrary tensor derivation. Then there exists a vector field X on M and $A \in \mathcal{T}^{1,1}(M)$ such that $\xi = \mathcal{L}_X + \zeta_A$. Thus the space of tensor derivations on M can be identified with $\mathfrak{X}(M) \times \Gamma(\mathrm{End}(TM))$.*

An explicit formula for the Lie derivative is given by the following result.

PROPOSITION 22.14. *Let $X \in \mathfrak{X}(M)$ with flow Φ_t. Then for any tensor field A, the Lie derivative $\mathcal{L}_X A$ is given by:*

$$\mathcal{L}_X A := \frac{d}{dt}\Big|_{t=0} \Phi_t^* A = \lim_{t \to 0} \frac{\Phi_t^* A - A}{t}. \tag{22.5}$$

The proof of Proposition 22.14 is given in the bonus section below.

We now switch our focus back to differential forms. The following statement is a minor variation of the the Tensor Criterion 21.5, and the proof is left as an exercise.

THEOREM 22.15 (The Differential Form Criterion). *Let M be a smooth manifold and let $U \subset M$ be a non-empty open set. Then there is a canonical identification between $\Omega^k(U)$ and* **alternating** *$C^\infty(U)$-multilinear functions*

$$\underbrace{\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U)}_{k \text{ copies}} \to C^\infty(U).$$

Since an alternating multilinear map is (in particular) a multilinear map, we see that any differential k-form may be regarded as a tensor of type $(0,k)$. But for $k \geq 2$, there are (many) multilinear maps that are not alternating, and thus not every tensor of type $(0,k)$ is a differential form. We define

$$\Omega(M) = \bigoplus_{0 \leq k \leq m} \Omega^k(M),$$

with $\Omega(U)$ defined similarly. Thus an element of $\Omega(M)$ is a sum $\sum_{i=0}^m \omega_i$ where $\omega_i \in \Omega^i(M)$.

DEFINITION 22.16. If $\omega \in \Omega^h(M)$ and $\theta \in \Omega^k(M)$ then the **wedge product** is the differential form $\omega \wedge \theta \in \Omega^{h+k}(M)$ defined pointwise by

$$(\omega \wedge \theta)_p = \omega_p \wedge \theta_p$$

Since $\bigwedge^k V = 0$ if $k > \dim V$, the wedge product $\omega \wedge \theta$ is zero if $h + k > \dim M$. Note that by part (ii) of Proposition 19.22, one has

$$\omega \wedge \theta = (-1)^{hk} \theta \wedge \omega, \qquad \omega \in \Omega^h(M), \theta \in \Omega^k(M)$$

The wedge product gives $\Omega(M)$ the structure of graded ring, and in fact, also a $C^\infty(M)$-graded skew-commutative algebra.

The "skew-commutative" refers to the sign $(-1)^{hk}$.

Here is a useful piece of linear algebra, whose proof is on Problem Sheet J. We let $\mathfrak{S}_k$ denote the group of all permutations on k letters.

Definition 22.17. Let $h, k \geq 0$. An (h,k)**-shuffle** is a permutation $\varrho \in \mathfrak{S}_{h+k}$ such that

$$\varrho(1) < \cdots < \varrho(h) \qquad \text{and} \qquad \varrho(h+1) < \cdots < \varrho(h+k).$$

We let $\mathrm{Shuffle}(h,k) \subset \mathfrak{S}_{h+k}$ denote the set of all (h,k)-shuffles.

Lemma 22.18. *Let V be a vector space and suppose $\omega \in \bigwedge^h V^*$ and $\theta \in \bigwedge^k V^*$. Let $v_i \in V$ for $i = 1, \dots, h+k$. Then if we identify ω with an element of $\mathrm{Alt}_h(V)$, θ with an element of $\mathrm{Alt}_k(V)$, and $\omega \wedge \theta$ with an element of $\mathrm{Alt}_{h+k}(V)$, one has:*

$$(\omega \wedge \theta)(v_1, \dots, v_{h+k}) = \frac{1}{h!k!} \sum_{\varrho \in \mathfrak{S}_{h+k}} \mathrm{sgn}(\varrho)\omega\big(v_{\varrho(1)}, \dots, v_{\varrho(h)}\big)\, \theta\big(v_{\varrho(h+1)}, \dots, v_{\varrho(h+k)}\big)$$

or equivalently

$$(\omega \wedge \theta)(v_1, \dots, v_{h+k}) = \sum_{\varrho \in \mathrm{Shuffle}(h,k)} \mathrm{sgn}(\varrho)\omega\big(v_{\varrho(1)}, \dots, v_{\varrho(h)}\big)\, \theta\big(v_{\varrho(h+1)}, \dots, v_{\varrho(h+k)}\big).$$

For low values of h and k this gives an easy method to compute the wedge product of two differential forms. For instance, we have:

Corollary 22.19. *Let $\omega, \theta \in \Omega^1(M)$ denote two 1-forms. Then*

$$(\omega \wedge \theta)_p(\xi, \zeta) = \omega_p(\xi)\theta_p(\zeta) - \omega_p(\zeta)\theta_p(\xi), \qquad \forall\, p \in M,\ \forall\, \xi, \zeta \in T_pM.$$

If $\varphi \colon M \to N$ is smooth we showed in Definition 21.8 how to pullback a tensor $A \in \mathscr{T}^{0,k}(N)$ to obtain a tensor $\varphi^* A \in \mathscr{T}^{0,k}(M)$. It is clear from the definition that if A is alternating then so is $\varphi^* A$. Thus φ^* restricts to define a map $\Omega(N) \to \Omega(M)$. For convenience, we state this again here:

Definition 22.20. Let $\varphi \colon M \to N$ denote a smooth map. Given $\omega \in \Omega^k(N)$, we define the **pullback** form $\varphi^*\omega \in \Omega^k(M)$ by

$$(\varphi^*\omega)_p(\xi_1, \dots, \xi_k) := \omega_{\varphi(p)}\big(D\varphi(p)\xi_1, \dots, D\varphi(p)\xi_k\big).$$

The next lemma tells us that φ^* is an algebra homomorphism.

Lemma 22.21. *If $\varphi \colon M \to N$ is a smooth map and $\omega, \theta \in \Omega(N)$ then*

$$\varphi^*(\omega \wedge \theta) = \varphi^*\omega \wedge \varphi^*\theta.$$

Proof. Immediate from Lemma 22.18 and the definition. ■

Note also that (just as with tensors), the pullback operation is functorial: if $\varphi \colon M \to N$ and $\psi \colon L \to M$ then

$$(\varphi \circ \psi)^* = \psi^* \circ \varphi^* \tag{22.6}$$

as maps $\Omega(N) \to \Omega(L)$.

Since any differential form $\omega \in \Omega^k(M)$ can be thought of a tensor of type $(0,k)$, we can apply the Lie derivative $\mathcal{L}_X$ to it to obtain another tensor of type $(0,k)$, denoted by $\mathcal{L}_X\omega$. In fact, from Lemma 22.9 the tensor $\mathcal{L}_X\omega$ is easily seen to be alternating, and hence $\mathcal{L}_X\omega$ is another differential k-form. Explicitly, by (22.4) one has for $\omega \in \Omega^1(M)$ and $X, Y \in \mathfrak{X}(M)$ that

$$(\mathcal{L}_X\omega)(Y) = X(\omega(Y)) - \omega([X,Y]), \tag{22.7}$$

and more generally

$$(\mathcal{L}_X\omega)(X_1,\dots,X_k) = X(\omega(X_1,\dots,X_k)) - \sum_{i=1}^{k} \omega(X_1,\dots,[X,X_i],\dots X_k) \tag{22.8}$$

for $\omega \in \Omega^k(M)$ and $X, X_1, \dots, X_k \in \mathfrak{X}(M)$.

Here is how the Lie derivative behaves with respect to the wedge product.

LEMMA 22.22. *Let $\omega, \theta \in \Omega(M)$. Then*

$$\mathcal{L}_X(\omega \wedge \theta) = \mathcal{L}_X(\omega) \wedge \theta + \omega \wedge \mathcal{L}_X(\theta).$$

Proof. Apply Proposition 22.29 with $\mathcal{A}(\omega, \theta) = \omega \wedge \theta$. ■

Strictly speaking, this is a slight modification of Proposition 22.29 for differential forms instead of tensors, but the proof is exactly the same.

REMARK 22.23. The Lie derivative $\mathcal{L}_X$ gives us a way to "differentiate" a tensor field (or a differential form) with respect to a vector field, but it does *not* allow us differentiate a tensor field (or differential form) with respect to a single tangent vector. Indeed, the value of $\mathcal{L}_X A$ at a point p depends on the values of X on a whole neighbourhood of p, not just on $X(p)$. This is because $X \mapsto \mathcal{L}_X$ is *not* $C^\infty(M)$-linear. For instance, if A is a 1-form ω then for $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$,

$$\begin{aligned}(\mathcal{L}_{fX}\omega)(Y) &= (fX)\omega(Y) - \omega([fX,Y]) \\ &= fX(\omega(Y)) - \omega(f[X,Y] - Y(f)X) \\ &= f(\mathcal{L}_X\omega)(Y) + Y(f)\omega(X),\end{aligned}$$

where the second line used Problem D.5. The presence of the "error term" $Y(f)\omega(X)$ shows that $\mathcal{L}_{fX} \neq f\mathcal{L}_X$.

The first topic we will cover next semester will be *connections* in vector bundles and principal bundles. A connection on the tangent bundle TM induces a *covariant derivative* ∇_X on tensor fields associated to every vector field X. As we will see, a covariant derivative ∇_X will have the nice property that $\nabla_{fX} = f\nabla_X$. The downside it that it requires a choice of extra structure (namely, a connection). Meanwhile the Lie derivative is canonical. See Remark 31.7.

We conclude today's lecture with the analogue of a tensor derivation on differential forms.

DEFINITION 22.24. Let M be a smooth manifold and let $h \in \mathbb{Z}$. A **graded derivation of degree** h on M is a local operator $\zeta \colon \Omega(M) \to \Omega(M)$ of degree h such that if $\omega \in \Omega^k(M)$ and $\theta \in \Omega(M)$ then

$$\zeta(\omega \wedge \theta) = \zeta\omega \wedge \theta + (-1)^{hk}\omega \wedge \zeta\theta. \tag{22.9}$$

We refer to (22.9) as the **graded derivation property**. As in Remark 22.8, if ζ is a graded derivation and $U \subset M$ is open then the induced map $\zeta^U : \Omega(U) \to \Omega(U)$ is another graded derivation.

EXAMPLE 22.25. The Lie derivative $\mathcal{L}_X$ is a graded derivation of degree 0 by Lemma 22.22.

DEFINITION 22.26. A 1-form $\omega \in \Omega^1(U)$ is called **exact** if $\omega = df$ for some $f \in C^\infty(U)$.

Just as a tensor derivation is entirely determined by what it does to functions and vector fields, a graded derivation is entirely determined by what it does to functions and exact 1-forms.

PROPOSITION 22.27. *Suppose ζ and ξ are two graded derivations of the same degree h. If ζ and ξ agree on functions and exact 1-forms then $\zeta = \xi$.*

Proof. Since a graded derivation is a local operator, by Problem J.1 it is entirely determined by all its restrictions ζ^U where $U \subset M$ is the domain of a chart (U, x). By (21.1), any $\omega \in \Omega^k(U)$ can be written as a sum of elements of the form

$$f dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Since ζ is $\mathbb{R}$-linear, ζ^U is determined by what it does to such a term. But by repeatedly applying (22.9), we see that $\zeta^U(f dx^{i_1} \wedge \cdots \wedge dx^{i_k})$ is determined by $\zeta^U(f)$ and $\zeta^U(dx^{i_j})$. Thus if two graded derivations agree on functions and exact 1-forms then they are identical. ∎

REMARK 22.28. It follows directly from the definition that the space of graded derivations of a given degree h forms a vector space (with the caveat that the zero operator has to be regarded as a graded derivation of all degrees). Moreover if ζ and ξ are two graded derivations of degrees h and k respectively then

Exercise: Check this!

$$\zeta \circ \xi - (-1)^{hk} \xi \circ \zeta$$

is another graded derivation of degree $h + k$.

Next lecture we will investigate a particularly important graded derivation of degree +1: the *exterior differential*. This will lead us to the *de Rham cohomology* of a manifold.

Bonus Material for Lecture 22

In this bonus section we prove Proposition 22.14. Let us temporarily denote the right-hand side of (22.5) by $\widetilde{\mathcal{L}}_X A$. Thus to prove Proposition 22.14 we must show that for any tensor A, one has:

$$\mathcal{L}_X A = \widetilde{\mathcal{L}}_X A. \qquad (22.10)$$

The proof of (22.10) uses the following result, which will be useful elsewhere.

Proposition 22.29. *Let* (h_0,k_0), (h_1,k_1) *and* (h_2,k_2) *be three pairs of non-negative integers. Suppose we are given a* $C^\infty(M)$*-bilinear operator*

$$\mathcal{A}\colon \mathscr{T}^{h_0,k_0}(M)\times\mathscr{T}^{h_1,k_1}(M)\to\mathscr{T}^{h_2,k_2}(M).$$

Assume in addition that $\mathcal{A}$ *has the property that if* $\varphi\colon U\to V$ *is a local diffeomorphism between open sets of* M *then the corresponding local operators*

$$\varphi^*\big(\mathcal{A}^V(A,B)\big)=\mathcal{A}^U(\varphi^*A,\varphi^*B). \tag{22.11}$$

Then for every vector field X *on* M*, one has*

$$\widetilde{\mathcal{L}}_X\big(\mathcal{A}(A,B)\big)=\mathcal{A}\big(\widetilde{\mathcal{L}}_XA,B\big)+\mathcal{A}\big(A,\widetilde{\mathcal{L}}_XB\big).$$

The proof of Proposition 22.29 is on Problem Sheet I.

Proof of Proposition 22.14. Since $\Phi_t^*=((\Phi_t)^{-1})_*=(\Phi_{-t})_*$ it follows from the definitions that $\widetilde{\mathcal{L}}_X=\mathcal{L}_X$ on functions and vector fields. Thus if we can show that $\widetilde{\mathcal{L}}_X$ is a tensor derivation, it will follow from the uniqueness part of of Theorem 22.12 that $\mathcal{L}_X=\widetilde{\mathcal{L}}_X$.

Thus we must show that $\widetilde{\mathcal{L}}_X$ is a derivation that commutes with contractions. For this we use Proposition 22.29. Taking $\mathcal{A}(A,B):=A\otimes B$ shows that $\widetilde{\mathcal{L}}_X$ is a derivation (note (22.11) is satisfied by (21.2)). Similarly taking for instance $\mathcal{A}(A,B)=C^{1,1}(A\otimes B)$ shows that

$$X(\omega(Y))=\big(\widetilde{\mathcal{L}}_X\omega\big)(Y)+\omega\big(\widetilde{\mathcal{L}}_XY\big).$$

More generally, taking $\mathcal{A}(A,B)=C^{h,k}(A\otimes B)$ shows that $\widetilde{\mathcal{L}}_X$ commutes with $C^{h,k}$. This completes the proof. ∎

From now on, we will just write $\mathcal{L}_X$ for both the operator $\mathcal{L}_X$ from Theorem 22.12 and the operator defined in (22.5).

LECTURE 23

The Exterior Differential

Differential forms are typically more important than tensors in geometry for two key reasons:

- We can differentiate them.
- We can integrate them.

We will discuss differentiation in this lecture. Integration will be covered next week. Let us motivate this by considering the special case of a 0-form, i.e. a smooth function. If $f \in C^\infty(M)$ then we have already defined the differential of f as the exact 1-form $df \in \Omega^1(M)$. Generalising this, we will define the differential of a k-form ω to be an $(k+1)$-form $d\omega$.

Here is the main result of today's lecture.

THEOREM 23.1 (The exterior differential). *Let M be a smooth manifold. There is a unique graded derivation $d\colon \Omega(M) \to \Omega(M)$ of degree 1 that extends the operator $f \mapsto df$ on $\Omega^0(M)$ and satisfies*

$$d \circ d = 0. \tag{23.1}$$

We call d the **exterior differential operator** and refer to $d\omega$ as the **exterior differential** of ω (often shortened to the just "the **differential** of ω"). The importance of the condition (23.1) will be explained shortly.

> **Warning!** The words "derivative" and "differential" are often used synonymously, but – at least as far as this course is concerned – there is an important difference. As explained in Remark 5.3, if f is a smooth function then the derivative Df and the differential df differ only by the dash-to-dot map. However if $\omega \in \Omega^k(M)$ for $k \geq 1$ then the derivative $D\omega$ and the differential $d\omega$ are very different objects. The latter is an element of $\Omega^{k+1}(M)$, whereas the former is a linear map $D\omega\colon TM \to T \bigwedge T^*M$!.

Our proof of Theorem 23.1 will construct d in coordinates. We will give a coordinate-free expression for d at the end of the lecture in Theorem 23.13.

Proof of Theorem 23.1. We prove the result in three steps.

1. We first deal with uniqueness. This is immediate from Proposition 22.27, since we have specified what d does to functions and to exact 1-forms (namely, $d(df) = 0$).

2. To construct d it suffices by Problem J.1 to show that for any chart (U, x) there is a an operator $d^U : \Omega^k(U) \to \Omega^{k+1}(U)$ satisfying the requirements of the theorem, which moreover has the property that if (V, y) is another chart with $U \cap V \neq \emptyset$ then

$$d^U = d^V, \qquad \text{on } U \cap V. \tag{23.2}$$

In this step we define an operator $d^U : \Omega^k(U) \to \Omega^{k+1}(U)$ which satisfies the requirements of the theorem. In the last step we will show that (23.2) is satisfied.

To ease the notation we adopt the following shorthand: if $I = (i_1, \dots, i_k)$ is a subset of $\{1, \dots, m\}$ with $i_j < i_{j+1}$ for each $j = 1, \dots, k-1$ then we set:

$$dx^I := dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

We also define $dx^I := 1$ if $I = \emptyset$. Thus any $\omega \in \Omega(U)$ can be written as a sum

$$\omega = \sum_I f_I \, dx^I.$$

We define

$$d^U \omega := \sum_I df_I \wedge dx^I.$$

If $\omega \in \Omega^k(U)$ then $d^U\omega \in \Omega^{k+1}(U)$. Moreover d^U is obviously $\mathbb{R}$-linear and satisfies the first bullet point by definition. Thus we need only check that $d^U(d^U\omega) = 0$ and that (22.9) holds.

To establish (22.9), we may assume $\omega = f\, dx^I$ and $\theta = g\, dx^J$. If either I or J are empty then (22.9) follows from the Leibniz rule $d(fg) = fdg + gdf$. In the general case we argue as follows. Assume ω has degree k. Then:

$$\begin{aligned}
d^U(\omega \wedge \theta) &= d^U(fg\, dx^I \wedge dx^J) \\
&= d(fg) \wedge dx^I \wedge dx^J \\
&= (fdg + gdf) \wedge dx^I \wedge dx^J \\
&= \big(df \wedge dx^I\big) \wedge (g\, dx^J) + (-1)^k (f\, dx^I) \wedge \big(dg \wedge dx^J\big) \\
&= d^U\omega \wedge \theta + (-1)^k \omega \wedge d^U\theta.
\end{aligned}$$

To see that $d^U(d^U\omega) = 0$ we first show that $d^U(df) = 0$ for any function $f \in C^\infty(U)$. For this write $df = \frac{\partial f}{\partial x^i}\, dx^i$ (cf. Definition 8.4). Then

$$d^U(df) = \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j,$$

where we abbreviate $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial}{\partial x^j}\big(\frac{\partial f}{\partial x^i}\big)$. But by elementary calculus, $\frac{\partial^2 f}{\partial x^i \partial x^j}$ is symmetric in i and j, whereas $dx^i \wedge dx^j$ is anti-symmetric. Thus the sum cancels.

Next, if $f, g \in C^\infty(U)$ then using the graded derivation property we have

$$d^U(df \wedge dg) = d^U(df) \wedge dg - df \wedge d^U(dg) = 0,$$

and more generally if $f, f_1, \dots f_k \in C^\infty(U)$ are any functions then

$$d^U(df \wedge df_1 \wedge \cdots \wedge df_k) = 0.$$

Applying this with $f_j = x^{i_j}$ and using $\mathbb{R}$-linearity shows that $d^U(d^U\omega) = 0$ for any $\omega \in \Omega^k(U)$.

3. Suppose (V, y) is another chart defined on an open set such that $U \cap V \neq \emptyset$. By Proposition 22.27 applied to the graded derivations d^U and d^V restricted $\Omega(U \cap V)$, to prove (23.2) it suffices to show they agree on functions and on exact 1-forms. But this is clear: if $f \in C^\infty(U \cap V)$ then one has

$$d^U f = d^V f = df,$$

and the argument we just gave above showed that

$$d^U(df) = d^V(df) = 0.$$

This completes the proof. ■

Definition 23.2. A differential form ω is said to be **closed** if $d\omega = 0$. A differential form ω is said to be **exact** if $\omega = d\theta$ for some θ (this extends Definition 22.26 to k-forms for $k > 1$. Since $d \circ d = 0$, any exact form is closed, but the converse is typically false. One denotes the quotient vector space by

$$H^k_{\mathrm{dR}}(M) := \frac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}},$$

where the "dR" stands for "de Rham". An element of $H^k_{\mathrm{dR}}(M)$ is written as $[\omega]$, where ω is a closed k-form. Thus by definition

$$[\omega] = [\omega + d\theta].$$

We call $H^k_{\mathrm{dR}}(M)$ the **kth de Rham cohomology group of M**.

Even though the de Rham groups are actually vector spaces (and not just abelian groups), it is still common to refer to them as the "de Rham cohomology groups". of M.

Do not be scared by the word "cohomology" if you are not familiar with algebraic topology. As far as this course is concerned, all that is important is that $H^k_{\mathrm{dR}}(M)$ is a quotient vector space.

In Lecture 27 we will see that the de Rham groups are a topological invariant of M. For now we content ourselves with computing the zeroth groups.

Lemma 23.3. *If M is connected then $H^0_{\mathrm{dR}}(M) \cong \mathbb{R}$.*

Proof. Since there are no differential forms of negative degree, there are in particular no exact 0-forms. Thus $H^0_{\mathrm{dR}}(M)$ is simply the space of closed 0-forms. A function f satisfies $df = 0$ if and only if f is locally constant. ■

The same argument shows that if M has n connected components then $H^0_{\mathrm{dR}}(M) \cong \mathbb{R}^n$.

Lemma 23.4. *Let $\varphi\colon M \to N$ be a smooth map and let $\omega \in \Omega(N)$. Then*

$$\varphi^*(d\omega) = d(\varphi^*\omega).$$

Thus φ^* commutes with the exterior differential operators:

$$\begin{array}{ccc} \Omega(M) & \xrightarrow{d} & \Omega(M) \\ \uparrow{\scriptstyle \varphi^*} & & \uparrow{\scriptstyle \varphi^*} \\ \Omega(N) & \xrightarrow[d]{} & \Omega(N) \end{array}$$

Proof. We first prove the lemma for the special case of 0-forms, i.e functions. Let $f \in C^\infty(N)$ and $X \in \mathfrak{X}(M)$. Then manipulating the definitions gives us

$$\begin{aligned} \varphi^*(df)(X) &= df(D\varphi(X)) \\ &= D\varphi(X)(f) \\ &= X(f \circ \varphi) \\ &= d(f \circ \varphi)(X) \\ &= d(\varphi^* f)(X). \end{aligned}$$

Each line in this block of equations is a function on M.

For the general case, by arguing as in the proof of Theorem 23.1 it suffices to work in a chart domain and assume ω is of the form $f\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. Then since we already know that φ^* is an algebra homomorphism (Lemma 22.21) and we already proved the result for functions:

$$\begin{aligned} \varphi^*(d\omega) &= \varphi^*\big(df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\big) \\ &= \varphi^*(df) \wedge \varphi^*(dx^{i_1}) \wedge \cdots \wedge \varphi^*(dx^{i_k}) \\ &= d(\varphi^*(f)) \wedge d(\varphi^* x^{i_1}) \wedge \cdots \wedge d(\varphi^* x^{i_k}) \\ &= d\big(\varphi^*(f) \wedge d(\varphi^* x^{i_1}) \wedge \cdots \wedge d(\varphi^* x^{i_k})\big) \\ &= d(\varphi^* \omega). \end{aligned}$$

This completes the proof. ∎

Corollary 23.5. *If $\varphi \colon M \to N$ is a smooth map then φ^* induces a well-defined map (also denoted by) $\varphi^* \colon H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$ via:*

$$[\omega] \mapsto [\varphi^* \omega].$$

Proof. By Lemma 23.4, φ^* maps closed forms to closed forms and exact forms to exact forms. ∎

We now relate the Lie derivative to the exterior differential.

Proposition 23.6. *Let M be a smooth manifold and fix $X \in \mathfrak{X}(M)$. Then d commutes with $\mathcal{L}_X$:*

$$\begin{array}{ccc} \Omega(M) & \xrightarrow{d} & \Omega(M) \\ {\scriptstyle \mathcal{L}_X}\downarrow & & \downarrow{\scriptstyle \mathcal{L}_X} \\ \Omega(M) & \xrightarrow[d]{} & \Omega(M) \end{array}$$

Proof. We first prove the result for functions. For a function f and a vector field Y, one has by (22.7) that

$$\begin{aligned}\mathcal{L}_X(df)(Y) &= X(Y(f)) - [X,Y](f)\\ &= Y(X(f))\\ &= d(X(f))(Y)\\ &= d(\mathcal{L}_X f)(Y).\end{aligned}$$

Since Y was arbitrary, this shows that $\mathcal{L}_X(df) = d(\mathcal{L}_X f)$. For the general case, by Remark 22.28, $d \circ \mathcal{L}_X - \mathcal{L}_X \circ d$ is a graded derivation of degree $+1$. By Proposition 22.27, if we can show it vanishes on functions and exact 1-forms then it is identically zero. We just did the case for functions, and for an exact 1-form we have

$$\begin{aligned}d(\mathcal{L}_X(df)) - \mathcal{L}_X(d(df)) &= d(\mathcal{L}_X(df)) - 0\\ &= d(d(\mathcal{L}_X f)\\ &= 0.\end{aligned}$$

This completes the proof. ∎

We now move onto defining a third operator on $\Omega(M)$, called the interior product. As usual, this begins at the linear algebra level.

Definition 23.7. Let V be a vector space, and fix $v \in V$. Define $i_v \colon \bigwedge^k V^* \to \bigwedge^{k-1} V^*$ by declaring that on decomposable elements $\lambda^1 \wedge \cdots \wedge \lambda^k$ that

$$i_v(\lambda^1 \wedge \cdots \wedge \lambda^k) = \sum_{i=1}^{k} (-1)^{i+1} \lambda^i(v) \cdot \lambda^1 \wedge \cdots \wedge \lambda^{i-1} \wedge \lambda^{i+1} \wedge \cdots \wedge \lambda^k,$$

and then extending by linearity.

Straight from the definition, we see that:

Lemma 23.8. *Let $\omega \in \bigwedge^h V^*$ and $\theta \in \bigwedge^k V^*$. Then*

$$i_v(\omega \wedge \theta) = i_v\omega \wedge \theta + (-1)^h \omega \wedge i_v\theta.$$

An alternative characterisation of the interior product is given by the following statement, whose proof is on Problem Sheet J.

Lemma 23.9. *Let $v \in V$ and let $\omega \in \bigwedge^k V^*$. If we regard both ω and $i_v\omega$ as elements of $\mathrm{Alt}_k(V)$ and $\mathrm{Alt}_{k-1}(V)$ respectively (via Proposition 19.24), then*

$$(i_v\omega)(v_1, \dots, v_{k-1}) = \omega(v, v_1, \dots, v_{k-1}).$$

Note this shows that $i_v \circ i_v = 0$. We now transfer this to manifolds.

Proposition 23.10. *Let M be a smooth manifold and let $X \in \mathfrak{X}(M)$. There is a graded derivation $i_X \colon \Omega(M) \to \Omega(M)$ of degree -1 defined by declaring that if $\omega \in \Omega^k(M)$ for $k \geq 1$ then*

$$(i_X\omega)(X_1, \dots, X_{k-1}) := \omega(X, X_1, \dots, X_{k-1}),$$

for $X_1, \dots, X_{k-1} \in \mathfrak{X}(M)$. Meanwhile for $k = 0$ we set $i_X f := 0$. One has $i_X \circ i_X = 0$. This is the unique graded derivation of degree -1 such that $i_X\omega = \omega(X)$ for ω a 1-form and $i_X f = 0$ for f a function.

The proof is immediate; uniqueness follows from Proposition 22.27.

Corollary 23.11. *Let $X, Y \in \mathfrak{X}(M)$. Then*

$$i_{[X,Y]} = \mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X$$

as operators on $\Omega(M)$.

Proof. Both sides are graded derivations of degree -1 by Remark 22.28. Thus it suffices to check on functions and exact 1-forms. For functions both sides are zero. For an exact 1-form df this follows from (22.7) applied with $\omega = df$. ∎

The exterior differential, Lie derivative, and interior product are related by the following useful formula that goes by the somewhat flamboyant name of **Cartan's Magic Formula**.

Theorem 23.12 (Cartan's Magic Formula). *Let $X \in \mathfrak{X}(M)$. Then*

$$\mathcal{L}_X = d \circ i_X + i_X \circ d.$$

Proof. Again, both sides are graded derivations of degree 0 by Remark 22.28. Thus by Proposition 22.27 we need only check they agree on functions and exact 1-forms. On functions this follows from Lemma 10.2 and on exact 1-forms this was Proposition 23.6. ∎

We conclude this lecture by using Cartan's Magic Formula to give a coordinate free definition of the exterior differential d.

Theorem 23.13. *Let M be a smooth manifold, $\omega \in \Omega^k(M)$ and $X_0, \dots X_k \in \mathfrak{X}(M)$. Then:*

$$\begin{aligned} d\omega(X_0, \dots, X_k) = &\sum_{i=0}^{k} (-1)^i X_i\big(\omega(X_0, \dots, \widehat{X}_i, \dots, X_k)\big) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k). \end{aligned}$$

Here and elsewhere, the caret $\widehat{X}_i$ means that the X_i term should be omitted.

Proof. One has $d\omega(X_0, \dots, X_k) = i_{X_0}(d\omega)(X_1, \dots, X_k)$, which by Cartan's Magic Formula is equal to

$$(\mathcal{L}_{X_0}\omega)(X_1, \dots, X_k) - d(i_{X_0}\omega)(X_1, \dots, X_k). \tag{23.3}$$

We now argue by induction on k. If $k = 1$ then by (22.7) this becomes

$$\begin{aligned} &\mathcal{L}_{X_0}(\omega(X_1)) - \omega([X_0, X_1]) - d(\omega(X_0))(X_1) \\ &\qquad = X_0(\omega(X_1)) - X_1(\omega(X_0)) - \omega([X_0, X_1]), \end{aligned}$$

which is what we want. Now assume $k \ge 2$ and that the result is true for all forms of degree $k - 1$. By (22.8) the first term of (23.3) is equal to

$$X_0(\omega(X_1, \dots, X_k)) - \sum_{i=1}^{k} \omega(X_1, \dots, [X_0, X_i], \dots, X_k).$$

By induction, we have that $d(i_{X_0}\omega)(X_1,\dots,X_k)$ is equal to

$$\sum_{i=1}^{k}(-1)^{i-1}X_i\big((i_{X_0}\omega)(X_1,\dots,\widehat{X}_i,\dots,X_k)\big) + \sum_{1\le i<j\le k}(-1)^{i+j}\big(i_{X_0}(\omega)([X_i,X_j],X_1,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_k)\big).$$

Putting this into (23.3) and checking the signs carefully gives the desired result. ■

LECTURE 24

Orientations and Manifolds With Boundary

We now move onto a somewhat different topic and discuss *orientations* of vector bundles. This is the first of two preliminary topics we need to cover (the second is manifolds with boundary) before we can state and prove *Stokes' Theorem*, which is about integrating differential forms on oriented manifolds with boundary.

As usual, let us start at the level of linear algebra. Of course, you have all known since kindergarten what an orientation of a vector space is, but perhaps you haven't seen it in the "right" language yet.

DEFINITION 24.1. Let V be a one-dimensional vector space. Then $V \setminus \{0\}$ has two components. An **orientation** of V is a choice of one of these components, which one then labels as "positive". The other component is then labelled "negative". A **positive basis** of V is a choice of any non-zero vector belonging to the positive component. A **negative basis** of V is a choice of any non-zero vector belonging to the negative component.

EXAMPLE 24.2. The **standard orientation** of $\mathbb{R}$ is given by declaring that the positive numbers are (surprise!) the positive component of $\mathbb{R} \setminus \{0\}$.

DEFINITION 24.3. Let V be a vector space. We will use the notation $\mathsf{det}\, V$ to mean $\bigwedge^n V$ where $n = \dim V$. One calls $\mathsf{det}\, V$ the **determinant** of V. From Lemma 19.27, the space $\mathsf{det}\, V$ is a one-dimensional vector space. Moreover if (e_i) is a basis for V then $e_1 \wedge \cdots \wedge e_n$ is a basis of $\mathsf{det}\, V$.

DEFINITIONS 24.4. Let V be a vector space of positive dimension. An **orientation** on V is a choice of orientation on $\mathsf{det}\, V$. Thus there are exactly two orientations. An **oriented vector space** is a vector space together with a choice of orientation. A basis (e_i) of an oriented vector space V is said to be **positive** if $e_1 \wedge \cdots \wedge e_n$ is a positive basis of $\mathsf{det}\, V$. If instead $e_1 \wedge \cdots \wedge e_n$ is a negative basis of $\mathsf{det}\, V$, then (e_i) is a negative basis of V.

EXAMPLE 24.5. If e_i denotes the standard ith basis vector in $\mathbb{R}^m$ then the **standard orientation** of $\mathbb{R}^m$ is given by declaring that $e_1 \wedge \cdots \wedge e_n$ is a positive basis of $\det \mathbb{R}^m$. Thus (e_i) is a positive basis of $\mathbb{R}^m$.

You are probably more used to thinking of the determinant of a linear transformation, rather than the determinant of a vector space itself. The motivation for this terminology goes as follows. Suppose that V and W are vector spaces of the same dimension n. A linear map $\ell \colon V \to W$ induces a linear map $\Phi_\ell \colon \mathsf{det}\, V \to \mathsf{det}\, W$, defined explicitly by

$$\Phi_\ell(v_1 \wedge \cdots \wedge v_n) := \ell v_1 \wedge \cdots \wedge \ell v_n.$$

This is a linear map between two one-dimensional vector spaces, and hence is multiplication by a scalar. This scalar is non-zero if and only if ℓ is an isomorphism. In general the precise value of this scalar depends on a choice of basis of V and W, but the linear map Φ_ℓ itself clearly does not. If ℓ is an isomorphism and V and W are oriented, then we say that ℓ is **orientation-preserving** if Φ_ℓ maps the positive component of $\mathsf{det}\, V$ to the positive component of $\mathsf{det}\, W$. Otherwise ℓ is **orientation-reversing**.

If $V = W$ and we choose the same basis for both the domain V and the target V then the value of the scalar is independent of the basis. In this case, it is common to call the scalar the **determinant of ℓ**. Explicitly, if (e_i) is our chosen basis then

$$\Phi_\ell(e_1 \wedge \cdots \wedge e_n) = (\det \ell) \cdot e_1 \wedge \cdots \wedge e_n.$$

It is convenient to extend the definition of Φ_ℓ to all linear maps by declaring that if $\ell \colon V \to W$ is a linear map with $\dim V \neq \dim W$ then $\Phi_\ell \colon \mathsf{det}\, V \to \mathsf{det}\, W$ is the zero map.

Exercise: Check this new definition of determinant coincides with the one you are used to from linear algebra. Use this to give slicker proofs of everything you learnt in your linear algebra course.

REMARK 24.6. If V is a vector space then an orientation on V canonically determines an orientation on the dual space V^* by declaring that the dual basis to a positive basis is positive.

Now we move to vector bundles. A vector bundle of rank one is often called a **line bundle**.

WARNING 24.7. This terminology is also popular in complex geometry and algebraic geometry too. But typically there people are working with *complex* vector bundles, not real vector bundles. A complex line bundle is (in particular) a two-dimensional real vector bundle. So when taken out of context, beware that the phrase “line bundle” may either refer to a one-dimensional real bundle or a one-dimensional complex bundle.

DEFINITION 24.8. Let E be a vector bundle over M. The **determinant line bundle** associated to E is the vector bundle $\mathsf{det}\, E \to M$ of rank one whose fibre over $x \in M$ is $\mathsf{det}\, E_p$.

Roughly speaking, a vector bundle $\pi \colon E \to M$ is oriented if each fibre E_p is given an orientation which depends smoothly on p. To make this precise, we prove the following result.

PROPOSITION 24.9. *Let $\pi \colon E \to M$ be a vector bundle of rank n over M. The following are equivalent:*

(i) *There is a smooth nowhere vanishing section $\mu \in \Gamma(\mathsf{det}\, E^*)$.*

(ii) *It is possible to reduce the structure group of E from $\mathrm{GL}(n)$ to $\mathrm{GL}^+(n)$.*

(iii) *The bundle* $\mathsf{det}\, E^* \to M$ *is a trivial bundle.*

Proof. We prove the result in three steps.

1. We first prove (i) implies (ii). Let $\{(U_a, \varepsilon_a) \mid a \in A\}$ be a vector bundle atlas for E. We may assume each U_a is connected. For each $a \in A$, we obtain a local frame $(e_1^a, \dots, e_n^a)$ for E over U_a via Lemma 20.6. Since μ is non-vanishing, for each $a \in A$ the function

Explicitly, this is the function given by $p \mapsto \mu_p(e_1(p), \dots, e_n(p))$, where we view $\mu_p \in \bigwedge^n E_p^*$ as an element of $\mathrm{Alt}_n(E_p)$ via Proposition 19.24.

$$\mu(e_1^a, \dots, e_n^a) \colon U_a \to \mathbb{R} \tag{24.1}$$

is either everywhere positive or everywhere negative. If for a given a one has that (24.1) is positive, we do nothing. If instead for a given a one has that (24.1) is negative, we replace the local frame $(e_1^a, \dots, e_n^a)$ with the new one $(-e_1^a, \dots, e_n^a)$, and then, using Lemma 20.6 again, replace ε_a with the vector bundle chart corresponding to this new frame. Having done this, we may assume that (24.1) is a positive function for every $a \in A$.

We claim that our new bundle atlas, which we still denote by $\{(U_a, \varepsilon_a) \mid a \in A\}$, has its structure group contained in $\mathrm{GL}^+(n)$. Indeed, if $U_a \cap U_b \neq \emptyset$ then we can write

$$e_j^b(p) = f_j^i(p) e_i^b(p), \qquad \forall\, p \in U_a \cap U_b,$$

for smooth functions $f_j^i \colon U_a \cap U_b \to \mathbb{R}$. In fact, unravelling the definition shows that $f_j^i(p)$ is the (i,j)th entry of the transition matrix $\varepsilon_{ba}(p)$. Thus for any $p \in U_a \cap U_b$, by Corollary 19.26 we have

$$\mu(e_1^a, \dots, e_n^b)(p) = \det \varepsilon_{ba}(p) \cdot \mu(e_1^a, \dots, e_n^a)(p).$$

Thus $\det \varepsilon_{ba}(p) > 0$ for all $p \in U_a \cap U_b$, which is what we wanted to prove.

2. Now we show that (ii) implies (i). For this we start with a vector bundle atlas $\{(U_a, \varepsilon_a) \mid a \in A\}$ with structure group in $\mathrm{GL}^+(n)$ and we have to build a section μ. Let $(e_a^1, \dots, e_a^n)$ denote the dual frame to the local frame $(e_1^a, \dots, e_n^a)$ associated to ε_a, and let $\{\kappa_a \mid a \in A\}$ denote a partition of unity subordinate to the open cover $\{U_a \mid a \in A\}$. We now define

$$\mu \colon M \to \mathsf{det}\, E^*, \qquad \mu := \sum_{a \in A} \kappa_a\, e_a^1 \wedge \dots \wedge e_a^n.$$

We need only check that μ is nowhere vanishing. Fix $p \in M$. Then there exists $b \in A$ such that $p \in U_b$. We evaluate μ on $e_1^b \wedge \dots \wedge e_n^b$ at p to discover

$$\mu_p\big(e_1^b(p), \dots, e_n^b(p)\big) = \sum_{a \in A} \Big(\det \varepsilon_{ba}(p)\Big) \kappa_a(p),$$

which is positive as desired.

3. Finally, since $\mathsf{det}\, E^*$ is a one-dimensional vector bundle, it is trivial if and only if it admits a nowhere vanishing section (this is a special case of Corollary 20.7). Thus (i) and (iii) are equivalent by Lemma 20.6. This completes the proof. ∎

We now use Proposition 24.9 to define precisely what it means for a vector bundle to be orientable.

Definition 24.10. Let $\pi\colon E \to M$ be a vector bundle. We say that E is **orientable** if either of the three conditions from Proposition 24.9 is satisfied.

Assume E is orientable. Define an equivalence relation on the space of non-vanishing sections of $\mathsf{det}\, E^*$ by declaring that μ_1 and μ_2 are equivalent if there exists a strictly positive smooth function f on M such that $\mu_2 = f\mu_1$. We call an equivalence class an **orientation** of E. An **oriented vector bundle** is an orientable vector bundle E together with a choice of orientation. If M is connected then there are exactly two possible orientations.

Notation. We typically denote an orientation with the symbol $\mathfrak{o}$; thus saying $\mu \in \mathfrak{o}$ means that μ is a non-vanishing section of $\mathsf{det}\, E^*$ belonging to the orientation $\mathfrak{o}$.

Definition 24.11. Suppose $(E, \mathfrak{o})$ is an oriented vector bundle. We assign an orientation to each vector space E_p as follows: a basis (v_i) of E_p is positive if and only if $\mu_p(v_1, \dots, v_n) > 0$ for some (and hence any) $\mu \in \mathfrak{o}$.

This clarifies the intuitive idea that an orientation of a vector bundle is an orientation of each fibre that depends smoothly on p. Similarly we say a local frame (e_i) is **positively oriented** if the function $\mu(e_1, \dots, e_n)$ is positive.

Specialising this to our favourite type of vector bundle tells what it means for a manifold to be orientable.

Definition 24.12. A manifold M is said to be **orientable** if $TM \to M$ is an orientable vector bundle.

In this case since $\Gamma(\det T^*M) = \Omega^m(M)$ is just the top-dimensional differential forms, an non-vanishing section of $\mathsf{det}\, T^*M$ is simply a nowhere vanishing differential m-form. This has its own special name:

Definition 24.13. A **volume form** on a smooth manifold M is a nowhere-vanishing differential m-form.

A manifold together with a choice of orientation $\mathfrak{o}$ is called an **oriented manifold**. By a slight abuse of notation we often refer to $\mathfrak{o}$ as an orientation of M itself, rather than an orientation of TM.

Definition 24.14. Let $(M, \mathfrak{o}_M)$ and $(N, \mathfrak{o}_N)$ be two oriented manifolds of the same dimension m. Suppose $\varphi\colon M \to N$ is a diffeomorphism. Let $\mu_M \in \mathfrak{o}_M$ and $\mu_N \in \mathfrak{o}_N$. Then $\varphi^*\mu_N = f\mu_M$ for a smooth nowhere vanishing function $f \in C^\infty(M)$. We say that φ is **orientation preserving** if f is everywhere positive and **orientation reversing** if f is everywhere negative. This does not depend on the choice of μ_M and μ_N.

In symbols we write this as $\varphi^*\mathfrak{o}_N = \pm\mathfrak{o}_M$.

Note that if M and N are not connected, it may be the case that φ is neither orientation preserving or reversing.

Definition 24.15. As a special case of this, a chart $x\colon U \to \mathcal{O}$ on an oriented manifold M is said to be **positively oriented** if x is an orientation preserving diffeomorphism between manifolds U and $\mathcal{O}$ (here U inherits the orientation from M and $\mathcal{O}$ inherits the standard orientation from $\mathbb{R}^m$).

We conclude this lecture by restating Proposition 24.9 in the special case of a tangent bundle, since this will more convenient to refer back to in the future.

Corollary 24.16 (Orientability of manifolds). *Let M be a smooth manifold. The following are equivalent:*

(i) *M admits a volume form.*

(ii) *There exists a smooth atlas $\{x_a \colon U_a \to \mathcal{O}_a \mid a \in A\}$ for M such that whenever $U_a \cap U_b \neq \emptyset$,*

$$\det D(x_a \circ x_b^{-1})(x_b(p)) > 0, \qquad \forall\, p \in U_a \cap U_b. \tag{24.2}$$

We call such an atlas a **positively oriented** *smooth atlas. Note that every chart x_a is then positively oriented.*

(iii) *The determinant line bundle of the cotangent bundle T^*M is a trivial bundle.*

On Problem Sheet J you will see some examples of orientable and non-orientable manifolds.

Let us now move on to defining manifolds with *boundary*. A serious defect of differential geometry so far (at least as we have defined it) is that many natural and interesting compact subsets of Euclidean space fail to be manifolds, and thus none of our results are applicable to them.

Two key examples are the *closed* unit ball D^m, or a closed interval $[a,b] \subset \mathbb{R}$. Neither of these are locally Euclidean spaces (of dimension m and 1 respectively), since points on their boundary do not have neighbourhoods that are homeomorphic to open subsets of $\mathbb{R}^m$ (or $\mathbb{R}$). But note in both cases their interior *is* a smooth manifold of the desired dimension. For the closed ball D^m, the interior is B^m which is an m-dimensional manifold, and for the interval $[a,b]$, the interior (a,b) is a one-dimensional manifold. Moreover the boundary in both cases is an $(m-1)$-dimensional manifold: for the closed ball, $\partial D^m = S^{m-1}$, and $\partial[a,b] = \{a,b\}$.

Warning 24.17. In Lecture 1 (cf. Remark 1.18) we noted that manifold theory had re-purposed the words "open" and "closed" and given them their own meanings, which in many cases were *not* the same as the topological definitions of open and closed. In these notes we elected not to use the "manifold" meanings, and thus for us the words "open" and "closed" should always be taken to have their standard topological meaning.

Unfortunately the same is true of the word "boundary". As we will shortly see, the "boundary" of a manifold does *not* necessarily coincide with the topological definition of the word boundary. This time we will favour the manifold definition of the word, and thus when we write ∂M this is always taken to mean the "manifold" definition of the boundary (which we will shortly introduce). We will use the phrase *topological boundary* to denote the boundary in the sense of point-set topology, and use the notation ∂^{top}. Thus for any subset Y of a topological space X,

$$\partial^{\text{top}} Y \coloneqq \overline{Y} \setminus \operatorname{int}(Y).$$

We will see several examples below where $\partial M \neq \partial^{\text{top}} M$ for M a manifold with boundary.

Definitions 24.18. A pair of **half-spaces** of $\mathbb{R}^m$ is specified by two things: a linear functional $\lambda \in (\mathbb{R}^m)^*$, and a real number c, which gives us the

$$\mathbb{R}^m_{\lambda \geq c} \coloneqq \{p \in \mathbb{R}^m \mid \lambda(p) \geq c\},$$
$$\mathbb{R}^m_{\lambda \leq c} \coloneqq \{p \in \mathbb{R}^m \mid \lambda(p) \leq c\}.$$

In a similar way we have **open half-spaces**

$$\mathbb{R}^m_{\lambda > c} \coloneqq \{p \in \mathbb{R}^m \mid \lambda(p) > c\},$$
$$\mathbb{R}^m_{\lambda < c} \coloneqq \{p \in \mathbb{R}^m \mid \lambda(p) < c\}.$$

The intersection

$$\mathbb{R}^m_{\lambda = c} = \mathbb{R}^m_{\lambda \geq c} \cap \mathbb{R}^m_{\lambda \leq c} = \{p \in \mathbb{R}^m \mid \lambda(p) = c\}$$

is called a **hyperplane**.

Example 24.19. Take $\lambda \colon \mathbb{R}^m \to \mathbb{R}$ denote the linear functional u^1, i.e.

$$\lambda(u^1, \dots, u^m) = u^1.$$

We define the **standard half-spaces** to be

$$\mathbb{R}^m_{u^1 \geq 0} \coloneqq \left\{(u^1, \dots, u^m) \in \mathbb{R}^m \mid u^1 \geq 0\right\},$$
$$\mathbb{R}^m_{u^1 \leq 0} \coloneqq \left\{(u^1, \dots, u^m) \in \mathbb{R}^m \mid u^1 \leq 0\right\},$$

which we will typically abbreviate by $\mathbb{R}^m_+$ and $\mathbb{R}^m_-$ respectively.

Warning 24.20. It is more common in the literature to define the "standard" half-spaces using $\lambda = u^m$ instead. For instance, $\mathbb{R}^m_{u^m \geq 0}$ is the "upper half-plane" $\mathbb{H}^m$ usually used in hyperbolic geometry. We prefer to use the standard half-spaces from Example 24.19 for two reasons:

(i) As we will see next lecture, using $\mathbb{R}^m_-$ as our "model" half-space leads to simpler formulae when discussing integration. The reason for this is explained in Problem J.7.

(ii) The symbol $\mathbb{H}^m$ is usually understood to denote the half-space $\mathbb{R}^m_{u^m \geq 0}$ which *in addition* has been endowed with its standard *hyperbolic metric* (a topic we will come back to extensively in Differential Geometry II). Since we are not making any statements about metrics here, to avoid confusion we prefer not to use the symbol $\mathbb{H}^m$.

Of course, at the end of the day it is essentially irrelevant which half-space we choose as our "standard" one; they all give rise to the same notion. We could equally as well set the entire theory up with our "standard" half-space being $\mathbb{R}^m_{\lambda \geq \pi}$, where

$$\lambda(u^1, \dots, u^m) := \sum_{i=1}^{m} (-1)^i u^i - \log \Gamma(m).$$

(This choice would be somewhat inconvenient when it came to computations though!)

With these considerations in mind, let us now define a topological manifold with boundary.

Definition 24.21. A separable metrisable space M is called a **topological manifold with boundary of dimension** m if every point $p \in M$ has a neighbourhood homeomorphic to an open subset of the standard half-space $\mathbb{R}^m_-$.

The assumptions "separable and metrisable" can be replaced with Hausdorff and second countable; cf. Proposition 1.32.

As with normal manifolds, by convention the dimension is usually understood to be the corresponding small letter.

Any topological manifold of dimension m is also a topological manifold with boundary of dimension m. This is because the intersection of an open set in $\mathbb{R}^m$ with $\mathbb{R}^m_-$ is open in $\mathbb{R}^m_-$. The converse is not necessarily true, however, since an open subset of $\mathbb{R}^m_-$ that intersects the hyperplane $\mathbb{R}^m_{u^1=0}$ is *not* an open subset of $\mathbb{R}^m$.

Definition 24.22. Let M be a topological manifold with boundary. We say a point $p \in M$ is an **interior point** if p admits a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^m$. We denote by $\operatorname{int}(M)$ the set of interior points. If p is not an interior point then we say p is a **boundary point**. We denote by ∂M the collection of boundary points.

This notion coincides with the topological one, see Proposition 24.24 below and Problem Sheet J.

The fact that the dimension is well-defined again requires us to invoke Brouwer's Invariance of Domain Theorem (cf. Remark 1.5). In the smooth case however this will be much easier.

Example 24.23. Here are some examples of topological manifolds with boundary:

(i) A topological space M is a topological manifold of dimension m if and only if it is a topological manifold with boundary of dimension m such that $\partial M = \emptyset$.

(ii) Any half-space $\mathbb{R}^m_{\lambda \geq c}$ is a topological manifold with boundary of dimension m. The boundary $\partial \mathbb{R}^m_{\lambda \geq c}$ is $\mathbb{R}^m_{\lambda = c}$. More generally any open subset Q of $\mathbb{R}^m_{\lambda \geq c}$ is a topological manifold with boundary, with $\partial Q = Q \cap \mathbb{R}^m_{\lambda = c}$.

(iii) The closed unit ball D^m is a topological manifold with boundary of dimension m. One has $\partial D^m = S^{m-1}$.

Exercise: Prove this!

(iv) The closed m-dimensional cube $\overline{\mathbb{I}}^m = [-1, 1]^m$ that we used in Lecture 14 is a topological manifold with boundary of dimension m. In this case $\partial\overline{\mathbb{I}}^m$ is homeomorphic to S^{m-1}.

(v) The punctured closed unit ball $D^m \setminus \{0\}$ is a topological manifold with boundary, since it is an open subset of the topological manifold with boundary D^m. This is an example where the manifold boundary is *not* the same as the topological boundary, since:

$$\partial(D^m \setminus \{0\}) = S^{m-1}, \qquad \partial^{\text{top}}(D^m \setminus \{0\}) = S^{m-1} \cup \{0\}.$$

(vi) More generally, any annulus which is half-open and half-closed, eg.

$$A^{\leq R}_{>r} := \{p \in \mathbb{R}^m \mid r < \|p\| \leq R\},$$
$$A^{<R}_{\geq r} := \{p \in \mathbb{R}^m \mid r \leq \|p\| < R\},$$

is a topological manifold with boundary whose boundary consists of the boundary circle for which one has the non-strict equality:

$$\partial A^{\leq R}_{>r} = \{\|p\| = R\}, \qquad \partial A^{<R}_{\geq r} = \{\|p\| = r\},$$

meanwhile

$$\partial^{\text{top}} A^{\leq R}_{>r} = \partial^{\text{top}} A^{<R}_{\geq r} = \{\|p\| = r\} \cup \{\|p\| = R\}.$$

PROPOSITION 24.24. *Let M be a topological manifold with boundary. Then $\operatorname{int}(M) \cap \partial M = \emptyset$. Moreover $\operatorname{int}(M)$ is a topological manifold without boundary of dimension m and ∂M is a topological manifold without boundary of dimension $m-1$.*

Proof. The fact that $\operatorname{int}(M) \cap \partial M = \emptyset$ uses Brouwer's Theorem as mentioned above (since $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^{m-1}$). The rest is clear, since an open subset Q of $\mathbb{R}^m_{\lambda \geq c}$ that does not intersect $\mathbb{R}^m_{\lambda = c}$ is also open in $\mathbb{R}^m$, and if Q is open in $\mathbb{R}^m_{\lambda \geq c}$ then $Q \cap \mathbb{R}^m_{\lambda = c}$ is open in $\mathbb{R}^m_{\lambda = c} \cong \mathbb{R}^{m-1}$. ■

COROLLARY 24.25. *If M is a topological manifold with boundary and $U \subset M$ is an open set then U is a topological manifold with boundary, and $\partial U = U \cap \partial M$.*

We now define smooth manifolds with boundary. We begin by extending by the definition of a diffeomorphism between open subsets of half-spaces. We already know (Definition 7.15) how to define what it means for a map to be smooth whose domain is not open, so it remains to extend this to the case when the range is also not open.

DEFINITION 24.26. Let $Q \subset \mathbb{R}^m_{\lambda \geq c}$ denote an open set and $f \colon Q \to \mathbb{R}^n_{\eta \geq d}$ a continuous map. We say that f is **smooth** if the composition $\iota \circ f \colon Q \to \mathbb{R}^n$ is smooth in the sense of Definition 7.15, where $\iota \colon \mathbb{R}^n_{\eta \geq d} \hookrightarrow \mathbb{R}^n$ is the inclusion. If both $f \colon Q \to f(Q)$ and $f^{-1} \colon f(Q) \to Q$ are homeomorphisms between open sets of half-spaces that are smooth in this sense, then we say that f is a **diffeomorphism**.

The next result is standard calculus; the proof is omitted.

PROPOSITION 24.27. *Here are some properties of smooth maps between open sets of half-spaces:*

(i) *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^m$ with non-empty intersection with $\mathbb{R}^m_{\lambda \geq c}$. Suppose $f, g \colon \mathcal{O} \to \mathbb{R}^n$ are smooth maps in the usual sense. If $f = g$ on $\mathcal{O} \cap \mathbb{R}^m_{\lambda \geq c}$ then $Df(p) = Dg(p)$ for all $p \in \mathcal{O} \cap \mathbb{R}^m_{\lambda \geq c}$.* i.e. in the sense of Definition 1.8.

(ii) *Let $\mathcal{O} \subset \mathbb{R}^m$ be open and $f \colon \mathcal{O} \to \mathbb{R}^m_{\eta \geq d}$ be smooth. If $f(p) \in \mathbb{R}^m_{\eta = d}$ for all $p \in \mathcal{O}$ then $Df(p)$ has image in $\mathcal{J}_p(\mathbb{R}^m_{\eta=d}) \cong \mathbb{R}^m_{\eta=0}$ for all $p \in \mathcal{O}$.* i.e. in the sense of Definition 24.26.

(iii) *Suppose $Q_1 \subset \mathbb{R}^m_{\lambda \geq a}$ and $Q_2 \subset \mathbb{R}^n_{\eta \geq d}$ are open sets, and suppose $f \colon Q_1 \to Q_2$ is a diffeomorphism. Assume $\partial Q_1 = Q_1 \cap \mathbb{R}^m_{\lambda = a}$ and $\partial Q_2 = Q_2 \cap \mathbb{R}^n_{\eta=d}$ are both non-empty. Then f induces diffeomorphisms $\partial Q_1 \to \partial Q_2$ and $\operatorname{int}(Q_1) \to \operatorname{int}(Q_2)$ in the sense of Definition 1.8, where we think of ∂Q_1 and ∂Q_2 as open subsets of $\mathbb{R}^{m-1}$ and $\mathbb{R}^{n-1}$ respectively.* i.e. in the sense of Definition 24.26.

We then have:

DEFINITION 24.28. Let M be a topological manifold with boundary. A **smooth atlas** on M is a collection $\mathcal{X} = \{x_a \colon U_a \to Q_a \mid a \in A\}$, where $\{U_a \mid a \in A\}$ is an open cover of M, each Q_a is an open subset of some m-dimensional half-space $\mathbb{R}^m_{\lambda_a \geq c_a}$ (the precise half-space may depend on a), and each $x_a \colon U_a \to Q_a$ is a homeomorphism such that the usual compatibility condition is satisfied: if $a, b \in A$ are such that $U_a \cap U_b \neq \emptyset$ then the composition

$$x_b \circ x_a^{-1} \colon x_a(U_a \cap U_b) \to x_b(U_a \cap U_b)$$

should be a diffeomorphism in the sense of Definition 24.26.

We call each such x_a a **half-space chart**. One then defines a smooth structure in exactly the same way as in Definition 1.11, and this gives us the definition of a smooth manifold with boundary.

DEFINITION 24.29. A **smooth manifold with boundary of dimension** m is a pair $(M, \mathcal{X})$ where M is a topological manifold with boundary of dimension m, and $\mathcal{X}$ is a smooth structure on M in the sense of Definition 24.28.

Just as with Proposition 24.24 we have:

PROPOSITION 24.30. *Let M be a smooth manifold with boundary. Then $\operatorname{int}(M) \cap \partial M = \emptyset$. Moreover $\operatorname{int}(M)$ naturally inherits the structure of a smooth manifold without boundary of dimension m, and ∂M naturally inherits the structure of a smooth manifold without boundary of dimension $m-1$.*

Proof. This follows from part (iii) of Proposition 24.27. ■

EXAMPLE 24.31. All the examples from Example 24.23 are naturally smooth manifolds with boundary, except for the unit cube $\overline{\mathbb{I}}^m$, which is *not* a smooth manifold with boundary when $m \geq 2$. (See Problem Sheet K.)

Although the definition of a smooth atlas does not require all the half-space charts to take values in the same half-space, it is often convenient to assume they do.

Definition 24.32. A **standard half-space chart** is a half-space chart $x \colon U \to Q$ with the property that Q is an open subset of our preferred standard half-space $\mathbb{R}^m_-$. A **standard smooth atlas** on a smooth manifold with boundary M is a smooth atlas as in Definition 24.28 all of whose charts are standard half-space charts.

It is easy to see that we may always assume this:

Lemma 24.33. *Every smooth manifold with boundary admits a standard smooth atlas.*

Remark 24.34. You might therefore ask what the point was in the more general definition. This is two-fold: firstly it is convenient when proving certain standard spaces are topological (resp. smooth) manifolds with boundary to be allowed more flexibility. Secondly, the distinction between good smooth atlases and normal smooth atlases is meaningful in dimension $m = 1$ when one in addition insists on orientability, as we will see in Proposition 24.40 below.

Many of the concepts we have covered so far in this course make sense for manifolds with boundary, and we don't have the time (or energy) to fill in the details, so let us just briefly summarise some of the important points:

- Partitions of unity still make sense for smooth manifolds with boundary, and they always exist.
- If M is a smooth manifold with boundary then T_pM is still an m-dimensional vector space for all $p \in M$. This is clear for $p \in \operatorname{int}(M)$, so suppose $p \in \partial M$. Let $x \colon U \to Q$ denote a half-space chart about p, where Q is an open set in some half-space $\mathbb{R}^m_{\lambda \geq c}$ and $x(p)$ lies in the hyperplane $\mathbb{R}^m_{\lambda = c}$. As before, a function f defined near p on M is smooth at p if and only if $f \circ x^{-1}$ is smooth near $z := x(p)$. Now recall by definition a function is smooth if and only if it admits a smooth extension to some open neighbourhood of z in $\mathbb{R}^m$. If g and h are any two such extensions of $f \circ x^{-1}$ then by part (i) of Proposition 24.27 the derivatives of g and h coincide on $\mathbb{R}^m_{\lambda = c}$. It follows that a derivation on the space of germs of smooth functions at p can be defined in exactly the same way as before, and thus the arguments from Lectures 2 and 3 go through without change to show that the tangent space T_pM at p is again an m-dimensional vector space.
- On the other hand, the tangent space to ∂M at $p \in \partial M$ can be identified with an $(m-1)$-dimensional subspace of T_pM. Indeed, if we let $\iota \colon \partial M \hookrightarrow M$ denote the inclusion then with the notation as above, $x \circ \iota|_{U \cap \partial M}$ is a chart on ∂M and thus

$$D\iota(p)(T_p\partial M) = Dx(p)^{-1}\big(T_z\mathbb{R}^m_{p=c}\big) \tag{24.3}$$

Exercise: Prove that the right-hand side of (24.3) does not depend on the choice of half-space chart x.

We usually suppress the $D\iota(p)$ map and thus think of $T_p\partial M$ as an actual subspace of T_pM.

- If N is a smooth manifold (with or without boundary) and $M \subset N$ is a subset endowed with a topology and a smooth structure making it into a smooth manifold with boundary such that the inclusion $M \hookrightarrow N$ is an embedding then M is said to be an **embedded submanifold with boundary**. Immersed submanifolds with boundary are defined similarly. If M is a smooth manifold with boundary then ∂M is an embedded submanifold of M – this follows immediately from the definition.

Exercise: Investigate how the Implicit Function Theorem 6.10 behaves with respect to manifolds with boundary. What is the correct notion of a slice chart in this setting?

- Both the Whitney Embedding Theorem 7.1 and the Whitney Approximation Theorem 7.13 still work for manifolds with boundary.
- A vector field X on a smooth manifold with boundary M is said to be **tangent to** ∂M if $X(p) \in T_p\partial M$ for each $p \in \partial M$. For vector fields that are tangent to M, Theorem 9.10 still works.
- The notion of a fibre bundle still makes sense if the base space is allowed to have boundary. In particular, vector bundles over manifolds with boundary are defined entirely analogously. Things go wrong however if the fibre is allowed to have boundary.

Exercise: Why?

- Tensors and differential forms are defined in exactly the same way.

We will however go through one aspect in detail, since this will be important in our treatment of the global Stokes' Theorem in Lecture 27. Suppose M is a manifold with boundary and $\pi\colon E \to M$ is a vector bundle over M. An orientation $\mathfrak{o}$ of E is, as before, determined by a non-vanishing section $\mu \in \Gamma(\mathsf{det}\, E^*)$. Given such a section μ, we can restrict it to obtain a section $\mu|_{\partial M}$ of the bundle $\mathsf{det}\, E|_{\partial M} \to \partial M$, where $E|_{\partial M} = \pi^{-1}(\partial M)$.

This is a subbundle of E since ∂M is an embedded submanifold of M, cf. Definition 17.17.

For the special case $E = TM$, this gives us an orientation of the bundle $TM|_{\partial M} \to \partial M$. This however is *not* the same thing as an orientation ∂M as a manifold – this would be an orientation of the bundle $T\partial M \to \partial M$.

DEFINITION 24.35. Let M be a smooth manifold with boundary of dimension m, and let $p \in \partial M$. A tangent vector $\xi \in T_pM$ is said to be **outward pointing** if for some half-space chart $x\colon U \to Q$ about p, with $Q \subset \mathbb{R}^m_{\lambda \geq c}$ an open set and $z := x(p) \in \mathbb{R}^m_{\lambda = c}$, one has

$$\lambda\big(\mathcal{J}_z^{-1}(Dx(p)\xi)\big) < 0.$$

To unwrap this: $Dx(p)$ is a linear map $T_pM \to T_z\mathbb{R}^m_{\lambda \geq c} = T_z\mathbb{R}^m$. Applying the dash-to-dot map $\mathcal{J}_z^{-1}$ we obtain a vector $\mathcal{J}_z^{-1}(Dx(p)\xi) \in \mathbb{R}^m$, which λ can then eat to produce a real number. It follows from part (iii) of Proposition 24.27 that the property of being outward pointing is independent of the choice of half-space chart x.

The definition is rather clearer if we take a standard half-space chart. Then the condition that $\xi \in T_pM$ is outward pointing is simply that

The sign change is because $\mathbb{R}^m_- = \mathbb{R}^m_{-u^1 \geq 0}$.

$$dx^1_p(\xi) > 0.$$

Similarly an **inward-pointing** vector is one for which $dx^1_p(\xi) < 0$.

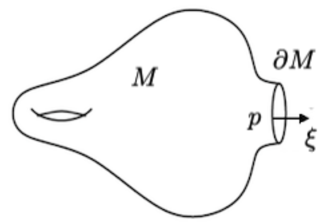


Figure 24.1: An outward pointing vector ξ

The notion of outward and inward pointing vectors allow us to decompose T_pM as:

$$T_pM = \{\text{outward pointing}\} \cup \{\text{inward pointing}\} \cup T_p\partial M,$$

since in such a chart x, one has

$$T_p\partial M = \big\{\xi \in T_pM \mid dx^1_p(\xi) = 0\big\}$$

by (24.3). Similarly a section X of $TM|_{\partial M}$ is said to be **outward pointing** if $X(p)$ is outward pointing for every p.

Example 24.36. Let M be a manifold with boundary and let $p \in \partial M$. Let (U, x) denote a standard half-space chart about p. Then $\frac{\partial}{\partial x^1}$ is an outward pointing section of $TM|_{\partial M}$ over $U \cap \partial M$.

In fact, via a standard partition of unity argument, one can produce outward pointing sections defined on the entire boundary.

Lemma 24.37. *Let M be a smooth manifold with boundary. Then there exists a section $X \in \Gamma(TM|_{\partial M})$ of the bundle $TM|_{\partial M} \to \partial M$ which is outward pointing at every $p \in \partial M$.*

Proof. We may assume M has a standard smooth atlas $\{x_a : U_a \to Q_a \mid a \in A\}$. Let $\{\kappa_a \mid a \in A\}$ denote a partition of unity subordinate to $\{U_a \mid a \in A\}$. Set

$$X := \sum_a \kappa_a \frac{\partial}{\partial x^1_a}.$$

This is outward pointing by Example 24.36. ∎

Let us now use this to define the induced orientation.

Definition 24.38. Let $(M, \mathfrak{o})$ be an oriented smooth manifold with boundary. Let $\mu \in \Omega^m(M)$ be a volume form representing $\mathfrak{o}$. Let X be an outward pointing section. Then we can view $i_X\mu$ as an element of $\Omega^{m-1}(\partial M)$. Since X is outward pointing, $i_X\mu$ is nowhere vanishing on ∂M. Thus $i_X\mu$ determines an orientation of ∂M, which we rather suggestively write as $\partial\mathfrak{o}$. We call $\partial\mathfrak{o}$ the **induced orientation**.

On Problem Sheet J you are asked to show that the induced orientation is well-defined, i.e. independent of the choice of X and μ. Note that if $(X_1, \dots, X_n)$ is a positively oriented frame for TM such that $U \cap \partial M$ is non-empty and such that $X_1|_{U\cap\partial M}$ is outward pointing. Then $(X_2, \dots, X_n)$ is a positively oriented local frame for $TM|_{\partial M}$ over $U \cap \partial M$ with respect to the induced orientation

i.e. $\mu(X_1, \dots, X_n) > 0$ over U.

Remark 24.39. In the case $m = 1$, the boundary ∂M is a discrete set of points. We only defined orientations for vector spaces of positive dimension, but this can still be made sense of if we simply think of a boundary point p being positively oriented if the function $\mu(X)$ is positive at p and negatively oriented otherwise.

Here is an extension of Corollary 24.16 for manifolds with boundary. This is where it is important to make the distinction between a standard atlas and a normal one.

PROPOSITION 24.40. *Let M be an oriented smooth manifold with boundary of dimension m. Then M admits a positively oriented smooth atlas (that is, one such that* (24.2) *holds). If $m \geq 2$ then M admits a positively oriented standard smooth atlas.*

Proof. If x is a chart with local coordinates $(x^1, \ldots, x^m)$ that is not positively oriented then we replace it with a new chart $(x^1, -x^2, \ldots, x^m)$. If $m \geq 2$ and x is a standard half-space chart then the new chart is also a standard half-space chart. This goes wrong for $m = 1$ however, since in this case it changes a $\mathbb{R}^1_-$ half-space chart into a $\mathbb{R}^1_+$ half-space chart. ∎

For the rest of these notes, **all** manifolds (topological or smooth) should be assumed **not** to have boundary, unless it is explicitly said that they do.

LECTURE 25

Smooth Singular Cubes

In this lecture we begin our discussion of *integration* on manifolds. We introduce the notion of a *singular cube* in a manifold and explain how a differential form can be integrated over such a cube. We then mirror the construction of singular homology and show how to create the (smooth) cubical chain complex of M. We will use this formalism next lecture en route to proving the manifold version of Stokes' Theorem.

Don't worry if you have not seen homology before. We will not use any algebraic topology in this course. Exception: the bonus section to Lecture 27.

DEFINITION 25.1. Let us abbreviate by C^k the closed cube $[0,1]^k$, thought of as sitting inside $\mathbb{R}^k$. For $k = 0$, $C^0 = \{0\}$ is a point. A **smooth singular k-cube** (often shorted to: a "singular k-cube" or just a "k-cube") in a smooth manifold M is a smooth map $c\colon C^k \to M$. Thus a singular 0-cube is simply a point $c(0)$ in M.

In the past we used $\mathbb{I}^k$ for the open cube $(-1,1)^k$; here it is more convenient to work on $[0,1]$ itself, so we choose different notation.

Recall that by Definition 7.15 a map $c\colon C^k \to M$ is smooth if there exists a neighbourhood U of C^k in $\mathbb{R}^k$ and a smooth map $\tilde{c}\colon U \to M$ such that $\tilde{c}|_{C^k} = c$. Of course the extension $\tilde{c}$ is not unique. For $k = 0$, we declare that any map $c\colon C^0 \to M$ is smooth.

REMARK 25.2. The adjective "singular" is meant to draw your attention to the fact that c need not be injective nor an immersion. Indeed, a valid smooth singular k-cube would be a constant map! Moreover if $k > \dim M$ then no singular k-cube can be an immersion.

The next example is more important than you would first guess.

EXAMPLE 25.3. We let $\mathsf{i}_k\colon C^k \hookrightarrow \mathbb{R}^k$ denote the inclusion and call i_k the **standard smooth singular k-cube**.

REMARK 25.4. We will often regard i_k as taking values in C^k, not $\mathbb{R}^k$, and hence identify i_k with the identity map $C^k \to C^k$. Strictly speaking however in this case i_k is not a smooth singular cube, since the range space C^k is not a smooth manifold. For the most part we shall ignore this pedantry.

REMARK 25.5. You might hope that the machinery of smooth manifolds with boundary that we developed last lecture would allow us to forego the tedious extension business. This works fine for $k = 1$: $C^1 = [0,1]$ is a smooth manifold with boundary, and a singular 1-cube is simply a smooth map $C^1 \to M$ between manifolds. However for $k = 2$ it goes wrong: C^2 is not a manifold with boundary (see Problem J.9). It is however a **smooth manifold with corners**, which is defined as you might expect: instead of a half space atlas one works with a *quarter space atlas*. If M is a smooth manifold with corners then its boundary ∂M is a smooth manifold with boundary, and the boundary of the boundary is then a smooth manifold without boundary.

For C^2, one has ∂C^2 equal to the union of the edges, and $\partial(\partial C^2)$ equal to the four vertices.

Sadly however this *still* isn't enough, since for $k \geq 3$ the space C^k is not a smooth manifold with corners either. The correct notion is that of a **stratified manifold**, which, roughly speaking is a manifold which is allowed to "boundary-like" pieces of arbitrarily high codimension. A manifold with boundary is a stratified manifold with only codimension one strata, and a manifold with corners is a stratified manifold with only codimension one and two strata. In general, C^k is a stratified manifold with k different stratas.

That said, developing the entire theory of stratified manifolds just to dispense with the need to talk about extensions is somewhat inefficient, even by our standards, so we will stick with the extensions. This will therefore be a minor annoyance throughout the lecture.

Definition 25.6. Let $k > 0$, and let $\omega \in \Omega^k(C^k)$ denote a k-form on C^k. We can write $\omega = f\, du^1 \wedge \cdots \wedge du^k$ for some $f \in C^\infty(C^k)$. We define the **integral of** ω to be the Riemann integral of f:

i.e. ω is a k-form on some neighbourhood U of C^k in $\mathbb{R}^k$.

$$\int_{C^k} \omega := \int_{C^k} f$$

We emphasise the right-hand side is the *normal Riemann integral* of the function h.

We now transfer this to manifolds:

Definition 25.7. Let $k > 0$ and let c be a smooth singular k-cube in M and let $\omega \in \Omega^k(M)$ denote a k-form. Then $c^*\omega$ is a k-form on C^k. We define the **integral of** ω **over** c to be the real number

$$\int_c \omega := \int_{C^k} c^*\omega.$$

It would be sufficient if ω was only defined on some neighbourhood of the image of c for this to make sense. For $k = 0$, the definition is simpler; in this case ω is just a function f, and

$$\int_c f := f(c(0)).$$

Remark 25.8. If we write $c^*\omega = f\, du^1 \wedge \cdots \wedge du^k$ then the function f is given explicitly by

This expression is well-defined since $c^*\omega$ is really defined on some open neighbourhood of C^k.

$$f = c^*\omega\left(\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^k}\right).$$

Thus an alternative formula is

$$\int_c \omega = \int_{C^k} c^*\omega\left(\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^k}\right),$$

where again the right-hand side is just a normal Riemann integral.

Remark 25.9. Since for any singular k-cube c we have $c = c \circ \mathsf{i}_k$, we have using (22.6) that

$$\begin{aligned}\int_c \omega &= \int_{c\circ \mathsf{i}_k} \omega \\ &= \int_{C^k} (c \circ \mathsf{i}_k)^*(\omega) \\ &= \int_{C^k} \mathsf{i}_k^*(c^*\omega) \\ &= \int_{\mathsf{i}_k} c^*\omega.\end{aligned}$$

DEFINITION 25.10. A singular k-cube $c\colon C^k \to M$ is said to be **degenerate** if there exists $1 \le i \le k$ such that c does not depend on u^i. Otherwise c is said to be **non-degenerate**. Thus a 0-cube is never degenerate, and a 1-cube is degenerate if and only if it is a constant map.

On Problem Sheet K you will prove.

LEMMA 25.11. *If $c\colon C^k \to M$ is a degenerate singular k-cube then $\int_c \omega = 0$ for any $\omega \in \Omega^k(M)$.*

The next result is also on Problem Sheet K. You should think of it as a version of the usual change of variables formula from multivariable calculus:

PROPOSITION 25.12 (Change of Variables). *Let $c\colon C^k \to M$ be a smooth singular k-cube in M and let $\varphi\colon C^k \to C^k$ be an orientation preserving diffeomorphism. Let $\tilde{c} := c \circ \varphi$. Then*

$$\int_c \omega = \int_{\tilde{c}} \omega.$$

As usual, think of this as meaning that φ is the restriction to C^k of an orientation preserving diffeomorphism of some neighbourhood.

Let us now consider formal sums of singular cubes.

DEFINITION 25.13. Let $\mathsf{Q}_k(M)$ denote the (infinite-dimensional) free vector space generated by the collection of all the smooth singular k-cubes in M. Thus an element of $\mathsf{Q}_k(M)$ is a formal finite sum $\mathsf{q} = \sum_i a_i\, c_i$ where $a_i \in \mathbb{R}$ and the c_i are smooth singular k-cubes. We call an element $\mathsf{q} \in \mathsf{Q}_k(M)$ a **smooth singular k-chain**, or (sometimes just a k-chain). A k-chain $\mathsf{q} = \sum_i a_i\, c_i$ is said to be **non-degenerate** if each cube c_i is non-degenerate.

To explain the notation: "cube" sounds like it begins with a "Q".

EXAMPLE 25.14. Since a 0-cube in M is just a point in M, the space $\mathsf{Q}_0(M)$ can be thought as the infinite-dimensional vector space with basis the points of M. In particular, if $p, q \in M$ then the expression $p - q$ makes sense in $\mathsf{Q}_0(M)$, even though it does not in M.

WARNING 25.15. The space $\mathsf{Q}_0(M)$ is a vector space with basis the points in M. Thus (by definition) there are no relations between different elements. This can be confusing, particularly if the manifold M happens to be a submanifold of Euclidean space where it *does* make sense to add points together. As an example, let us take $M = \mathbb{R}^m$. Let $v, w \in \mathbb{R}^m$ be two vectors. Then in $\mathbb{R}^m$, we can add v and w together to get a new vector $v + w$. However in $\mathsf{Q}_0(\mathbb{R}^m)$, the three elements v, w and $v + w$ are *linearly independent* and thus it is *not* true that $v + w = (v + w)$! A similar issue occurs with scalar multiplication. If this confuses you, consider writing the addition and multiplication operations in $\mathsf{Q}_0(\mathbb{R}^m)$ with a different colour, for instance red. Thus if $v, w \in \mathbb{R}^m$ and $a \in \mathbb{R}$ then

$$v + w \ne v + w, \qquad a\, v \ne 1(av).$$

Luckily most of the time this shouldn't be confusing, since typically on manifolds one cannot add points together, and thus the notation is unambiguous.

Definition 25.16. We define the integral of a k-form over a k-chain in M by linearity: if $\mathsf{q} = \sum_i a_i\, c_i$ then

$$\int_{\mathsf{q}} \omega := \sum_i a_i \int_{c_i} \omega.$$

We will also need the concept of the *boundary* of a chain.

Definition 25.17. Let $1 \le i \le k$. Define the **ith front face map**

$$\mathsf{f}_{i,k} \colon C^{k-1} \to C^k, \qquad (u^1, \dots, u^{k-1}) \mapsto (u^1, \dots, u^{i-1}, 0, u^i, \dots u^{k-1})$$

and the **ith back face map**

$$\mathsf{b}_{i,k} \colon C^{k-1} \to C^k, \qquad (u^1, \dots, u^{k-1}) \mapsto (u^1, \dots, u^{i-1}, 1, u^i, \dots u^{k-1})$$

When k is clear from the context, we write simply f_i and b_i instead.

Definition 25.18. Fix $1 \le i \le k$ and let $c \colon C^k \to M$ denote a singular k-cube. The ith **front face** of c is the smooth singular $(k-1)$-cube $c \circ \mathsf{f}_i \colon C^{k-1} \to M$. Similarly the **$i$th back face** of c is the smooth singular $(k-1)$-cube $c \circ \mathsf{b}_i \colon C^{k-1} \to M$. We abbreviate

$$\mathsf{f}_i c := c \circ \mathsf{f}_{\mathsf{i}}, \qquad \mathsf{b}_i c := c \circ \mathsf{b}_i.$$

Writing out the definitions yields the following statement.

Lemma 25.19. *Let $c \colon C^k \to M$ be a smooth singular k-cube. Let $1 \le i < j \le k$. Then:*

$$\begin{aligned} \mathsf{f}_i(\mathsf{f}_j c) &= \mathsf{f}_{j-1}(\mathsf{f}_i c), \\ \mathsf{b}_i(\mathsf{b}_j c) &= \mathsf{b}_{j-1}(\mathsf{b}_i c), \\ \mathsf{f}_i(\mathsf{b}_j c) &= \mathsf{b}_{j-1}(\mathsf{f}_i c), \\ \mathsf{b}_i(\mathsf{f}_j c) &= \mathsf{f}_{j-1}(\mathsf{b}_i c). \end{aligned} \tag{25.1}$$

Thus by definition

$$\mathsf{f}_{i,k} = \mathsf{f}_i \mathsf{i}_k, \qquad \mathsf{b}_{i,k} = \mathsf{b}_i \mathsf{i}_k.$$

Definition 25.20. Let $c \colon C^k \to M$ be a smooth singular k-cube for $k > 0$. We define the **boundary** of c, written ∂c, to be the element of $\mathsf{Q}_{k-1}(M)$ given by

$$\partial c := \sum_{i=1}^{k} (-1)^i (\mathsf{f}_i c - \mathsf{b}_i c).$$

We define the boundary of a 0-cube to be the real number 1. Note that if a cube c is non-degenerate then so is ∂c. We then extend ∂ to arbitrary k-chains by linearity. Thus we may think of ∂ as a linear map $\mathsf{Q}_k(M) \to \mathsf{Q}_{k-1}(M)$ for all $k \ge 1$.

This works for $k = 0$ too if we define $\mathsf{Q}_{-1}(M) := \mathbb{R}$.

Remark 25.21. Thus this is yet another meaning of the symbol ∂. This one is not as confusing as the topological boundary and the manifold boundary (cf. Remark 24.17), since c is a function, and thus there can be no ambiguity about what is meant.

Example 25.22. Let $c\colon [0,1] \to M$ be a 1-cube. Then $\mathsf{f}_1 c$ is the 0-cube $c(0)$ and $\mathsf{b}_1 c$ is the 0-cube $c(1)$. Thus $\partial c = c(1) - c(0)$. Remember the subtraction is taking place in $\mathsf{Q}_0(M)$, not M itself!

Proposition 25.23. *The boundary operator squares to zero:* $\partial^2 = 0$.

Proof. Since ∂ is linear, it suffices to show that $\partial(\partial c) = 0$ for any cube c. For this we compute:

$$\begin{aligned}\partial(\partial c) &= \partial\left(\sum_{i=1}^{k}(-1)^i(\mathsf{f}_i c - \mathsf{b}_i c)\right)\\ &= \sum_{i=1}^{k}\sum_{j=1}^{k-1}(-1)^{i+j}\Big(\mathsf{f}_j(\mathsf{f}_i c) - \mathsf{f}_j(\mathsf{b}_i c) - \mathsf{b}_j(\mathsf{f}_i c) + \mathsf{b}_j(\mathsf{b}_i c)\Big)\end{aligned}$$

Using the face relations (25.1), we see that the first and fourth terms cancel in pairs, and the second and third terms cancel each other. ■

Proposition 25.23 allows us to play a similar game to the definition (Definition 23.2) of the de Rham cohomology groups.

Using the terminology from algebraic topology, Proposition 25.23 tells us that $(Q_\bullet(M), \partial)$ is a **chain complex**. We can thus take its homology.

Definition 25.24. We say a chain q is **closed** if $\partial \mathsf{q} = 0$ and a chain q is **exact** if $\mathsf{q} = \partial \mathsf{p}$ for some $(k+1)$-chain p. Then every exact chain is also closed (as $\partial^2 = 0$), and thus we can form the quotient vector space:

$$H_k^{\text{cube}}(M;\mathbb{R}) := \frac{\{\text{closed non-degenerate } k\text{-chains}\}}{\{\text{exact non-degenerate } k\text{-chains}\}}.$$

We call $H_k^{\text{cube}}(M;\mathbb{R})$ the **kth cubical singular homology group** of M.

The word "homology" is used because ∂ decreases the degree: $\partial\colon \mathsf{Q}_k \to \mathsf{Q}_{k-1}$. In the de Rham construction the exterior differential d increases the degree: $d\colon \Omega^k \to \Omega^{k+1}$. Thus we use the word "cohomology".

If q is a closed k-chain, we denote by $[\mathsf{q}]$ its equivalence class in $H_k^{\text{cube}}(M;\mathbb{R})$. The reason for insisting on non-degeneracy will be explained in the bonus section to Lecture 27. Unlike the groups $H^k_{\text{dR}}(M)$, which are certainly zero for $k > \dim M$, a priori the groups $H_k^{\text{cube}}(M;\mathbb{R})$ could be non-zero for arbitrarily high k. However this is not the case. In fact, as we will explain in the bonus section to Lecture 27, there is an isomorphism

$$H_k^{\text{cube}}(M;\mathbb{R}) \cong H^{n-k}_{\text{dR}}(M), \qquad \forall k \geq 0. \tag{25.2}$$

The isomorphism (25.2) is one way of stating **Poincaré Duality**. This is one of the cornerstones of modern algebraic topology.

LECTURE 26

Stokes' Theorem

In this lecture we state and prove Stokes' Theorem, which is one of the cornerstones of modern differential geometry. In fact, we will prove two versions of Stokes' Theorem: a local version (Theorem 26.2) using the language of smooth singular cubes from the last lecture, and a global version (Theorem 26.16) which concerns integration over the entire manifold.

Fun Fact: The only connection Stokes has with "Stokes' Theorem" is that he decided to set it as a problem on an exam.

As a warm up, we translate the Fundamental Theorem of Calculus into the language of chains. This can be thought of as the one-dimensional version of the Local Stokes' Theorem.

THEOREM 26.1 (The Fundamental Theorem of Calculus for Singular 1-Chains). *Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable and let q be a singular 1-chain in $\mathbb{R}$. Then*

$$\int_{\mathsf{q}} df = \int_{\partial \mathsf{q}} f$$

Proof. By linearity of the integral, we may assume that q is a single singular 1-cube c. Then we compute

$$\begin{aligned}\int_c df &= \int_0^1 c^*(df)\left(\frac{\partial}{\partial t}\right) dt \\ &= \int_0^1 (f \circ c)'(t) dt \\ &= f(c(1)) - f(c(0)) \\ &= \int_{c(1)} f - \int_{c(0)} f \\ &= \int_{\partial c} f,\end{aligned}$$

where the third equality used the usual Fundamental Theorem of Calculus that you have known since Kindergarten. ■

Here is the general Local Stokes' Theorem.

THEOREM 26.2 (The Local Stokes' Theorem). *Let M be a smooth manifold. Let $q \in \mathsf{Q}_k(M)$ and $\omega \in \Omega^{k-1}(M)$. Then*

$$\int_{\mathsf{q}} d\omega = \int_{\partial \mathsf{q}} \omega.$$

Note Theorem 26.1 is the special case $M = \mathbb{R}$ and $k = 1$. The proof is not particularly difficult, but it is somewhat fiddly, and hence it is deferred to the bonus section below.

DEFINITION 26.3. Let M be a smooth manifold. Then for $0 \le k \le m$ we can think of integration as defining a bilinear map

$$\int : \mathsf{Q}_k(M) \times \Omega^k(M) \to \mathbb{R}, \qquad (\mathsf{q}, \omega) \mapsto \int_{\mathsf{q}} \omega.$$

Corollary 26.4. *The bilinear form* $\int$ *is also well-defined on the (co)homology level, that is, the map*

$$\int : H_k^{\mathrm{cube}}(M;\mathbb{R}) \times H_{\mathrm{dR}}^k(M) \to \mathbb{R}, \qquad ([q],[\omega]) \mapsto \int_{\mathsf{q}} \omega$$

is well-defined.

Proof. We already know that $\int_{\mathsf{q}} \omega$ vanishes whenever q is degenerate (Lemma 25.11). Thus we need only show that if q is a closed non-degenerate k-chain and ω is a closed k-form, then for any non-degenerate $(k+1)$-chain p and any $(k-1)$-form θ, one has

$$\int_{\mathsf{q}+\partial\mathsf{p}} (\omega + d\theta) = \int_{\mathsf{q}} \omega.$$

To see this we compute

$$\begin{aligned}\int_{\mathsf{q}+\partial\mathsf{p}} (\omega + d\theta) &= \int_{\mathsf{q}} \omega + \int_{\partial\mathsf{p}} \omega + \int_{\mathsf{q}} d\theta + \int_{\partial\mathsf{p}} d\theta \\ &= \int_{\mathsf{q}} \omega + \int_{\mathsf{p}} d\omega + \int_{\partial\mathsf{q}} \theta + \int_{\mathsf{p}} (d^2\theta) \\ &= \int_{\mathsf{q}} \omega + 0,\end{aligned}$$

where the second equality used Stokes' Theorem and the last used the assumption that q and ω are closed. ■

We move onto the global version of Stokes' Theorem, starting with an explanation of how to make sense of the expression $\int_{M,\mathfrak{o}} \omega$. Unlike the local version, this will only work when M is oriented and ω is a compactly supported differential form of top degree $m = \dim M$.

Definition 26.5. Let $(M, \mathfrak{o})$ be an oriented manifold. A singular cube $c \colon C^m \to M$ is said to be **orientation preserving** if there exists a neighbourhood U of C^m in $\mathbb{R}^m$ (or $\mathbb{R}^m_+$ when M has boundary) and an orientation preserving embedding $\tilde{c} \colon U \to M$ such that $\tilde{c}|_{C^m} = c$. Note that $\tilde{c}$ is thus a diffeomorphism onto its image.

Here we always implicitly assume $C^m \subset \mathbb{R}^m$ carries its standard orientation.

Remark 26.6. If $(M, \mathfrak{o})$ is an oriented manifold, we can always find an open cover of M such that each open set U in that cover is contained in the interior of the image of an orientation preserving singular cube $c \colon C^m \to M$. Indeed suppose $x \colon V \to \mathcal{O}$ is a positively oriented chart (cf. Definition 24.15) such that $C^m \subset \mathcal{O}$. Set $c := x^{-1}|_{C^m}$ and let $U := x^{-1}\big((\tfrac{1}{3}, \tfrac{2}{3})^m\big)$. The collection of all open sets U as (V, x) ranges over all such charts forms an open cover with the desired properties.

Definition 26.7. Let M be a smooth manifold, and let $\omega \in \Omega(M)$. The **support** of ω is defined in the same way as normal:

$$\operatorname{supp}(\omega) := \overline{\{p \in M \mid \omega_p \neq 0\}}.$$

A differential form ω is said to have **compact support** if $\operatorname{supp}(\omega)$ is compact. We denote by $\Omega_c(M) \subset \Omega(M)$ the subset of differential forms with compact support, and $\Omega_c^k(M)$ the differential k-forms with compact support. Note that by definition of the exterior differential, we have

$$\operatorname{supp}(d\omega) \subseteq \operatorname{supp}(\omega).$$

The next lemma is the reason why global integration only works on oriented manifolds.

LEMMA 26.8. *Let $(M, \mathfrak{o})$ be an orientated manifold and $\omega \in \Omega^m(M)$. Let $c_1, c_2 \colon C^m \to M$ be two orientation preserving singular cubes, and assume that*

$$\operatorname{supp}(\omega) \subset \operatorname{int}(\operatorname{im} c_1) \cap \operatorname{int}(\operatorname{im} c_2)$$

Then

$$\int_{c_1} \omega = \int_{c_2} \omega.$$

Proof. This almost follows from Proposition 25.12, since $c_2^{-1} \circ c_1$ is almost an orientation preserving diffeomorphism of C^m. The only issue is that $c_2^{-1} \circ c_1$ may not be defined on all of C^m. However, since $\operatorname{supp}(\omega) \subset \operatorname{int}(\operatorname{im} c_1) \cap \operatorname{int}(\operatorname{im} c_2)$, the proof of Proposition 25.12 goes through without change to give

$$\int_{c_2} \omega = \int_{c_2 \circ c_2^{-1} \circ c_1} \omega = \int_{c_1} \omega$$

as required. ∎

Thus we can unambiguously make the following definition.

DEFINITION 26.9. Let $(M, \mathfrak{o})$ be an oriented manifold and $\omega \in \Omega^m(M)$. Assume that ω has support in the interior of the image of some orientation preserving singular cube c. We define

$$\int_{M, \mathfrak{o}} \omega := \int_c \omega.$$

The following lemma is immediate from the definitions.

LEMMA 26.10. *If c is an orientation reversing singular cube and ω has support in $\operatorname{im} c$ then*

$$\int_{M, \mathfrak{o}} \omega = -\int_c \omega.$$

Thus

$$\int_{M, \mathfrak{o}} \omega = -\int_{M, -\mathfrak{o}} \omega.$$

We can use a partition of unity to extend this to an arbitrary $\omega \in \Omega_c^m(M)$. We first give the definition, and then prove it is well defined.

DEFINITION 26.11. Let $(M, \mathfrak{o})$ be an oriented manifold and let $\omega \in \Omega_c^m(M)$. Let $\{U_a \mid a \in A\}$ be an open cover with the property that each U_a is contained in the interior of the image of some orientation preserving singular cube (cf. Remark 26.6). Let $\{\kappa_a \mid a \in A\}$ be a partition of unity subordinate to this cover. We define

$$\int_{M, \mathfrak{o}} \omega := \sum_{a \in A} \int_{M, \mathfrak{o}} \kappa_a \, \omega. \tag{26.1}$$

Note this is a finite sum since ω has compact support and $\operatorname{supp}(\kappa_a)$ is locally finite.

Lemma 26.12. *The sum (26.1) is well defined: if $\{V_b \mid b \in B\}$ is another open cover with the property that each V_b is contained in the interior of the image of some orientation preserving singular cube and $\{\nu_b \mid b \in B\}$ is a partition of unity subordinate to that cover then for any $\omega \in \Omega_c^m(M)$ one has:*

$$\sum_{a\in A} \int_{M,\mathfrak{o}} \kappa_a \, \omega = \sum_{b\in B} \int_{M,\mathfrak{o}} \nu_b \, \omega$$

Proof. Since

$$\sum_{a\in A} \kappa_a(p) = \sum_{b\in B} \nu_b(p) = 1, \qquad \forall\, p \in M,$$

we have using linearity of the integral that

$$\begin{aligned} \sum_{a\in A} \int_{M,\mathfrak{o}} \kappa_a \, \omega &= \sum_{a\in A} \int_{M,\mathfrak{o}} \left(\sum_{b\in B} \nu_b \right) \kappa_a \, \omega \\ &= \sum_{a\in A} \sum_{b\in B} \int_{M,\mathfrak{o}} \nu_b \, \kappa_a \, \omega \\ &= \sum_{b\in B} \int_{M,\mathfrak{o}} \left(\sum_{a\in A} \kappa_a \right) \nu_b \, \omega \\ &= \sum_{b\in B} \int_{M,\mathfrak{o}} \nu_b \, \omega \end{aligned}$$

where the rearrangement of the sums is justified as everything is a finite sum. ■

We now know how to integrate a top-dimensional differential form with compact support on an oriented manifold. Let us extend this to oriented manifolds with boundary. For this we use the following trick:

Definition 26.13. Let $(M, \mathfrak{o})$ be an oriented smooth manifold with boundary. An orientation preserving singular cube $c \colon C^m \to M$ is said to be **adapted to the boundary** if either $\operatorname{im} c \subset \operatorname{int}(M)$ or

$$\partial M \cap \operatorname{im} c = \operatorname{im}(\mathsf{f}_1 c),$$

where as usual $\mathsf{f}_1 c \colon C^{m-1} \to M$ is the first front face.

Lemma 26.14. *Let $(M, \mathfrak{o})$ be an oriented smooth manifold with boundary, and let $c \colon C^m \to M$ be a singular cube which is adapted to the boundary such that $\operatorname{im} c \cap \partial M \neq \emptyset$. Then $\mathsf{f}_1 c$ is an orientation reversing singular cube for $(\partial M, \partial\mathfrak{o})$.*

Recall $\partial\mathfrak{o}$ denote the induced orientation of ∂M.

This is *not* a typo – we really do want $\mathsf{f}_1 c$ to reverse orientation! As we shall see, the minus sign will eventually cancel, since the coefficient of $\mathsf{f}_1 c$ in ∂c is -1.

Proof. We need only check that $\mathsf{f}_1 c$ is orientation reversing with respect to the induced orientation. Let (u^i) denote the standard coordinates on C^m. Since c is a diffeomorphism, we may take c^{-1} as a $\mathbb{R}^m_+$

Remember c is really defined on an open neighbourhood of C^m.

half-space chart on M. Let $x^i := u^i \circ c^{-1}$ denote the local coordinates of this chart. Since c is orientation preserving, $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\right\}$ is a positive oriented local frame of TM. Note that $\frac{\partial}{\partial x^1}$ is an *inward* pointing section, and thus the frame $\left\{\frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^m}\right\}$ is a negatively oriented frame for $T(\partial M)$. This shows that $\mathsf{f}_1 c$ is orientation reversing. ■

The reason why $\frac{\partial}{\partial x^1}$ is inward pointing (in contrast to Example 24.36) is that x is an $\mathbb{R}^m_+$ chart not a $\mathbb{R}^m_-$ chart.

Just as in Remark 26.6, if M is a smooth manifold with boundary then we can always find an open cover of M with the property that each open set is contained in the interior of the image of a orientating preserving singular cube which is adapted to the boundary. We use this to extend the definition of integration to manifolds with boundary.

Definition 26.15. Let M be an oriented smooth manifold with boundary, and let $\omega \in \Omega^m_c(M)$. Let $\{U_a \mid a \in A\}$ be an open cover with the property that each U_a is contained in the image of some orientation preserving singular cube which is adapted to the boundary. Let $\{\kappa_a \mid a \in A\}$ be a partition of unity subordinate to this cover. We define

$$\int_{M,\mathfrak{o}} \omega := \sum_{a \in A} \int_{M,\mathfrak{o}} \kappa_a \, \omega.$$

The same proof as Lemma 26.12 shows this is well-defined. We are finally ready to state and prove the Global Stokes' Theorem.

Theorem 26.16 (The Global Stokes' Theorem). *Let $(M, \mathfrak{o})$ be an oriented smooth manifold with boundary, and let $\omega \in \Omega^{m-1}_c(M)$. Then*

$$\int_{M,\mathfrak{o}} d\omega = \int_{\partial M, \partial\mathfrak{o}} \omega.$$

Proof. We prove the result in two steps. Let $\{U_a \mid a \in A\}$ be an open cover with the property that each U_a is contained in the image of some orientation preserving singular cube which is adapted to the boundary.

1. First assume that $\operatorname{supp}(\omega)$ is contained in one of the sets U_a, which itself is contained in the image of some orientation preserving singular cube c which is adapted to the boundary. If $\operatorname{im} c \cap \partial M = \emptyset$ then the result is immediate from the Local Stokes' Theorem 26.2, since

$$\int_{M,\mathfrak{o}} d\omega = \int_c d\omega = \int_{\partial c} \omega = 0,$$

since $\operatorname{supp}(\omega)$ does not intersect the image of ∂c. But also clearly $\int_{\partial M, \partial\mathfrak{o}} \omega = 0$ since $\operatorname{supp}(\omega)$ does not intersect ∂M.

Now assume that $\operatorname{im} c \cap \partial M \neq \emptyset$. Then we have:

$$\begin{aligned} \int_{M,\mathfrak{o}} d\omega &= \int_c d\omega \\ &= \int_{\partial c} \omega \\ &= \sum_{i=1}^m (-1)^i \left(\int_{\mathsf{f}_i c} \omega - \int_{\mathsf{b}_i c} \omega \right) \\ &= - \int_{\mathsf{f}_1 c} \omega, \end{aligned}$$

since $\operatorname{supp}(\omega)$ misses all faces apart from $\mathsf{f}_1 c$ as c is adapted to the boundary. Thus by Lemma 26.10 and Lemma 26.14 we have

$$\begin{aligned}\int_{M,\mathfrak{o}} d\omega &= -\int_{\mathsf{f}_1 c} \omega \\ &= (-1)^2 \int_{\partial M, \partial\mathfrak{o}} \omega \\ &= \int_{\partial M, \partial\mathfrak{o}} \omega.\end{aligned}$$

2. Now we prove the general case. Let $\{\kappa_a \mid a \in A\}$ be a partition of unity subordinate to the open cover $\{U_a \mid a \in A\}$. Then by definition,

$$\begin{aligned}\int_{\partial M,\partial\mathfrak{o}} \omega &= \sum_{a\in A} \int_{\partial M,\partial\mathfrak{o}} \kappa_a\, \omega \\ &= \sum_{a\in A} \int_M d(\kappa_a\, \omega) \\ &= \sum_{a\in A} \int_M d\kappa_a \wedge \omega + \kappa_a\, d\omega \\ &= \int_{M,\mathfrak{o}} d\omega + \sum_{a\in A} \int_M d\kappa_a \wedge \omega \\ &= \int_{M,\mathfrak{o}} d\omega + \int_{M,\mathfrak{o}} d\left(\sum_{a\in A} \kappa_a\right) \wedge \omega \\ &= \int_{M,\mathfrak{o}} d\omega + 0,\end{aligned}$$

where the second equation used Step 1 and the fourth and last equalities used he fact that $\sum_{a\in A} \kappa_a \equiv 1$, and the interchange of summation and integral is always justified as these are always finite sums as $\operatorname{supp}(\omega)$ is compact. This completes the proof. ■

Bonus Material for Lecture 26

In this bonus section we prove the Local Stokes' Theorem, which for convenience we restate here.

THEOREM 26.17 (The Local Stokes' Theorem). *Let M be a smooth manifold. Let $q \in \mathsf{Q}_k(M)$ and $\omega \in \Omega^{k-1}(M)$. Then*

$$\int_{\mathsf{q}} d\omega = \int_{\partial\mathsf{q}} \omega.$$

Proof. We prove the result in three steps.

1. Let us first consider the case where $M = \mathbb{R}^k$ and $c = \mathsf{i}_k$ is the standard cube from Example 25.3. This actually represents most of the work. By linearity we may assume that ω is of the form

The Local Stokes' Theorem in the case $M = \mathbb{R}^k$ and $c = \mathsf{i}_k$ is in fact just a fancy way of expressing Fubini's Theorem.

$$\omega = g\, du^1 \wedge \cdots \wedge \widehat{du^j} \wedge \cdots \wedge du^k,$$

where the carat indicates we skip the term du^j. In this first step, we come up with a nice formula for the right-hand side $\int_{\partial \mathsf{i}_k} \omega$.

We have by definition that

$$\int_{\partial \mathsf{i}_k} \omega = \sum_{i=1}^{k} (-1)^i \left(\int_{\mathsf{f}_i} \omega - \int_{\mathsf{b}_i} \omega \right). \tag{26.2}$$

We now claim that:

$$\int_{\mathsf{f}_i} \omega = \begin{cases} \int_{C^{k-1}} g \circ \mathsf{f}_i, & i = j, \\ 0, & i \neq j. \end{cases} \tag{26.3}$$

The proof of (26.3) is a little fiddly. One way to argue this is as follows: from Remark 25.8 and Corollary 19.26 we have that

$$\int_{\mathsf{f}_i} \omega = \int_{C^{k-1}} (g \circ \mathsf{f}_i) \cdot \det A,$$

where $A = (A^h_l)$ is the $(k-1) \times (k-1)$ matrix whose entries are given by

$$A^h_l = D_l(u^h \circ \mathsf{f}_i), \qquad \text{for } 1 \leq l \leq k-1 \text{ and } 1 \leq h \leq k,\ h \neq j.$$

The function $u^h \circ \mathsf{f}_i$ is given by

$$u^h \circ \mathsf{f}_i \mathsf{i}_k(u^1, \dots, u^{k-1}) = u^h(u^1, \dots, u^{i-1}, 0, u^i, \dots u^{k-1}).$$

Thus if $i = j$ then $A^h_l = \delta^h_l$ and thus $\det A = 1$. However if $i \neq j$ then the entire ith row (A^i_l) is zero (since $u^i \circ \mathsf{f}_i$ is the zero function), and thus $\det A = 0$. This proves (26.3). Together with a similar formula for the back face, we see that (26.2) reduces to

$$\int_{\partial \mathsf{i}_k} \omega = (-1)^j \int_{C^{k-1}} g \circ \mathsf{f}_j - g \circ \mathsf{b}_j. \tag{26.4}$$

By Fubini's Theorem and the Fundamental Theorem of Calculus:

$$\begin{aligned} &\int_{C^{k-1}} g \circ \mathsf{b}_j - g \circ \mathsf{f}_j \\ &\qquad = \int_0^1 \cdots \int_0^1 \Big(g(u^1, \dots, 1, \dots, u^k) - g(u^1, \dots, 0, \dots, u^k) \Big) du^1 \cdots \widehat{du^j} \cdots du^k \\ &\qquad = \int_{C^k} \frac{\partial g}{\partial u^j}. \end{aligned}$$

Thus we conclude from (26.4) that

$$\int_{\partial \mathsf{i}_k} \omega = (-1)^{j-1} \int_{C^k} \frac{\partial g}{\partial u^j}.$$

2. We now consider the term $\int_{\mathsf{i}_k} d\omega$. Since $dg = \frac{\partial g}{\partial u^j}\, du^j$ we have (writing the summation signs for clarity)

$$\begin{aligned} \int_{\mathsf{i}_k} d\omega &= \int_{\mathsf{i}_k} dg \wedge du^1 \wedge \cdots \wedge \widehat{du^j} \wedge \cdots \wedge du^k \\ &= \int_{\mathsf{i}_k} \sum_{i=1}^{k} \frac{\partial g}{\partial u^i}\, du^i \wedge du^1 \wedge \cdots \wedge \widehat{du^j} \wedge \cdots \wedge du^k) \\ &= (-1)^{j-1} \int_{\mathsf{i}_k} \frac{\partial g}{\partial u^j}\, du^1 \wedge \cdots \wedge du^k \\ &= (-1)^{j-1} \int_{C^k} \frac{\partial g}{\partial u^j} \end{aligned}$$

This completes the proof for $M = \mathbb{R}^k$ and $c = \mathsf{i}_k$.

3. In the general case, again by linearity we may assume $\mathsf{q} = c$ is a singular k-cube. Then

The general case follows from the special case simply by unravelling the formalism. Thus the Local Stokes' Theorem really is just Fubini's Theorem in disguise.

$$\begin{aligned}
\int_{\partial c} \omega &= \sum_{i=1}^{k} (-1)^i \left(\int_{\mathsf{f}_i c} \omega - \int_{\mathsf{b}_i c} \omega \right) \\
&= \sum_{i=1}^{k} (-1)^i \left(\int_{c \circ \mathsf{f}_i} \omega - \int_{c \circ \mathsf{b}_i} \omega \right) \\
&= \sum_{i=1}^{k} (-1)^i \left(\int_{C^{k-1}} (c \circ \mathsf{f}_i)^* \omega - \int_{C^{k-1}} (c \circ \mathsf{b}_i)^* \omega \right) \\
&= \sum_{i=1}^{k} (-1)^i \left(\int_{C^{k-1}} \mathsf{f}_i^* (c^* \omega) - \int_{C^{k-1}} \mathsf{b}_i^* (c^* \omega) \right) \\
&= \sum_{i=1}^{k} (-1)^i \left(\int_{\mathsf{f}_i} c^* \omega - \int_{\mathsf{b}_i} c^* \omega \right) \\
&= \int_{\partial \mathsf{i}_k} c^* \omega \\
&= \int_{\mathsf{i}_k} d(c^* \omega)
\end{aligned}$$

by the previous step. But since $c^* \circ d = d \circ c^*$ by Lemma 23.4, we have

$$\int_{\mathsf{i}_k} d(c^* \omega) = \int_{\mathsf{i}_k} c^*(d\omega) = \int_c d\omega,$$

where we used Remark 25.9 at the end. This completes the proof. ∎

LECTURE 27

The Poincaré Lemma and the de Rham Theorem

In this final lecture we return to the de Rham cohomology of a smooth manifold. We show that de Rham cohomology is a *homotopy invariant*, and use this to prove the Poincaré Lemma, which states that any closed form is locally exact. In the bonus section we prove the de Rham Theorem, which can be thought of as a massive generalisation of the homotopy invariance property.

We begin with the following application of the global version of Stokes' Theorem.

PROPOSITION 27.1. *Let* $(M, \mathfrak{o})$ *be a oriented manifold, and let* $\omega \in \Omega_c^{m-1}(M)$. *Then* $\int_{M,\mathfrak{o}} d\omega = 0$.

Recall by the convention at the end of Lecture 24, all manifolds are assumed not to have boundary unless explicitly stated to the contrary.

Proof. M is also a smooth manifold with boundary whose boundary is the empty set. The claim is now thus immediate from the Global Stokes' Theorem 26.16. ∎

COROLLARY 27.2. *Let* M *be an oriented connected compact smooth manifold. Then* $H^m_{\mathrm{dR}}(M) \neq 0$.

Proof. Let μ be a volume form. Then for any orientation preserving cube, we have $\int_c \mu > 0$. Thus $\int_{M,\mathfrak{o}} \mu > 0$. The form μ is closed (as $d\mu = 0$ for dimension reasons). If μ was exact then $\int_{M,\mathfrak{o}} \mu = 0$ by Proposition 27.1. Thus μ is a closed non-exact form, and hence defines a non-zero element in $H^m_{\mathrm{dR}}(M)$. ∎

REMARK 27.3. In the bonus section below we will improve Corollary 27.2 and show that for an oriented connected compact smooth manifold M the class $[\mu]$ actually generates $H^m_{\mathrm{dR}}(M)$, and hence $H^m_{\mathrm{dR}}(M) \cong \mathbb{R}$. Together with Lemma 23.3, this implies that

$$H^0_{\mathrm{dR}}(M) \cong H^m_{\mathrm{dR}}(M).$$

This is not an accident. It is a special case of **Poincaré Duality** (together with Universal Coefficients Theorem for Cohomology). We briefly discuss this at the end of the bonus section.

The main step in proving the homotopy invariance property of de Rham cohomology is the following innocuous looking statement.

PROPOSITION 27.4. *Let* M *be a smooth manifold. Define for* $t \in [0,1]$ *a smooth map*

Here we view $M \times [0,1]$ as a smooth manifold with boundary.

$$e_t \colon M \to M \times [0,1], \qquad e_t(x) := (x,t).$$

Then the induced maps on de Rham cohomology

$$e_0^*,\ e_1^* \colon H^k_{\mathrm{dR}}(M \times [0,1]) \to H^k_{\mathrm{dR}}(M)$$

coincide for all $0 \le k \le m+1$.

Proof. We prove the result in two steps.

1. Fix $1 \le k \le m+1$. In this step we construct a map

$$\mathsf{h}\colon \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M)$$

such that for every differential k-form $\omega \in \Omega^k(M \times [0,1])$, one has

$$\mathsf{h}(d\omega) + d(\mathsf{h}\omega) = e_1^*\omega - e_0^*\omega$$

as elements of $\Omega^k(M)$. Let Y denote the vector field on $M \times [0,1]$ whose value at (p,t) is

$$Y(p,t) = \left(0, \frac{\partial}{\partial t}\Big|_t\right).$$

(Compare (9.3) in Lecture 9 – we are using slightly different notation to simplify the formulae to come). The desired map h is then given by

$$\mathsf{h}(\omega) := \int_0^1 e_t^* i_Y \omega \, dt, \tag{27.1}$$

for $\omega \in \Omega^k(M \times [0,1])$. That is, for any $p \in M$, $\mathsf{h}(\omega)_p \in \bigwedge^{k-1} T_p^* M$ given by

$$\mathsf{h}(\omega)_p = \int_0^1 e_t^*(i_Y\omega)_{(p,t)} \, dt,$$

where the integrand is thought of as a function of t on the vector space $\bigwedge^{k-1} T_p^* M$. By choosing local coordinates, we see that the integral defines a smooth $(k-1)$-form on M. To compute $d(\mathsf{h}\omega)$ it suffices to work locally. In local coordinates (x^i) we can express $\mathsf{h}(\omega)$ as a sum

This is just a normal Riemann integral on a vector space, not an integral on a manifold!

$$\mathsf{h}(\omega) = \sum_I \left(\int_0^1 f_I(p,t)\, dt\right) dx^I, \tag{27.2}$$

using the notation introduced in the proof of Theorem 23.1. Applying d to such a term and differentiating under the integral sign gives

This is sometimes referred to as the Leibniz integral rule. In this expression we include the summation signs for clarity.

$$\begin{aligned} d(\mathsf{h}\omega) &= \sum_I \sum_j \frac{\partial}{\partial x^j}\left(\int_0^1 f_I(p,t)\, dt\right) dx^j \wedge dx^I \\ &= \sum_I \left(\int_0^1 \sum_j \frac{\partial f_I}{\partial x^j}(p,t)\, dt\right) dx^j \wedge dx^I. \end{aligned}$$

By comparing (27.1) and (27.2), we see from this last expression that

$$d(\mathsf{h}\omega) = \int_0^1 d\big(e_t^* i_Y \omega\big) dt.$$

Thus using Lemma 23.4 and Cartan's Magic Formula (Theorem 23.12) we see that

$$\begin{aligned} \mathsf{h}(d\omega) + d(\mathsf{h}\omega) &= \int_0^1 \Big(e_t^* i_Y d\omega + d\big(e_t^* i_Y \omega\big)\Big) dt \\ &= \int_0^1 e_t^*\big(i_Y d\omega + d(i_Y\omega)\big) dt \\ &= \int_0^1 e_t^* \mathcal{L}_Y \omega \, dt. \end{aligned}$$

Let Φ_t denote the flow of Y. Then $\Phi_t(x,s) = (x, t+s)$, and thus $e_t = \Phi_t \circ e_0$ and we can compute the Lie derivative as

$$\begin{aligned} e_t^* \mathcal{L}_Y \omega &= e_0^* \Phi_t^* (\mathcal{L}_Y \omega) \\ &= e_0^* \left(\frac{d}{dt} \Phi_t^* \omega \right) \\ &= \frac{d}{dt} e_0^* \Phi_t^* \omega \\ &= \frac{d}{dt} e_t^* \omega. \end{aligned}$$

where the second equation used Problem K.5. Thus by the (normal) Fundamental Theorem of Calculus we obtain

$$\begin{aligned} \mathsf{h}(d\omega) + d(\mathsf{h}\omega) &= \int_0^1 \frac{d}{dt} e_t^*(\omega)\, dt \\ &= e_1^*\omega - e_0^*\omega. \end{aligned}$$

2. We now complete the proof. Suppose $[\omega] \in H^k_{\mathrm{dR}}(M \times [0,1])$. Fix a representative ω of $[\omega]$. Then

$$e_1^*[\omega] - e_0^*[\omega] = [\mathsf{h}(d\omega)] + [d(\mathsf{h}\omega)].$$

Since ω is closed, the first term is zero. The second term is exact, and hence zero in $H^k_{\mathrm{dR}}(M)$. This completes the proof. ■

Why is Proposition 27.4 useful? The next statement makes this abundantly clear.

THEOREM 27.5. *Let M and N be two smooth manifolds and suppose φ and ψ are two homotopic smooth maps from M to N. Then the induced maps φ^* and ψ^* on the de Rham cohomology groups are the same.*

Proof. To say that φ and ψ are homotopic means there is a *continuous* map $H\colon M \times [0,1] \to N$ such that $H(\cdot,0) = \varphi$ and $H(\cdot,1) = \psi$. In fact, by the Homotopy Whitney Approximation Theorem 7.17, we may assume H is a *smooth* map. Then the induced maps on de Rham cohomology satisfy

$$\begin{aligned} \varphi^* &= (H \circ e_0)^* \\ &= e_0^* \circ H^* \\ &= e_1^* \circ H^* \\ &= (H \circ e_1)^* \\ &= \psi^*, \end{aligned}$$

where the third equality used Proposition 27.4. This completes the proof. ■

Two topological spaces X and Y are said to be **homotopy equivalent** if there exist continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that both $f \circ g$ and $g \circ f$ are homotopic to the respective identity maps.

COROLLARY 27.6 (Homotopy invariance of de Rham cohomology). *Let M and N be smooth manifolds that are homotopy equivalent. Then M and N have isomorphic de Rham cohomology groups.*

Proof. Let $f: M \to N$ and $g: N \to M$ be continuous maps such that $f \circ g$ and $g \circ f$ are homotopic to the identity maps. By the Whitney Approximation Theorem 7.13 we can find smooth maps $\varphi: M \to N$ and $\psi: N \to M$ such that φ is homotopic to f and ψ is homotopic to g. Then $\varphi \circ \psi$ and $\psi \circ \varphi$ are homotopic to the identity maps. By Theorem 27.5, $(\varphi \circ \psi)^*$ and $(\psi \circ \varphi)^*$ coincide with the maps induced by the identity. Since id^* is clearly the identity, we see that φ^* is an inverse to ψ^*. The claim follows. ∎

Remark 27.7. A particular case of Corollary 27.6 tells us that the de Rham cohomology cannot see the smooth structure on a topological manifold M. This is surprising, since the data used to define it (the differential forms) very much depend on the choice of smooth structure.

A topological space is **contractible** if it is homotopy equivalent to a point.

Corollary 27.8. *Let M be contractible. Then $H^k_{\mathrm{dR}}(M) = 0$ for all $k \geq 1$.*

Proof. It is clear this is true for M equal to a point. Now apply Corollary 27.6. ∎

Remark 27.9. This shows that de Rham cohomology cannot distinguish Euclidean spaces: $H^k_{\mathrm{dR}}(\mathbb{R}^m)$ is independent of m (since all Euclidean spaces are contractible). Thus a lot of information is lost when passing to de Rham cohomology.

Perhaps the most useful corollary of this is the following statement, which is classically called the **Poincaré Lemma**.

Corollary 27.10 (The Poincaré Lemma). *Let M be a smooth manifold and let $\omega \in \Omega^k(M)$ be a closed differential form of degree $k > 0$. For any point $p \in M$ there exists a neighbourhood U of p such that $\omega|_U$ is an exact form in $\Omega^k(U)$.*

Proof. Every point in a manifold admits a contractible neighbourhood. ∎

And with that we have reached the end Differential Geometry I.

Enjoy your winter vacation, and see you next semester!

Bonus Material for Lecture 27

In this bonus section we prove that the de Rham cohomology agrees with singular cohomology – this is usually referred to as the **de Rham**

Theorem. There are many different ways to prove this result. Perhaps the neatest is via *sheaf cohomology*, but this is a little bit too far afield.

This bonus section assumes you are familiar with singular (co)homology and some basic homological algebra.

Convention. All our homology and cohomology groups should be understood to have coefficients in $\mathbb{R}$ for the remainder of this section. We will not comment on this further.

Let X be a topological space. You are hopefully familiar with the singular chain complex of X. This is normally defined by looking at singular *simplices* (i.e. continuous maps $\Delta^k \to X$, where Δ^k is the kth standard simplex). However one can equally well carry out the construction using singular *cubes* instead. The resulting algebraic invariant is the same (more on this later). Let us recall the definitions in the continuous category.

Definition 27.11. Let X be a topological space. A **singular k-cube** in X is a *continuous* map $c\colon C^k \to X$. We let $Q_k(X)$ denote the (infinite-dimensional) vector space with basis all the singular k-cubes in X.

Remark 27.12. Note in the continuous category there is no need to require c to extend to a map on an open neighbourhood.

Definition 27.13. A singular k-cube $c\colon C^k \to X$ is said to be **degenerate** if there exists $1 \le i \le k$ such that c does not depend on u^i. Otherwise c is said to be **non-degenerate**. We let $D_k(X)$ denote the subspace of $Q_k(X)$ generated by the degenerate cubes, and we let

$$\bar{Q}_k(X) := Q_k(X) \big/ D_k(X)$$

denote the quotient space.

Thus for instance

$$c\colon C^3 \to \mathbb{R}, \qquad c(u^1, u^2, u^3) := u^1 + u^3$$

is a degenerate singular 3-cube in $\mathbb{R}$. The front and back faces of a cube are defined in the same way as Definition 25.18, and this allows us to define the boundary operator as before:

Definition 27.14. Let $c\colon C^k \to X$ be a singular k-cube for $k > 0$. We define the **boundary** of c, written ∂c, to be the element of $Q_{k-1}(X)$ given by

$$\partial c := \sum_{i=1}^{k} (-1)^i (\mathsf{f}_i c - \mathsf{b}_i c).$$

We define the boundary of a 0-cube to be the real number 1. We then extend ∂ to arbitrary k-chains by linearity. Thus we may think of ∂ as a linear map $Q_k(X) \to Q_{k-1}(X)$ for all $k \ge 1$.

As before, this works for $k = -1$ too if we define $Q_{-1}(X) := \mathbb{R}$.

Note that if a cube c is non-degenerate then so is ∂c. Thus we can also regard ∂ as a linear map

$$\partial \colon \bar{Q}_k(X) \to \bar{Q}_{k-1}(X), \qquad k \geq 0.$$

The same argument from Proposition 25.23 gives:

PROPOSITION 27.15. *The boundary operator squares to zero:* $\partial^2 = 0$. *Thus* $(\bar{Q}_\bullet(X), \partial)$ *is a chain complex of vector spaces.*

DEFINITION 27.16. The **cubical singular homology groups** $H_k^{\mathrm{cube}}(X;\mathbb{R})$ are defined to be the homology of this chain complex.

REMARK 27.17. Why bother with quotienting out by the degenerate cubes? After all, $(Q_\bullet(X), \partial)$ is also a chain complex, so we could just take its homology instead. To see this why this quotienting out the degenerate cubes is superior, consider the case where X is a one point space $\{*\}$. It is easy to see that $H_k^{\mathrm{cube}}(\{*\}) = 0$ for $k > 0$ and $H_0^{\mathrm{cube}}(\{*\}) = \mathbb{R}$, as one would hope – this is a necessary requirement in order for $H_\bullet^{\mathrm{cube}}$ to be "a homology theory" in the sense of Eilenberg-Steenrod. However if one does not quotient out by degenerate cubes, this ceases to be the case.

Exercise: Why?

All the properties of the singular chain complex (built with singular simplices) continue to hold without change (homotopy invariance, long exact sequence, excision. Mayer-Vietoris, etc).

REMARK 27.18. In fact, sometimes the proof gets *easier* for singular cubes. For instance, one of the key steps in establishing excision for singular simplices is the concept of **barycentric subdivision**, which allows one to chop up a singular simplex into smaller ones whose diameter can be made arbitrarily small. This is quite involved, and rather unpleasant. On the other hand, it is not very hard to work out how to chop up a singular cube into smaller ones!

REMARK 27.19. For us the main reason we preferred cubes on simplices in Lecture 22 is that it is much easier to define an integral over a cube (it is just a nested sequence of integrals $\int_0^1 \cdots \int_0^1$), whereas this is messier for simplices.

REMARK 27.20. At a much more advanced level, there are compelling reasons both to prefer using simplices and to prefer using cubes. This concerns **cubical sets** and **simplicial sets** in homotopy theory. However this all goes way beyond the course, so we won't discuss it.

It is not completely obvious why the resulting cubical singular homology groups agree with the normal singular homology groups. It can be proved directly using the *Acyclic Models Theorem*, or it can deduced from the uniqueness result for Eilenberg-Steenrod homology theories.

Let us now return to manifolds. If M is a smooth manifold then the vector spaces $Q_k(M)$ defined today do *not* coincide with the vector spaces $Q_k(M)$ defined in Lecture 22. This is because in Lecture 22 we insisted on *smooth* maps. Let us temporarily write $Q_k^\infty(M)$ for the smooth singular k-cubes, and $H_k^{\mathrm{cube},\infty}(M)$ for the homology of the chain complex $(\bar{Q}_k^\infty(M), \partial)$. It is not obvious that the two groups coincide, but luckily they do:

THEOREM 27.21. *Let M be a smooth manifold. Then*

$$H_k^{\mathrm{cube}}(M;\mathbb{R}) \cong H_k^{\mathrm{cube},\infty}(M;\mathbb{R}), \qquad \forall\, k \geq 0.$$

Proof (Sketch). The proof is a standard induction-type argument on the complexity of M. We proceed in six steps.

1. Suppose M is a single point. This is trivial.

2. Suppose M is an open contractible subspace of $\mathbb{R}^m$. This follows from Step 1 and the Whitney Approximation Theorem 7.13, which allows us to assume the contraction of M is smooth.

3. Suppose $M = U \cup V$, where U and V are open in M and the theorem is assumed to be true for U, V and $U \cap V$. We apply naturality of the Mayer-Vietoris sequence to see that the following diagram commutes, where we omit the coefficient group $\mathbb{R}$ so that the diagram fits on the page:

$$\begin{array}{ccccccc} H_k^{\mathrm{cube}}(U\cap V) & \longrightarrow & H_k^{\mathrm{cube}}(U) \oplus H_k^{\mathrm{cube}}(V) & \longrightarrow & H_k^{\mathrm{cube}}(M) & \longrightarrow & H_{k-1}^{\mathrm{cube}}(U\cap V) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_k^{\mathrm{cube},\infty}(U\cap V) & \longrightarrow & H_k^{\mathrm{cube},\infty}(U) \oplus H_k^{\mathrm{cube},\infty}(V) & \longrightarrow & H_k^{\mathrm{cube},\infty}(M) & \longrightarrow & H_{k-1}^{\mathrm{cube},\infty}(U\cap V) \end{array}$$

The Five Lemma then completes the proof.

4. Now assume $M = \bigcup_i U_i$, where $U_i \subset U_{i+1}$ is an open set, and the theorem is true for each U_i. Then the theorem follows for M via an abstract argument using filtered colimits:

$$\begin{aligned} H_k^{\mathrm{cube}}(M;\mathbb{R}) &\cong \varinjlim H_k^{\mathrm{cube}}(U_i;\mathbb{R}) \\ &= \varinjlim H_k^{\mathrm{cube},\infty}(U_i;\mathbb{R}) \\ &\cong H_k^{\mathrm{cube},\infty}(M;\mathbb{R}). \end{aligned}$$

5. Now assume M is an arbitrary open subset of $\mathbb{R}^m$. Then we can write M as a countable union of convex open subsets. For any finite union, the theorem holds by applying Step 3 and induction, and then Step 4 gives the result for M itself.

6. The general case: since we can cover M by charts, it follows from Step 5 and Zorn's Lemma that there exists a maximal open subset $U \subset M$ for which the theorem is true. If $U \neq M$, then we an find a chart domain V such that V is not contained in U. Then by Step 3 and Step 5, the theorem is true for $U \cup V$. This contradicts maximality of U. ∎

With this out of the way, we shall drop the ∞ from the notation and just write $\bar{Q}_k(M)$ for the groups defined in Lecture 22. Let us now recall how one constructs the *cohomology groups* from the homology groups.

DEFINITION 27.22. Let X be a topological space. Set

$$Q^k(X) := \operatorname{Hom}(\bar{Q}_k(X); \mathbb{R})$$

and define $d\colon Q^k(X) \to Q^{k+1}(X)$ by

$$d(\mathsf{a})(\mathsf{q}) := \mathsf{a}(\partial \mathsf{q}), \qquad \alpha \in Q^k(X),\ \mathsf{q} \in \bar{Q}_{k+1}(X).$$

Then $d^2 = 0$, and hence $(Q^\bullet(X), d)$ is cochain complex. Its homology is denoted by $H^k_{\mathrm{cube}}(X;\mathbb{R})$ and referred to as the **cubical singular cohomology** of X.

The next lemma is just a restatement of Corollary 26.4.

LEMMA 27.23. *Let M be a smooth manifold. Then integration induces a cochain map*

$$\Phi\colon \Omega^\bullet(M) \to Q^\bullet(M), \qquad \Phi[\omega][q] := \int_q \omega.$$

We can now state the main result of the lecture.

THEOREM 27.24 (The de Rham Theorem). *The integration cochain map Φ induces a natural isomorphism $H^k_{\mathrm{dR}}(M) \to H^k_{\mathrm{cube}}(M;\mathbb{R})$.*

Singular cohomology is a topological invariant. Thus Corollary 27.6 is an immediate consequence of the de Rham Theorem. Moreover the de Rham Theorem implies that when M is compact and oriented, $H^m_{\mathrm{dR}}(M) \cong \mathbb{R}$, since the same is true of singular cohomology (cf. Remark 27.3).

Proof of the de Rham Theorem (Sketch). The proof proceeds in the same fashion as Theorem 27.21.

1. Suppose M is an open convex subset of $\mathbb{R}^m$. Then the theorem follows from Corollary 27.8 and standard properties of singular cohomology.

2. Suppose $M = U \cup V$, where U and V are open in M and the theorem is assumed to be true for U, V and $U \cap V$. The theorem will gain follow for M via a standard argument using the Mayer-Vietoris sequences and the Five Lemma, apart from the fact that we have not constructed the Mayer-Vietoris sequence in de Rham cohomology. Let us rectify this. We denote by

$$\imath_U\colon U \cap V \hookrightarrow U, \qquad \imath_V\colon U \cap V \hookrightarrow V$$

and

$$e_U\colon U \hookrightarrow M, \qquad e_V\colon V \hookrightarrow M$$

the inclusions. One then defines

$$\alpha\colon \Omega^\bullet(M) \to \Omega^\bullet(U) \oplus \Omega^\bullet(V), \qquad \alpha(\omega) := (e_U^*(\omega), e_V^*(\omega))$$

and

$$\beta\colon \Omega^\bullet(U) \oplus \Omega^\bullet(V) \to \Omega^\bullet(U \cap V), \qquad \beta(\omega_1, \omega_2) := \imath_U^*(\omega_1) - \imath_V^*(\omega_2).$$

Then we claim that

$$0 \to \Omega^\bullet(M) \xrightarrow{\alpha} \Omega^\bullet(U) \oplus \Omega^\bullet(V) \xrightarrow{\beta} \Omega^\bullet(U \cap V) \to 0$$

is exact. The only claim that isn't clear is why β should be surjective. To see this, let $\{\kappa_U, \kappa_V\}$ be a partition of unity subordinate to the open cover $\{U, V\}$. Then if ω is a k-form on $U \cap V$, we can think of $\kappa_U \omega$ and $\kappa_V \omega$ as k-forms on U and V respectively, and $\beta(\kappa_U\, \omega, -\kappa_V\, \omega) = \omega$.

3. Now assume $M = \bigcup_i U_i$, where $U_i \subset U_{i+1}$ is an open set such that $\bar{U}_i$ is compact for each i, and assume the theorem is true for each U_i. Then the theorem is also true for M. The proof of this is considerably harder than the proof of Step 4 of Theorem 27.21, since now we are working with cohomology, and thus instead of filtered colimits we have filtered *limits*. It is a sad fact of life that limits are less well behaved than colimits, and are not exact functors from diagrams of vector spaces to diagrams of vector spaces. Consequently we need to worry about the first right *derived* functors $R^1 \varprojlim$. But this is not too bad: since $\bar{U}_i$ is compact one has $R^1 \varprojlim \Omega^k(U_i) = 0$ for all k and i, and thus we have a natural short exact sequence

$$0 \to R^1 \varprojlim H^{k-1}_{\mathrm{dR}}(U_i)) \to H^k_{\mathrm{dR}}(M) \to \varprojlim H^k_{\mathrm{dR}}(U_i) \to 0.$$

A similar sequence holds for $H^k_{\mathrm{cube}}(M)$, and naturality of the two sequences allow us to conclude this step.

4. Now assume M is an arbitrary open subset of $\mathbb{R}^m$. Then we can write M as a countable union of convex open subsets. For any finite union, the theorem holds by applying Step 2 and induction, and then Step 3 gives the result for M itself.

5. The general case: this follows from the previous step, since M has a countable basis of open sets diffeomorphic to open sets of Euclidean space. The proof is complete. ∎

From a purely topological point of view, the thing that makes manifolds "special" about manifolds is the following duality between homology and cohomology.

THEOREM 27.25 (Poincaré Duality). *Let M be a compact connected orientable manifold M. Then*

$$H^k_{\mathrm{cube}}(M; \mathbb{R}) \cong H^{\mathrm{cube}}_{m-k}(M; \mathbb{R}),$$

and hence there is a non-degenerate pairing

$$H^k_{\mathrm{cube}}(M; \mathbb{R}) \times H^{m-k}_{\mathrm{cube}}(M; \mathbb{R}) \to \mathbb{R}.$$

Poincaré Duality is false if we move outside the category of manifolds (eg. to finite cell complexes). For de Rham cohomology, this pairing is particularly easy to understand: it is induced from the pairing

$$\Omega^k(M) \times \Omega^{m-k}(M) \to \mathbb{R}, \qquad (\omega, \theta) \mapsto \int_M \omega \wedge \theta.$$

We leave it to you to investigate how to prove this.

LECTURE 28

Connections

Welcome to Differential Geometry II!

Differential Geometry I introduced the basics of smooth manifolds and bundle theory. Differential Geometry II will primarily be concerned with two extra pieces of data one can endow a manifold or bundle with: a **connection** and a **Riemannian metric**. The study of connections on bundles is usually called **gauge theory**, and the study of Riemannian manifolds – that is, smooth manifolds equipped with a Riemannian metric – is referred to as **Riemannian geometry**.

We begin with connections and gauge theory. To motivate the notion of a connection, let consider the following rather simple idea.

Let M be a smooth manifold. Suppose $f \in C^\infty(M)$ is a smooth function and $X \in \mathfrak{X}(M)$ is a vector field. We can feed f to X to get another smooth function $X(f) = df(X)$. Now consider the trivial one-dimensional vector bundle $M \times \mathbb{R} \to M$ over M. Recall from part (iii) of Examples 20.2 that there is a bijective correspondence between smooth functions f on M and sections $s \in \Gamma(M \times \mathbb{R})$. Explicitly, any section s can be uniquely written as

$$s(p) = s_f(p) = (p, f(p))$$

for a smooth function f. Thus the operation $f \mapsto X(f)$ can also be thought of as an operator on the space of sections of the trivial bundle. We write this operator as ∇_X:

$$\begin{aligned} \nabla_X \colon \Gamma(M \times \mathbb{R}) &\to \Gamma(M \times \mathbb{R}), \\ s_f &\mapsto s_{X(f)}. \end{aligned}$$

The operator ∇_X is local operator in the sense of Definition 20.16 but – provided X is not identically zero – it is not a point operator.

Next, note that the value of $\nabla_X s$ at a point p depends on X only via the tangent vector $X(p)$. Indeed, if $\gamma \colon (-\varepsilon, \varepsilon) \to M$ is any smooth curve with $\gamma(0) = p$ and $\dot\gamma(0) = X(p)$ then (up to identifying s_f with s) we have

$$\begin{aligned} (\nabla_X s_f)(p) &= df_p(X(p)) \\ &= \frac{d}{dt}\Big|_{t=0} f(\gamma(t)) \\ &= \lim_{t \to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t}. \end{aligned} \tag{28.1}$$

This shows that we can think of $\nabla_X s$ as "differentiating s in the direction of X".

Let us now see what goes wrong with extending this idea to an arbitrary vector bundle. Let $\pi \colon E \to M$ be a vector bundle. As before, let X be a vector field on M and let $s \in \Gamma(E)$. Fix a point $p \in M$ and let γ denote a smooth curve with $\gamma(0) = p$ and $\dot\gamma(0) = X(p)$. We can again attempt to "define" a new section via (28.1)

$$\text{“}(\nabla_X s)\text{”}(p) = \lim_{t \to 0} \frac{s(\gamma(t)) - s(\gamma(0))}{t}. \tag{28.2}$$

A moment's thought reveals that (28.2) is nonsense: the vector $s(\gamma(t))$ belongs to the vector space $E_{\gamma(t)}$, and for different values of t these are *different* vector spaces. Thus is simply *does not make sense* to add or subtract them from one another. One could use a local trivialisation of the bundle around $\gamma(0)$ to identify all the fibres with one fixed vector space, but unfortunately the resulting vector would depend on the choice of trivialisation.

Compare this to our original problem right at the beginning of Lecture 1 when we motivated manifolds: on an arbitrary topological space one cannot simply "add" points together. On a vector bundle whilst each fibre has a linear structure, in general each fibre is a different vector space, and thus we cannot add points.

The reason this worked on the trivial bundle $M \times \mathbb{R} \to M$ was that in this case each fibre $\{p\} \times \mathbb{R}$ was *canonically* isomorphic to $\mathbb{R}$ via the second projection. Equivalently, the identification $s = s_f$ of sections of $M \times \mathbb{R}$ with smooth functions on M was canonical – no choices were needed. This is also reflected in the fact that on the trivial bundle ∇_X can be identified with the Lie derivative $\mathcal{L}_X$.

More generally, the process we described at the start of the lecture works on any trivial bundle, and this leads us to the first definition of the course.

Definition 28.1. Let M be a smooth manifold and let $E = M \times \mathbb{R}^n$ denote the trivial bundle over M. The **trivial connection** on E associates to every vector field X on M the operator

$$\nabla_X \colon \Gamma(E) \to \Gamma(E)$$

given by

$$(\nabla_X s_f)(p) := \lim_{t \to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t},$$

where we identify each section $s = s_f$ with a smooth function $f \colon M \to \mathbb{R}^n$.

Thus (by definition) any trivial vector bundle admits a trivial connection. In fact, the converse is true: if E admits a trivial connection then E is necessarily a trivial bundle, although this will take us some time to prove (see Proposition 33.2), and will require us to give an alternative definition of a connection that does not explicitly reference the trivialisation.

We first define the weaker notion of a *preconnection*, which will work in an arbitrary fibre bundle. As with many of the concepts we've seen in Differential Geometry, the relation between the formal definition of a connection and Definition 28.1 will at first sight not be so obvious. We will rectify this in Lecture 32.

Definition 28.2. Let $L \to E \xrightarrow{\pi} M$ be a fibre bundle over a smooth manifold M with fibre L. A **preconnection** on E is a distribution Δ on E (i.e. a vector subbundle of the tangent bundle TE) with the additional property that for every $u \in E$ the map $D\pi(u)|_{\Delta_u} \colon \Delta_u \to T_{\pi(u)}M$ is a linear isomorphism.

This terminology is not standard. Some authors call what we call a preconnection an *Ehresmann connection*. However this won't matter, since we will shortly upgrade a preconnection to a genuine connection, and then will not have cause to speak about preconnections anymore.

Let us unpack this a bit. Requiring that $D\pi(u)|_{\Delta_u} \colon \Delta_u \to T_{\pi(u)}M$ is a linear isomorphism for every $u \in E$ is the same thing as saying that the restriction of $D\pi \colon TE \to TM$ to Δ is a vector bundle isomorphism along $\pi \colon E \to M$ in the sense of Definition 17.9.

Recall the notion of the pullback bundle from Problem H.4. We pullback the tangent bundle of M along the footpoint map of E:

When there are multiple bundles in play, we will label the various footpoint maps where needed.

$$\begin{array}{ccc} \pi^*TM & \xrightarrow{\mathrm{pr}_2} & TM \\ \Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow{\scriptstyle \pi_{TM}} \\ E & \xrightarrow[\pi]{} & M \end{array}$$

Since $TM \to M$ is a vector bundle, by part (iv) of Problem H.4 the bundle $\pi^*TM \to E$ is a vector bundle (even when E itself is only a fibre bundle). Explicitly:

$$\pi^*TM = \left\{(u, \xi) \in E \times TM \mid \xi \in T_{\pi(u)}M\right\},$$

and hence $(\pi^*TM)_u \cong T_{\pi(u)}M$.

Yet another way to express Definition 28.2 uses a notion from Problem Sheet I instead. Recall from Problem I.5 that the **vertical bundle** of E is the vector bundle $VE = \ker D\pi$. If M has dimension m and L has dimension l then E has dimension $m + \ell$ and $VE \to E$ is a vector subbundle of TE of rank l. If Δ is preconnection on E then $\Delta \to E$ is a vector subbundle of TE of rank m. Since $\Delta_u \cap \ker D\pi(u) = \{0\}$ we therefore have

$$TE \cong \Delta \oplus VE.$$

We summarise this with the following:

Lemma 28.3. *Let $L \to E \xrightarrow{\pi} M$ be a fibre bundle, and suppose Δ is a distribution on E. The following are equivalent:*

(i) *Δ is a preconnection,*

(ii) *$D\pi$ induces a vector bundle isomorphism $\Delta \cong \pi^*TM$,*

(iii) *$TE \cong \Delta \oplus VE$.*

So much for preconnections. If we instead start with a vector bundle, we can impose an additional condition on a preconnection, which gives rise to a *connection*. Recall from part (ii) of Examples 10.9 that $\mathbb{R} \setminus \{0\}$ is a Lie group under multiplication. There is a canonical action of $\mathbb{R} \setminus \{0\}$ on any vector bundle.

Definition 28.4. Let $\pi\colon E \to M$ be a vector bundle. Fix $p \in M$ and $c \in \mathbb{R} \setminus \{0\}$. We define $\mu_c\colon E_p \to E_p$ by

$$\mu_c(v) \coloneqq cv,$$

where the right-hand side denotes scalar multiplication in the vector space E_p. It is immediate that μ defines a fibre preserving action which we simply call **scalar multiplication.**

It is convenient to extend this definition to work for $c = 0$ by setting μ_0 to be the zero map in each fibre. Note however that μ is *not* an action of the Lie group $\mathbb{R}$ under addition.

Definition 28.5. Let $\pi\colon E \to M$ be a vector bundle over a smooth manifold M. A **connection** Δ on E is a preconnection with the additional property that for every $v \in E$ and every $c \in \mathbb{R}$, one has

Recall for a vector bundle we typically write a generic point with the letter v rather than u.

$$D\mu_c(v)(\Delta_v) = \Delta_{cv}. \tag{28.3}$$

The true significance of this condition won't become apparent until we discuss connections on principal bundles in Lecture 39 (see Proposition 39.10 in particular), although see Problem L.1 for one key consequence of (28.3).

For now, let us now prove that (pre)connections always exist.

Theorem 28.6. *Every fibre bundle admits a preconnection. Every vector bundle admits a connection.*

Proof. We prove the result in two steps.

1. We first prove the result when $E = M \times L$ is the trivial bundle. Let $\iota_q\colon M \to M \times L$ denote the map $p \mapsto (p, q)$, and set

$$\Delta_{(p,q)} \coloneqq D\iota_q(p)(T_pM).$$

This is a preconnection. If in addition $L = V$ is a vector space then this is a connection, since $\mu_c \circ \iota_v = \iota_{cv}$ and thus

$$\begin{aligned} D\mu_c(p,v)(\Delta_{(p,v)}) &= D\mu_c(p,v) \circ D\iota_v(p)(T_pM) \\ &= D\iota_{cv}(p)(T_pM) \\ &= \Delta_{(p,cv)}. \end{aligned}$$

2. For the general case, we use a partition of unity argument. Let $\{U_a \mid a \in A\}$ denote an open cover of M such that E is trivial over each U_a, and let $\{\kappa_a \mid a \in A\}$ denote a partition of unity subordinate to this cover. Let Δ^a denote a (pre)connection on $\pi^{-1}(U_a)$, whose existence is guaranteed by Step 1. Given $p \in M$ and $u \in E_p$, define

$$\ell_u\colon T_pM \to T_uE, \qquad \ell_u(\xi) \coloneqq \sum_{\{a \in A \mid p \in U_a\}} \kappa_a(p)\zeta_a,$$

where ζ_a is the unique vector in Δ^a_u such that $D\pi(u)\zeta_a = \xi$. Then ℓ_u is a linear map such that $D\pi(u) \circ \ell_u = \mathrm{id}_{T_pM}$. We then define

$$\Delta_p \coloneqq \ell_u(T_pM).$$

This is a (pre)connection. ■

We can use (pre)connections to lift vectors from TM to TE.

DEFINITION 28.7. Let $\pi\colon E \to M$ be a fibre bundle, and let Δ be a preconnection on E. The splitting $TE = \Delta \oplus VE$ allows us to uniquely express any vector $\zeta \in TE$ as

$$\zeta = \zeta^{\mathsf{h}} + \zeta^{\mathsf{v}}$$

where if $\zeta \in T_pE$ then $\zeta^{\mathsf{h}} \in \Delta_p$ and $\zeta^{\mathsf{v}} \in V_pE$. We call ζ^{h} the **horizontal component** of ζ and ζ^{v} the **vertical component** of ζ. A vector is **horizontal** if $\zeta = \zeta^{\mathsf{h}}$ and **vertical** if $\zeta = \zeta^{\mathsf{v}}$.

The property of being horizontal depends on the specific choice of preconnection, but the property of being vertical does not.

DEFINITION 28.8. Let $\pi\colon E \to M$ be a fibre bundle, and let Δ be a preconnection on E. Let $p \in M$, $u \in E_p$ and $\xi \in T_pM$. The **horizontal lift of** ξ **at** p is the unique horizontal vector $\overline{\xi} \in T_uE$ such that $D\pi(u)\overline{\xi} = \xi$.

Any horizontal lift is a horizontal vector. Conversely any horizontal vector is the horizontal lift of some tangent vector on M.

Since $u \mapsto \Delta_u$ is smooth (this is true of any distribution) we can also lift vector fields.

DEFINITION 28.9. Let $\pi\colon E \to M$ be a fibre bundle, and let Δ be a preconnection on E. If $X \in \mathfrak{X}(M)$ is a vector field then the **horizontal lift** of X is the unique vector field $\overline{X} \in \mathfrak{X}(E)$ such that $\overline{X}(p)$ is the horizontal lift of $X(\pi(p))$ at p for each $p \in E$.

The following result is almost immediate.

LEMMA 28.10. *Let $\pi\colon E \to M$ be a fibre bundle and let Δ be a preconnection on E. Given $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, we have:*

(i) $\overline{X+Y} = \overline{X} + \overline{Y}$,

(ii) $\overline{fX} = (f \circ \pi)\overline{X}$,

(iii) $\overline{[X,Y]} = [\overline{X}, \overline{Y}]^{\mathsf{h}}$.

Proof. The first two statements are obvious. For the third we observe that

$$D\pi[\overline{X}, \overline{Y}] = [X, Y] = D\pi\overline{[X,Y]},$$

and thus $\overline{[X,Y]} = [\overline{X}, \overline{Y}]^{\mathsf{h}}$ by definition of a preconnection. ■

We conclude this lecture with a brief discussion of the bigger picture:

What is a connection? Connections on vector bundles can be defined in many different ways. In this course we will see at least five:

(i) as distributions $\Delta \subset TE$,

(ii) as parallel transport systems $\mathbb{P}$ on E,

(iii) as covariant derivative operators ∇_X on $\Gamma(E)$,

(iv) in local coordinates via Christoffel symbols Γ_{ij}^k,

(v) via principal bundles $\varpi \in \Omega^1(P, \mathfrak{g})$

Today we did method (i). This is perhaps the cleanest and quickest way to define connections, but it is not always the most useful. We will cover method (ii) over the next two lectures. The motivational discussion at the start of the lecture concerns method (iii). We will come to this in Lecture 32. Arguing in local coordinates (method (iv)) is ugly, but useful for computations. We will eventually prove the equivalence of methods (i)–(iv). The principal bundle approach is more general; however, for matrix Lie groups method (v) is equivalent to all the others. We will prove this in Lecture 40.

Compare this to how we initial defined tangent vectors as derivations on the space of germs

Compare Problem I.1.

because principal bundles are more general than vector bundles, cf. the discussion at the end of Lecture 18.

Bonus Material for Lecture 28

Here is another take on Lemma 28.3, that uses a little bit of homological algebra.

Associated to any fibre bundle $\pi\colon E \to M$ is a short exact sequence of bundles

$$0 \to VE \xrightarrow{\iota} TE \xrightarrow{D\pi} \pi^*TM \to 0, \tag{28.4}$$

where the first map is inclusion and the second map is induced by $D\pi$. It follows from basic homological algebra (actually, in this case, linear algebra) that this sequence splits. That is, there exists a vector bundle homomorphism

$$\Phi\colon \pi^*TM \to TE \tag{28.5}$$

such that $D\pi \circ \Phi = \mathrm{id}_{\pi^*TM}$. If we set $\Delta := \operatorname{im}\Phi$ then Δ is a preconnection on E. In this way, we see that a choice of preconnection corresponds to a choice of splitting of (28.4). In general there is no canonical choice of splitting of (28.4) – and hence also no canonical choice of connection.

Finally, the existence of a splitting on the right-hand map of (28.4) is equivalent to the existence of a splitting on the left-hand map. That is, a choice of preconnection Δ is also equivalent to a bundle homomorphism $\Psi\colon TE \to VE$ such that $\iota \circ \Psi = \mathrm{id}_{VE}$; i.e. an idempotent bundle homomorphism $\Psi\colon TE \to TE$ such that $\operatorname{im}\Psi = VE$; this means that Ψ is a projection operator $TE \to VE$.

Using the the notion of a bundle-valued form (defined in Lecture 36), we arrive at what is arguably the single cleanest definition.

LEMMA 28.11. *A preconnection Δ on E is equivalent to a bundle-valued 1-form $\Psi \in \Omega^1(E, VE)$ such that $\Psi \circ \Psi = \Psi$ and $\operatorname{im}\Psi = VE$.*

LECTURE 29

Parallel Transport

In this lecture we define **parallel transport** axiomatically. Next week we will prove that the existence of a parallel transport system is equivalent to the existence of a connection, and thus, going forward we will view the two interchangeably.

We begin by showing that (pre)connections behave nicely under pullbacks. Let $\varphi\colon M \to N$ be a smooth map, and suppose $L \to E \xrightarrow{\pi} N$ is a fibre bundle. Recall from part (ii) of Problem H.4 that the tangent space of the pullback bundle φ^*E is given by

$$T_{(p,u)}(\varphi^*E) = \Big\{(\xi, \zeta) \in T_pM \times T_uE \mid D\varphi(p)\xi = D\pi(u)\zeta\Big\}. \qquad (29.1)$$

Suppose Δ is a (pre)connection on E. We define

$$\varphi^*\Delta := (D\,\mathrm{pr}_2)^{-1}(\Delta)$$

that is,

$$(\varphi^*\Delta)_{(p,u)} := \big\{(\xi, \zeta) \in T_{(p,u)}(\varphi^*E) \mid D\,\mathrm{pr}_2(p,u)(\xi,\zeta) \in \Delta_u\big\}.$$

PROPOSITION 29.1. *$\varphi^*\Delta$ is a preconnection on φ^*E. If E is a vector bundle and Δ is a connection on E then $\varphi^*\Delta$ is a connection on the vector bundle φ^*E.*

Proof. It follows from (29.1) that $V(\varphi^*E) = \{0\} \times VE$ and that $\varphi^*\Delta$ is given by

$$(\varphi^*\Delta)_{(p,u)} := \{(\xi, \zeta) \in T_pM \times \Delta_u \mid D\varphi(p)\xi = D\pi(u)\zeta\}.$$

Since any $\zeta \in TE$ decomposes uniquely as $\zeta^{\mathsf{h}} + \zeta^{\mathsf{v}} \in \Delta \oplus VE$, any $(\xi, \zeta) \in T(\varphi^*E)$ decomposes uniquely as

$$(\xi, \zeta) = (\xi, \zeta^{\mathsf{h}}) + (0, \zeta^{\mathsf{v}}) \in \varphi^*\Delta \oplus V(\varphi^*E).$$

This shows that $\varphi^*\Delta$ is complementary to $V(\varphi^*E)$, and thus $\varphi^*\Delta$ is a preconnection on φ^*E. If E is a vector bundle and Δ is a connection then so is $\varphi^*\Delta$, since if $(\xi, \zeta) \in (\varphi^*\Delta)_{(p,u)}$ we have

$$D\mu_c(p,u)(\xi,\zeta) = \big(\xi, D\mu_c(u)\zeta\big) \in \Delta_{(p,cu)}.$$

This completes the proof. ∎

Now for some more terminology. In the last lecture we defined what it meant for a tangent vector (or a vector field) to be horizontal. Now we explain what it means for a *section* to be horizontal.

DEFINITION 29.2. Let $\pi\colon E \to M$ be a fibre bundle with preconnection Δ. A section $s \in \Gamma(E)$ is said to be **horizontal** if

$$Ds(p)(T_pM) = \Delta_{s(p)}, \qquad \forall\, p \in M. \qquad (29.2)$$

Similarly a local section $s \in \Gamma(U, E)$ is horizontal if the above holds for all $p \in U$.

REMARK 29.3. If s is any section, then differentiating the equation $\pi \circ s = \mathrm{id}$ tells us that $Ds(p)(T_pM)$ is a subspace of dimension $m = \dim M$ inside $T_{s(p)}E$. Since also $\dim \Delta_{s(p)} = m$, we see that

$$Ds(p)(T_pM) \subseteq \Delta_{s(p)} \qquad \Rightarrow \qquad Ds(p)(T_pM) = \Delta_{s(p)}.$$

Thus we can replace the equality sign in (29.2) with $\subseteq$.

Next, we introduce the idea of a section *along* a map.

DEFINITION 29.4. Suppose $\pi\colon E \to N$ is a fibre bundle over a smooth manifold and $\varphi\colon M \to N$ is a smooth map. A **section of E along φ** is a smooth map $s\colon M \to E$ such that $s(p) \in E_{\varphi(p)}$. We denote by $\Gamma_\varphi(E)$ the space of such sections. If $U \subset M$ is an open set then we can also speak of the space $\Gamma_\varphi(U, E)$ of smooth maps $s\colon M \to E$ such that $s(p) \in E_{\varphi(p)}$ for all $p \in U$; we refer to these as **local sections along φ**.

Sections along a map are not really anything new:

LEMMA 29.5. *Suppose $\pi\colon E \to N$ is a fibre bundle over a smooth manifold and $\varphi\colon M \to N$ is a smooth map. There is a bijective correspondence between sections of the pullback bundle $\varphi^*E \to M$ and sections of E along φ. Thus:*

$$\Gamma_\varphi(E) \cong \Gamma(\varphi^*E).$$

The same is true for local sections.

Proof. Let $\mathrm{pr}_2\colon \varphi^*E \to E$ denote the second projection:

$$\begin{array}{ccc} \varphi^*E & \xrightarrow{\ \mathrm{pr}_2\ } & E \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[\ \pi\ ]{} & N \end{array}$$

If $\tilde{s} \in \Gamma(\varphi^*E)$ then

$$s = \mathrm{pr}_2 \circ \tilde{s}$$

is a section of E along φ. Conversely a section s of E along φ uniquely determines a section $\tilde{s} \in \Gamma(\varphi^*E)$ by the same equation. ∎

As a result of Lemma 29.5, we will often simply identify elements of $\Gamma_\varphi(E)$ and $\Gamma(\varphi^*E)$, and write them both with the same letter.

DEFINITION 29.6. Let $\pi\colon E \to N$ be a fibre bundle and let Δ be a preconnection on E. Suppose $\varphi\colon M \to N$ is a smooth map and $s \in \Gamma_\varphi(E)$ is a section of E along φ. We say that s is **horizontal along φ** if the corresponding section of φ^*E is horizontal with respect to the pullback connection $\varphi^*\Delta$. Explicitly, this means that

$$Ds(p)(T_pM) \subseteq \Delta_{s(p)}, \qquad \forall p \in M. \tag{29.3}$$

The fact that (29.2) has an $=$ and (29.3) has an $\subseteq$ is not a typo!

If we take $M = N$ and φ to be the identity Remark 29.3 tells us that Definition 29.2 and 29.6 coincide. At the opposite extreme, if we take M to be a point $q \in N$ then a section of the pullback bundle can be identified with an element of $T_q N$, and all such elements are horizontal. A more useful case arises when M has dimension 1, as we now explain.

Example 29.7. Take M to be an interval (a, b) and $\varphi = \gamma\colon (a, b) \to N$ to be a smooth curve in N. We will usually use the special letter ρ (instead of s) to denote a section along a curve. Thus a section $\rho \in \Gamma_\gamma(E)$ is simply a smooth curve in E such that $\rho(t) \in E_{\gamma(t)}$ for all $t \in (a, b)$. Moreover ρ is horizontal along γ if

$$\dot\rho(t) \in \Delta_{\rho(t)}, \qquad \forall\, t \in (a, b).$$

Note that if $\rho \in \Gamma_\gamma(E)$ then $\pi \circ \rho = \gamma$ and hence

$$D\pi(\rho(t))\dot\rho(t) = \dot\gamma(t).$$

Remark 29.8. It will often be convenient to work with smooth curves defined on a *closed* interval $[a, b]$. Here "smooth" can be interpreted as either requiring that there exists a smooth extension to some interval $(a - \varepsilon, b + \varepsilon)$, or just by considering $[a, b]$ as a smooth manifold with boundary. Note however that if $\gamma\colon [a, b] \to N$ is a smooth curve then $\gamma^* E \to [a, b]$ is a vector bundle over a smooth manifold with boundary.

Proposition 29.9. *Let $\pi\colon E \to M$ be a fibre bundle, and let Δ be a preconnection on E. Let $\gamma\colon [a, b] \to M$ be a smooth curve and let $t_0 \in [a, b]$. Then for any $u \in E_{\gamma(t_0)}$, there exists $\varepsilon > 0$ and unique horizontal section ρ of E along $\gamma|_{(t_0 - \varepsilon, t_0 + \varepsilon)}$ such that $\rho(t_0) = u$.*

Proof. We consider the pullback bundle $\gamma^* E$ over $[a, b]$:

$$\begin{array}{ccc} \gamma^* E & \xrightarrow{\ \mathrm{pr}_2\ } & E \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ [a, b] & \xrightarrow[\ \gamma\ ]{} & M \end{array}$$

We abbreviate by T the vector field $\frac{\partial}{\partial t}$ on $[a, b]$. Let Let $\overline{T} \in \mathfrak{X}(\gamma^* E)$ denote the horizontal lift of with respect to the pullback connection $\gamma^*\Delta$. Let δ denote the integral curve of $\overline{T}$ in $\gamma^* E$ such that $\delta(t_0) = (t_0, p)$, which is defined on some interval $I := (t_0 - \varepsilon, t_0 + \varepsilon)$. We claim that $\eta := \mathrm{pr}_1 \circ \delta$ is an integral curve of T. To see this we compute

$$\begin{aligned} \dot\eta(t) &= D\,\mathrm{pr}_1(\delta(t))\dot\delta(t) \\ &= D\,\mathrm{pr}_1(\delta(t))\overline{T}(\delta(t)) \\ &= T(\eta(t)). \end{aligned}$$

Since $\eta(t_0) = t_0$ and η is an integral curve of T, we must have $\eta(t) = t$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. Thus δ is a section of $\gamma^* E$ over I: $\delta \in \Gamma(I, \gamma^* E)$. Moreover it follows from the definition of δ and T that δ is a horizontal section of $\gamma^* E$. Thus by Lemma 29.5, $\rho := \mathrm{pr}_2 \circ \delta$ is an element of $\Gamma_\gamma(I, E)$ which is horizontal and satisfies $\rho(t_0) = u$. Finally, uniqueness is immediate from the uniqueness of integral curves. ■

For connections on vector bundles, Proposition 29.9 can be strengthened. On Problem Sheet L you will show:

PROPOSITION 29.10. *Let $\pi: E \to M$ be a vector bundle, and let Δ be a connection on E. Let $\gamma: [a,b] \to M$ be a smooth curve and let $t_0 \in [a,b]$. Then for any $v \in E_{\gamma(t_0)}$, there exists a unique horizontal section ρ of E along γ such that $\rho(t_0) = v$.*

The difference is that for connections, the corresponding section ρ is defined on all of $[a,b]$. This is because in this case the vector field $\overline{T}$ is complete.

From now on we will focus solely on vector bundles and connections, rather than fibre bundles and preconnections. Here is main definition of today's lecture.

DEFINITION 29.11. Let $\pi: E \to M$ be a vector bundle over a smooth manifold. A **parallel transport system** $\mathbb{P}$ on E assigns to every point $v \in E$ and every curve $\gamma: [a,b] \to M$ with $\gamma(a) = \pi(v)$, a unique section $\mathbb{p}_{\gamma;v} \in \Gamma_\gamma(E)$ with initial condition v, i.e. such that $\mathbb{p}_{\gamma;v}(a) = v$. One calls $\mathbb{p}_{\gamma;v}$ the **parallel lift** of γ starting at v. This association should satisfy the following five axioms:

(i) **(Linear isomorphism)**: For every smooth curve $\gamma: [a,b] \to M$ the map

$$\mathbb{P}_\gamma : E_{\gamma(a)} \to E_{\gamma(b)}, \qquad \mathbb{P}_\gamma(v) := \mathbb{p}_{\gamma;v}(b)$$

is a linear isomorphism. Moreover

$$\mathbb{P}_\gamma^{-1} = \mathbb{P}_{\gamma^-}$$

where $\gamma^-: [a,b] \to M$ is the reverse curve $t \mapsto \gamma(a - t + b)$.

(ii) (**Concatenation**): If $\gamma: [a,b] \to M$ is a smooth path and $c \in (a,b)$. then if we abbreviate

$$\gamma_1 := \gamma|_{[a,c]}, \qquad \gamma_2 := \gamma|_{[c,b]}, \qquad w := \mathbb{p}_{\gamma;v}(c),$$

then

$$\mathbb{p}_{\gamma;v}(t) = \begin{cases} \mathbb{p}_{\gamma_1;v}(t), & t \in [a,c], \\ \mathbb{p}_{\gamma_2;w}(t), & t \in [c,b]. \end{cases}$$

This implies that

$$\mathbb{P}_\gamma = \mathbb{P}_{\gamma_2} \circ \mathbb{P}_{\gamma_1}.$$

(iii) (**Independence of parametrisation**): If $\gamma: [a,b] \to M$ is a smooth curve and $h: [a_1,b_1] \to [a,b]$ is a diffeomorphism such that $h(a_1) = a$ and $h(b_1) = b$ then for every point $v \in E_{\gamma(a)}$ and every $t \in [a_1, b_1]$, we have

$$\mathbb{p}_{\gamma \circ h;v}(t) = \mathbb{p}_{\gamma;v}(h(t)).$$

(iv) **(Smooth dependence on initial conditions)**: The section $\mathbb{p}_{\gamma;v}$ depends smoothly on both γ and v.

The precise statement of this axiom is given in the bonus section below.

(v) **(Initial uniqueness)**: Suppose $\gamma, \delta: [a,b] \to M$ are two curves such that $\gamma(a) = \delta(a)$ and $\dot\gamma(a) = \dot\delta(a)$. Then for each $v \in E_{\gamma(a)}$, the two curves $\mathbb{p}_{\gamma;v}$ and $\mathbb{p}_{\delta;v}$ have the same initial tangent vector:

$$\dot{\mathbb{p}}_{\gamma;v}(a) = \dot{\mathbb{p}}_{\delta;v}(a)$$

Remark 29.12. In general if $\gamma\colon [a,b] \to M$ is a smooth curve on M and $\rho \in \Gamma_\gamma(E)$ is any section along γ then we say ρ is **parallel along** γ if $\rho = \mathbb{p}_{\gamma;v}$ for some $v \in E_{\gamma(a)}$. By uniqueness, this is only possible for $v = \rho(0)$.

Example 29.13. Let $E = M \times \mathbb{R}^n$ be a trivial bundle. We define the **trivial parallel transport system** on E by declaring that constant sections are parallel. Explicitly, if $\gamma\colon [a,b] \to M$ is any smooth curve with $\gamma(a) = p$ then we define

$$\mathbb{p}_{\gamma;v}(t) := (\gamma(t), v), \qquad v \in \mathbb{R}^n.$$

We will see in Lecture 32 that this is consistent with Definition 28.1.

Remark 29.14. We will explore this further in Lecture 32, but for now note that a parallel transport system gives us a way to identify two different fibres E_p and E_q of a vector bundle over M: simply take a curve γ from p to q and consider the linear isomorphism $\mathbb{P}_\gamma\colon E_p \to E_q$. This will allow us to make sense of (28.2) from the last lecture, and thus let us differentiate sections along vector fields for non-trivial vector bundles.

Next lecture we will prove that a parallel transport system $\mathbb{P}$ determines and is uniquely determined by a connection Δ. We conclude today's lecture by introducing a special type of chart on a manifold , which will be useful in several places during the course, including in the aforementioned proof.

More precisely: the inverse of a chart.

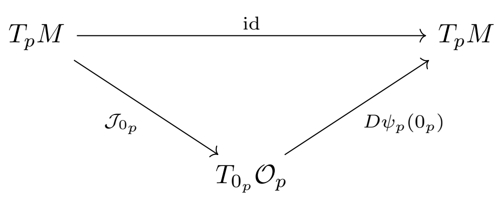
In order to reduce the number of π's floating around we adopt the convention that for a given vector bundle $\pi\colon E \to M$ and a subset $U \subset M$ we abbreviate

$$E|_U := \pi^{-1}(U) = \bigsqcup_{p \in U} E_p.$$

Definition 29.15. Let M be a smooth manifold and fix $p \in M$. Let $\mathcal{O}_p \subset T_pM$ be an open set which is star-shaped with respect to 0_p, and let $U_p \subset M$ be a neighbourhood of p. A diffeomorphism $\psi_p\colon \mathcal{O}_p \to U_p$ such that $\psi_p(0_p) = p$ is said to be a **ray parametrisation** at p. We say that the ray parametrisation ψ_p is **complete** if $\mathcal{O}_p = T_pM$. Finally we say that the ray parametrisation is **adapted** if

$$D\psi_p(0_p) \circ \mathcal{J}_{0_p} = \mathrm{id}_{T_pM}, \tag{29.4}$$

i.e. such that the following commutes:



A ray parametrisation is not really any new; it is simply a new name. Indeed, if $x\colon U \to \mathcal{O} \subset \mathbb{R}^m$ is a chart centred at p such that

$\mathcal{O}$ is star-shaped with respect to 0 then after choosing an isomorphism $T_pM \cong \mathbb{R}^m$ the map $\psi_p := x^{-1}$ is a ray parametrisation. Note also that since any such star-shaped set $\mathcal{O} \subset \mathbb{R}^m$ is diffeomorphic to all of $\mathbb{R}^m$ via a map fixing the origin, we can without loss of generality always assume that our ray parametrisations are complete.

We have used such charts throughout the course, starting with Lemma 3.7.

The adapted condition (29.4) should remind you of Theorem 12.3. This is no coincidence – see Remark 29.17 below. Let us now explain why (adapted) ray parametrisations are useful. Suppose $\psi_p \colon \mathcal{O}_p \to U_p$ is a ray parametrisation at p, and fix $\xi \in T_pM$. Then since $\mathcal{O}_p$ is star-shaped, there exists $\varepsilon > 0$ such that $t\xi \in \mathcal{O}_p$ for all $|t| \le \varepsilon$. This means that the curve

$$\gamma_{p,\xi} \colon (-\varepsilon, \varepsilon) \to U_p, \qquad \gamma_{p,\xi}(t) := \psi_p(t\xi) \tag{29.5}$$

is a well-defined smooth curve with $\gamma_{p,\xi}(0) = p$. This curve is the image of the ray segment $\{t\xi \mid |t| < \varepsilon\}$ in M (hence the name "ray parametrisation"). If ψ_p is complete then each $\gamma_{p,\xi}$ is defined for all $t \in \mathbb{R}$. If ψ_p is adapted then these curves also satisfy $\dot\gamma_{p,\xi}(0) = \xi$, since

$$\begin{aligned}\dot\gamma_{p,\xi}(0) &= \frac{d}{dt}\Big|_{t=0} \psi_p(t\xi) \\ &= D\psi_p(0_p) \circ \mathcal{J}_{0_p}(\xi) \\ &= \xi.\end{aligned}$$

This in itself is nothing new: the existence of *some* curve $\gamma \colon (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\dot\gamma(0) = \xi$ was proved all the way back in Lecture 4. The point of an adapted ray parametrisation is that it allows us to choose a family $\gamma_{p,\xi}$ of curves such that $(\xi, t) \mapsto \gamma_{p,\xi}(t)$ is *also* smooth in ξ.

Moreover the interval of definition of $\gamma_{p,\xi}$ also depends smoothly on ξ.

As a result, it is not immediately obvious that adapted ray parametrisations exist. We will prove this shortly, but first let us complicate the definition a bit by allowing p to vary.

DEFINITION 29.16. Let $\mathcal{O} \subset TM$ be an open set. Set $V = \pi(\mathcal{O})$ and assume that $\mathcal{O}_p := \mathcal{O} \cap T_pM$ is star-shaped with respect to $\mathcal{O}_p$ for all $p \in V$. Let U be an open set in M with $V \subset U$. A **moving parametrisation** is a smooth map $\psi \colon \mathcal{O} \to U$ such that $\psi(p, 0_p) = p$ for all $p \in V$. Thus $\psi_p := \psi|_{T_pM}$ is a ray parametrisation at p for each $p \in V$. As before, we say a moving parametrisation is **complete** if $\mathcal{O} = TM|_V$, and we say that ψ is **adapted** if

$$D\psi_p(0_p) \circ \mathcal{J}_{0_p} = \mathrm{id}_{T_pM}, \qquad \forall\, p \in V. \tag{29.6}$$

As in (29.5), we can associate to a moving parametrisation a whole smooth family of curves $\gamma_{p,\xi}$ for $p \in V$ and $\xi \in T_pM$ via the formula

$$\gamma_{p,\xi}(t) := \psi(p, t\xi),$$

which is well-defined for t small enough (depending on both p and ξ). This family of curves has the property that $\gamma_{p,\xi}(0) = p$ for all $p \in V$ and $\xi \in T_pM$. In the adapted case one also has $\dot\gamma_{p,\xi}(0) = \xi$.

REMARK 29.17. As remarked above, the adapted condition (29.6) for a moving parametrisation is very similar to the corresponding property of the exponential map of a Lie group (Theorem 12.3). In Lecture 44 we will see that a choice of spray $\mathbb{S}$ on M determines an adapted (but typically not complete) moving parametrisation $\exp\colon \mathcal{O} \to M$ over the entire manifold M. As the notation suggests, this map is called the **exponential map** of the spray $\mathbb{S}$.

Since it will be several lectures before we construct the exponential map associated to a Riemannian metric, let us give a direct proof that complete adapted moving parametrisations exist.

LEMMA 29.18. *Let M be a smooth manifold and let $p \in M$. Then there exists a neighbourhood U of p and an adapted moving parametrisation $\psi\colon TM|_U \to M$.*

Proof. Let $x\colon U \to \mathbb{R}^m$ be a chart on M such that $x(p) = 0$. We now define $\psi\colon TM|_U \to M$ by

$$\psi_q(\xi) := x^{-1}\Big(x(q) + \mathcal{J}^{-1}_{x(q)}\big(Dx(q)\xi\big)\Big).$$

Since $Dx(q)$ is linear we have $\psi(q, 0_q) = x^{-1}(x(q) + 0) = q$. Moreover for $\xi \in T_qM$ we compute

$$\begin{aligned} D\psi_q(0_q) \circ \mathcal{J}_{0_q}(\xi) &= \frac{d}{dt}\Big|_{t=0} x^{-1}\big(x(q) + \mathcal{J}^{-1}_{x(q)}(Dx(q)(0_q + t\xi)\big) \\ &= \frac{d}{dt}\Big|_{t=0} x^{-1}\big(x(q) + t\mathcal{J}^{-1}_{x(q)}(Dx(q)(\xi)\big) \\ &= Dx^{-1}(x(q)) \circ \mathcal{J}_{x(q)} \circ \mathcal{J}^{-1}_{x(q)} \circ Dx(q)\xi \\ &= Dx^{-1}(x(q)) \circ Dx(q)\xi \\ &= \xi. \end{aligned}$$

This completes the proof. ∎

Bonus Material for Lecture 29

In the bonus section we give a precise formulation of Axiom (iv) of Definition 29.11.

It follows from Axiom (i) that $v \mapsto \mathbb{P}_{\gamma;v}$ is also linear (where now addition and scalar multiplication take place in the vector space of sections $\Gamma_\gamma(E)$). Thus in particular $v \mapsto \mathbb{P}_{\gamma;v}$ is smooth. Therefore the only content of Axiom (iv) is the smooth dependence on γ. But what exactly does this mean? Since the space of all curves γ on M is itself infinite-dimensional, this is a little tricky to express precisely. Here we present one possible way, using moving parametrisations.

Let us temporarily write by $\pi_\oplus\colon E \oplus TM \to M$ for the footpoint map from the direct sum bundle $E \oplus TM$. Thus $\pi_\oplus^{-1}(p) = E_p \oplus T_pM$.

(iv)' **(Smooth dependence on initial conditions)**: For every open set $U \subset M$ and every moving parametrisation $\psi\colon TM|_U \to M$, the map

$$\pi_{\oplus}^{-1}(U) \to E, \qquad (p, \xi, v) \mapsto \mathbb{p}_{\gamma_{p,\xi};v}(1),$$

is smooth, where $\gamma_{p,\xi}$ is defined as in (29.6).

One could alternatively demand that this held for every adapted moving parametrisation.

LECTURE 30

The Equivalence of Connections and Parallel Transport

The goal of this lecture is to prove that a parallel transport system $\mathbb{P}$ determines and is uniquely determined by a connection Δ. This result is quite involved, and we prove each direction separately.

THEOREM 30.1. *Let $\pi\colon E \to M$ be a vector bundle, and let $\mathbb{P}$ be a parallel transport system on E. Then $\mathbb{P}$ determines a connection Δ on E. This connection has the property that a section ρ along a curve γ is parallel in the sense of Remark 29.12 if and only if ρ is horizontal with respect to Δ in the sense of Definition 29.2.*

Proof. Throughout we assume that M has dimension m, and that E is a vector bundle of rank n. Given $v \in E$, define $\Delta_v \subset T_vE$ to be the set of all tangent vectors ζ such that there exists a smooth curve $\gamma\colon [0,1] \to M$ with

$$\zeta = \dot{\mathbb{P}}_{\gamma;v}(0).$$

The fact that we choose a fixed interval $[0,1]$ is just for convenience; by Axiom (iii) of Definition 29.11 any interval will work.

Then set

$$\Delta := \bigsqcup_{p\in E} \Delta_p \subset TE.$$

We will prove that Δ is a connection in five steps. In the sixth and final step we prove the last sentence of the theorem.

1. Fix a point $p \in M$, and let $\psi\colon TM|_U \to M$ denote an adapted moving parametrisation about p. As in the paragraph after the statement of Lemma 29.18, for $(q,\xi) \in TM|_U$ set

$$\gamma_{q,\xi}(t) := \psi(q, t\xi),$$

so that $\gamma_{q,\xi}\colon \mathbb{R} \to M$ is a smooth curve with $\gamma_{q,\xi}(0) = q$ and $\dot\gamma_{q,\xi}(0) = \xi$, which moreover depends smoothly on (q,ξ). Next, we increase the numbers of subscripts from two to three by taking into account an element $v \in E_q$: set

$$\rho_{q,\xi,v}(t) := \mathbb{P}_{\gamma_{q,\xi};v}(t),$$

so that $\rho_{q,\xi,v}\colon \mathbb{R} \to E$ is a smooth curve such that

$$\rho_{q,\xi,v}(0) = v, \qquad \rho_{q,\xi,v}(t) \in E_{\gamma_{q,\xi}(t)}.$$

Finally we define

$$\varphi\colon (TM \oplus E)|_U \times \mathbb{R} \to E, \qquad (q,\xi,v,t) \mapsto \rho_{q,\xi,v}(t).$$

By Axiom (iv)' of Definition 29.11, φ is a smooth map. By Axiom (iii), $\rho_{q,\xi,v}(t) = \rho_{q,t\xi,v}(1)$, and thus

$$\begin{aligned}\dot\rho_{q,\xi,v}(0) &= \frac{d}{dt}\Big|_{t=0} \rho_{q,t\xi,v}(1) \\ &= D\varphi(0_q, v, 1)\big(\mathcal{J}_{0_p}(\xi), 0, 0\big).\end{aligned}$$

Thus the map $T_pM \to T_vE$ that sends ξ to $\dot\rho_{q,\xi,v}(0)$ is linear, as it is the composition of linear maps. If we call this map $\ell_{q,v}\colon \xi \mapsto \dot\rho_{q,\xi,v}(0)$ then Δ_v is equal (by definition) to $\operatorname{im}\ell_{q,v}$. Thus Δ_v is a vector space, as claimed.

2. In this step we show that $D\pi(v)|_{\Delta_v}$ is a linear isomorphism. We already know that Δ_v is a vector space of dimension at most m by the previous step. With $\ell_{q,v}$ as before, we have

$$\begin{aligned} D\pi(v)\circ \ell_{q,v}(\xi) &= D\pi(v)\dot\rho_{q,\xi,v}(0) \\ &= \frac{d}{dt}\Big|_{t=0}\pi(\rho_{q,\xi,v}(t)) \\ &= \frac{d}{dt}\Big|_{t=0}\gamma_{q,\xi}(t) \\ &= \frac{d}{dt}\Big|_{t=0}\psi(q,t\xi) \\ &= \xi, \end{aligned}$$

where the last equality used (29.6). Thus ℓ is a linear isomorphism, with inverse $D\pi(v)|_{\Delta(v)}$.

3. In this step, we prove that Δ is a distribution on E. For this, consider the pullback bundle $\pi^*TM \to E$. Using (28.5) as inspiration, we will show that locally Δ can be written as the image of an injective vector bundle morphism Φ from the pullback bundle π^*TM to TE. Part (i) of Problem H.8 then implies that Δ is a vector subbundle of TE, and hence a distribution.

This argument is due to Joscha Gillessen.

For this note that as smooth manifolds, one has

$$(\pi^*TM)|_{\pi^{-1}(U)} \cong (TM\oplus E)|_U.$$

These are the same manifold, but *not* the same bundle!

Thus we may alternatively regard φ as a map

$$\varphi\colon (\pi^*TM)|_{\pi^{-1}(U)}\times\mathbb{R}\to E.$$

This implies that the fibrewise map

$$\Phi\colon (\pi^*TM)|_{\pi^{-1}(U)} \to TE|_{\pi^{-1}(U)}, \qquad \Phi_{q,v}(\xi) := \dot\rho_{q,\xi,v}(0)$$

is smooth. Moreover the argument above shows that $\Phi_{q,v}$ is homogeneous:

$$\Phi_{q,v}(t\xi) = t\Phi_{q,v}(\xi).$$

Problem M.1 then implies that $\Phi_{q,v}$ is linear. Thus Φ is a vector bundle homomorphism, which moreover is injective by argument from Step 2.

4. Now that we know that Δ is a distribution, the fact that Δ is a preconnection follows directly from Step 2. In this step we prove that Δ is actually a connection.

Fix $p\in M$ and $v\in E_p$. We need to show that for any $c\in\mathbb{R}$,

$$D\mu_c(v)(\Delta_v) = \Delta_{cv}$$

where μ_c is scalar multiplication in the fibres, as in Definition 28.4. Let $\gamma\colon[0,1]\to M$ denote a smooth curve with $\gamma(0)=p$, and let $\rho :=$

$\mathbb{p}_{\gamma;v}$. By linearity of parallel transport (this is Axiom (i) of Definition 29.11), $\mu_c \circ \rho$ is also parallel along γ. Since

$$D\mu_c(v)\dot\rho(0) = \frac{d}{dt}\Big|_{t=0}(\mu_c \circ \rho)(t)$$

we see that $D\mu_c(v)(\Delta_v) \subset \Delta_{cv}$. Then since

$$D\pi(cv) \circ D\mu_c(v)\dot\rho(0) = D\pi(v)\dot\rho(0)$$

and $D\pi(cv)$ maps Δ_{cv} isomorphically onto T_pM, it follows that $D\mu_c(v)(\Delta_v) = \Delta_{cv}$. This completes the proof that Δ is a connection.

5. Finally we prove that a section ρ along a curve γ is parallel in the sense of Remark 29.12 if and only if ρ is horizontal with respect to Δ in the sense of Definition 29.2. One direction is clear by definition of Δ, so it suffices to show that if γ is a smooth curve and $\rho \in \Gamma_\gamma(E)$ is horizontal along γ then ρ is also parallel. Let $v = \rho(0)$ and let $\rho_1(t) := \mathbb{p}_{\gamma;v}$. Since both ρ and ρ_1 are horizontal and

$$D\pi(\rho_1(t))\dot\rho_1(t) = \dot\gamma(t) = D\pi(\rho(t))\dot\rho(t),$$

we have by the defining condition of a preconnection that $\dot\rho(t) = \dot\rho_1(t)$. Thus ρ and ρ_1 are two curves with the same initial condition and the same derivative, whence they are equal. This at last completes the proof of Theorem 30.1. ■

We now prove the opposite direction: how to go from a connection to a parallel transport system.

Theorem 30.2. *Let $\pi\colon E \to M$ be a vector bundle, and let Δ be a connection on E. The system of all horizontal lifts to E of smooth curves in M defines a parallel transport system $\mathbb{P}$ in E. Moreover the connection on E determined by $\mathbb{P}$ from Theorem 30.1 is just Δ again.*

Proof. As the statement of the theorem indicated, given a smooth curve $\gamma\colon [a,b] \to M$ and $v \in E_{\gamma(a)}$, we define $\mathbb{p}_{\gamma;v} \in \Gamma_\gamma(E)$ to be the horizontal lift of γ with respect to Δ, whose existence and uniqueness is guaranteed by Proposition 29.9. We must check that the five axioms of a parallel transport system are satisfied. We will do this in three steps.

1. In this step we check that our proposed parallel transport system satisfies Axiom (i) from Definition 29.11. Let $\gamma\colon [a,b] \to M$ be a smooth curve. Set $p = \gamma(a)$ and $q = \gamma(b)$. If ρ is a horizontal lift of γ to E then for any $c \in \mathbb{R}$ the curve $t \mapsto \mu_c(\rho(t)) = c\rho(t)$ is also horizontal since

$$\frac{d}{dt}(\mu_c(\rho))(t) = D\mu_c(c\rho(t))\dot\rho(t) \in \Delta_{\mu_c(\rho(t))}$$

by (28.3). This shows that the map $\mathbb{P}_\gamma\colon E_p \to E_q$ is homogeneous. Moreover it follows from the proof of Proposition 29.9 and the smooth dependence on initial conditions of integral curves that $\mathbb{P}_\gamma$ is differentiable as a map from the vector space E_p to the vector space E_q. Problem M.1 then implies once more that $\mathbb{P}_\gamma$ is actually linear.

That is, Theorem 9.1 applied to the vector field $\overline{T}$ from the proof of Proposition 29.9.

If $\gamma^-(t) := \gamma(a+b-t)$ is the reverse curve from q to p then $\rho^-(t) := \rho(b-t)$ is a horizontal section along γ^- with initial condition $\rho(b))$. It follows that $\mathbb{P}_\gamma$ is invertible, with inverse $\mathbb{P}_{\gamma^-}$. This proves that Axiom (i) from Definition 29.11 holds.

2. Axiom (ii) follows from the group property of the flow of a complete vector field (cf. Definition 9.14 and the marginal note next to Proposition 29.10.)

3. Let us now verify Axiom (iii) from Definition 29.11. Let $\gamma\colon [a,b] \to M$ be a smooth curve and $h\colon [a_1,b_1] \to [a,b]$ is a diffeomorphism such that $h(a_1) = a$ and $h(b_1) = b$. Set $\delta := \gamma \circ h$. Fix $v \in E_{\gamma(a)}$. Let ρ be the horizontal section of E along γ with $\rho(a) = v$ and let σ be the horizontal section along δ such that $\sigma(a_1) = v$. We claim that $\sigma = \rho \circ h$. Indeed, $\rho \circ h$ is certainly a lift of δ (as $\pi \circ \rho \circ h = \gamma \circ h = \delta$) and

$$\frac{d}{dt}\rho(h(t)) = h'(t)\dot{\rho}(h(t)) \in \Delta_{\rho(h(t))}$$

by the chain rule. Thus by the uniqueness part of Proposition 29.9, we have $\sigma = \rho \circ h$ as desired.

4. We now address the final two axioms, Axiom (iv) and Axiom (v). We will not say much about Axiom (iv) (given that we relegated the precise statement of this Axiom to the bonus section), other than that it essentially boils once again down to the fact that integral curves depend smoothly on initial conditions. Axiom (v) on the other hand is immediate, since if γ is a smooth curve in M, $v \in E_{\gamma(0)}$ and ρ is the horizontal section of E along γ with initial condition v then $\dot{\rho}(0)$ is the unique element of Δ_v which is mapped to $\dot{\gamma}(0)$ by $D\pi(v)$.

Thus $\mathbb{P}$ is indeed a parallel transport system. To complete the proof we must show that the connection obtained from $\mathbb{P}$ by applying Theorem 30.1 is simply Δ again. This however is immediate from Axiom (v) of Definition 29.11. ∎

Remark 30.3. From now on we will usually work with connections, rather than parallel transport systems. Thus if a connection is specified and we refer to a section being "parallel", it should always be implicitly assumed that the parallel transport system in question is the one associated via Theorem 30.2 to the given connection.

This convention has the somewhat amusing consequence that the words "parallel" and "horizontal" can now often be used interchangeably. In general we will (usually) favour the word "parallel" when talking about sections, and "horizontal" when talking about vectors.

LECTURE 31

Covariant Derivatives

In this lecture we introduce the *connection map* of a connection, and use this to define *covariant derivative* operators.

We begin with some remarks about the vertical bundle of a vector bundle. The vertical bundle is defined for any fibre bundle, but for a vector bundle the vertical bundle has extra structure, as we now explain. Let $\pi\colon E \to M$ be a vector bundle. We form the pullback bundle $\pi^*E \to E$:

$$\begin{array}{ccc} \pi^*E & \xrightarrow{\mathrm{pr}_2} & E \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow[\pi]{} & M \end{array}$$

This is the bundle whose fibre over $v \in E$ is $E_{\pi(v)}$. By part (ii) of Problem I.5 (or more accurately, its solution), the dash-to-dot maps assemble together to define a vector bundle isomorphism from π^*E to VE:

$$\mathcal{J}\colon \pi^*E \to VE, \qquad (u, v) \mapsto \mathcal{J}_u(v) = \frac{d}{dt}\Big|_{t=0} u + tv$$

$$\begin{array}{ccc} \pi^*E & \xrightarrow{\mathcal{J}} & VE \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi_{VE}} \\ E & \xrightarrow[\mathrm{id}]{} & E \end{array}$$

If we denote by

$$\widetilde{\mathrm{pr}}_2 := \mathrm{pr}_2 \circ \mathcal{J}^{-1}, \qquad \mathcal{J}_u(v) \mapsto v$$

If we suppress the dash-to-dot map from our notation, the map $\widetilde{\mathrm{pr}}_2$ is simply the identity map $v \mapsto v$. This abuse of notation is very common in introductory Differential Geometry texts.

then $\widetilde{\mathrm{pr}}_2$ is a vector bundle isomorphism along π and the following commutes

$$\begin{array}{ccc} VE & \xrightarrow{\widetilde{\mathrm{pr}}_2} & E \\ {\scriptstyle \pi_{VE}}\downarrow & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow[\pi]{} & M \end{array}$$

Definition 31.1. Let $\pi\colon E \to M$ be a vector bundle and let Δ be a connection on E. Define a map

$$K\colon TE \to E, \qquad K(\zeta) := \widetilde{\mathrm{pr}}_2(\zeta^{\mathsf{v}}).$$

This makes sense, since $\zeta^{\mathsf{v}} \in VE$. We call K the **connection map** of the connection Δ.

Remark 31.2. We can use the connection map K and the parallel transport system $\mathbb{P}$ associated to Δ to give a new way to express the horizontal-vertical splitting of a tangent vector. Indeed, if $v \in E$ and $\zeta \in T_v E$ and $\gamma\colon [0,1] \to M$ is any smooth curve with $\gamma(0) = \pi(v)$ and $D\pi(v)\zeta = \dot{\gamma}(0)$, then it follows from Theorem 30.1 and Definition 31.1 that

$$\zeta^{\mathsf{h}} = \dot{\mathbb{P}}_{\gamma;v}(0) \qquad \text{and} \qquad \zeta^{\mathsf{v}} = \mathcal{J}_v(K(\zeta)). \tag{31.1}$$

It is immediate that K is a vector bundle morphism along π, i.e. that K is linear on the fibres of TE and that the following commutes:

$$\begin{array}{ccc} TE & \xrightarrow{\;K\;} & E \\ \Big\downarrow{\scriptstyle \pi_{TE}} & & \Big\downarrow{\scriptstyle \pi} \\ E & \xrightarrow[\;\pi\;]{} & M \end{array}$$

In fact, if we combine K with $D\pi$ we can build a vector bundle isomorphism along π:

Lemma 31.3. *Let $\pi\colon E \to M$ be a vector bundle and let Δ be a connection on E with connection map K. Then $(D\pi, K)$ is a vector bundle isomorphism along π:*

$$\begin{array}{ccc} TE & \xrightarrow{\;(D\pi,K)\;} & TM \oplus E \\ \Big\downarrow{\scriptstyle \pi_{TE}} & & \Big\downarrow{\scriptstyle (\pi_{TM},\pi)} \\ E & \xrightarrow[\;\pi\;]{} & M \end{array}$$

Proof. Since TE and $TM \oplus E$ have the same fibre dimension, it suffices to check that $\ker(D\pi, K) = 0$. This is immediate from Lemma 28.3. ■

Lemma 31.4. *Let $\pi\colon E \to M$ be a vector bundle of rank n. Then $D\pi\colon TE \to TM$ is a vector bundle of rank $2n$.*

This result was part (iii) of Problem I.5. However we include the proof here again in order to fix the notation.

Proof. Roughly speaking, the vector bundle structure on $TE \to TM$ is obtained by differentiating the vector addition and scalar multiplication on $E \to M$. Here are the details.

The fibre of the bundle $D\pi\colon TE \to TM$ over a point $\xi \in T_p M$ is the set of all vectors $\zeta \in T_v E$, as v ranges over E_p, such that $D\pi(v)\zeta = \xi$. We endow each fibre with a vector space structure as follows: let $A\colon E \oplus E \to E$ denote the vector bundle homomorphism

$$A\colon E \oplus E \to E, \qquad A(u,v) = u + v \tag{31.2}$$

given by fibrewise addition. Then if $\zeta \in T_u E$ and $\eta \in T_v E$ belong to the same fibre we define

We use the special notation $\boldsymbol{+}$ and $\bullet$ in an attempt to minimise confusion later on.

$$\zeta \boldsymbol{+} \eta := DA(u,v)(\zeta, \eta), \tag{31.3}$$

which belongs to the fibre $T_{u+v}E$. Similarly if $c \in \mathbb{R}$ then we define

$$c \bullet (v, \zeta) := D\mu_c(v)\zeta, \tag{31.4}$$

where μ_c is the fibrewise scalar multiplication as in Definition 28.4. With these definitions, if $\varepsilon\colon \pi^{-1}(U) \to \mathbb{R}^n$ is a vector bundle chart on E then the bundle $D\varepsilon \colon T(\pi^{-1}(U)) \to T\mathbb{R}^n = \mathbb{R}^{2n}$ is a linear isomorphism on each fibre, and hence may serve as a vector bundle chart. The result now follows from Proposition 16.15. ■

Exercise: Check this!

Far less obviously, K is also a vector bundle morphism from TE to E along $\pi \colon TM \to M$.

THEOREM 31.5. *Let $\pi \colon E \to M$ be a vector bundle and let Δ be a connection on E with connection map $K \colon TE \to E$. Then*

$$K(\zeta \mathbin{\text{+}} \eta) = K(\zeta) + K(\eta), \qquad K(c \bullet \zeta) = cK(\zeta), \tag{31.5}$$

and hence K is a vector bundle morphism along π_{TM}:

$$\begin{array}{ccc} TE & \xrightarrow{K} & E \\ {\scriptstyle D\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ TM & \xrightarrow[\pi_{TM}]{} & M \end{array}$$

An alternative way of expressing (31.5) is via commutativity of the following two diagrams:

$$\begin{array}{ccc} TE \oplus TE & \xrightarrow{DA} & TE \\ {\scriptstyle (K,K)}\downarrow & & \downarrow{\scriptstyle K} \\ E \oplus E & \xrightarrow[A]{} & E \end{array} \qquad \begin{array}{ccc} TE & \xrightarrow{D\mu_c} & TE \\ {\scriptstyle K}\downarrow & & \downarrow{\scriptstyle K} \\ E & \xrightarrow[\mu_c]{} & E \end{array} \tag{31.6}$$

This innocuous looking result is actually the lynchpin needed to define covariant derivatives, as we will see in the proof of Theorem 31.8 below.

Proof. Fix $p \in M$, $u \in E_p$ and $\zeta \in T_uE$. Let $\gamma \colon [0,1] \to M$ be a smooth curve with $\gamma(0) = p$ and $\dot\gamma(0) = D\pi(u)\zeta$. By Remark 31.2 we can write

$$\zeta^{\mathsf{h}} = \dot{\mathbb{P}}_{\gamma;u}(t) \qquad \text{and} \qquad \zeta^{\mathsf{v}} = \mathcal{J}_u(K(\zeta)).$$

Now fix $w \in E_p$ and let

$$f_{u,w} \colon [0,1] \times (-1,1) \to M, \qquad f_{u,w}(s,t) := \mathbb{P}_{\gamma;u+tw}(s).$$

By the chain rule,

$$\begin{aligned} Df_{u,w}(0,0)\left(\frac{\partial}{\partial s}\Big|_0, \frac{\partial}{\partial t}\Big|_0\right) &= D_1 f_{u,w}(0,0)\frac{\partial}{\partial s}\Big|_0 + D_2 f_{u,w}(0,0)\frac{\partial}{\partial t}\Big|_0 \\ &= \dot{\mathbb{P}}_{\gamma;u}(0) + \frac{d}{dt}\Big|_{t=0} \mathbb{P}_{\gamma;u+tw}(0) \\ &= \zeta^{\mathsf{h}} + \frac{d}{dt}\Big|_{t=0} u + tw \\ &= \zeta^{\mathsf{h}} + \mathcal{J}_u(w). \end{aligned}$$

Applying this with $w = K(\zeta)$ yields the formula

$$\zeta = Df_{u,K(\zeta)}(0,0)\left(\frac{\partial}{\partial s}\Big|_0, \frac{\partial}{\partial t}\Big|_0\right).$$

Why is this helpful? Suppose v is another element of E_p, and $\eta \in T_v E$ satisfies $D\pi(v)\eta = D\pi(u)\zeta$ so the addition $\zeta \mathbin{\boldsymbol{+}} \eta$ makes sense. Then the same argument tells us

$$\eta = Df_{v,K(\eta)}(0,0)\left(\frac{\partial}{\partial s}\Big|_0, \frac{\partial}{\partial t}\Big|_0\right).$$

We can use these expressions to compute

$$\begin{aligned}
\zeta \mathbin{\boldsymbol{+}} \eta &:= DA(u,v)(\zeta,\eta) \\
&= \frac{d}{dt}\Big|_{t=0}\Big(f_{u,K(\zeta)}(t,t) + f_{v,K(\eta)}(t,t)\Big) \\
&= \frac{d}{dt}\Big|_{t=0} f_{u+v,K(\zeta)+K(\zeta)}(t,t) \\
&= (\zeta+\eta)^{\mathsf{h}} + \mathscr{J}_{u+v}\big(K(\zeta)+K(\eta)\big).
\end{aligned}$$

But Remark 31.2 also tells us that the vertical component of $\zeta \mathbin{\boldsymbol{+}} \eta$ is $\mathscr{J}_{u+v}\big(K(\zeta \mathbin{\boldsymbol{+}} \eta)\big)$. Comparing this to the expression above and using the fact that $\mathscr{J}_{u+v}$ is an isomorphism, we see that

$$K(\zeta \mathbin{\boldsymbol{+}} \xi) = K(\zeta) + K(\xi).$$

The proof that $K(c \bullet \zeta) = cK(\zeta)$ goes along similar lines, and is left as an exercise. ■

We now use the connection map to give a third interpretation of connections, via *covariant derivatives*. This point of view is the "usual" one, and many introductory accounts of connections only define them this way.

Definition 31.6. Let $\pi\colon E \to N$ be a vector bundle and let $\varphi\colon M \to N$ be a smooth map. An operator

$$\nabla^{\varphi}\colon \mathfrak{X}(M) \times \Gamma_{\varphi}(E) \to \Gamma_{\varphi}(E),$$

written

$$(X,s) \mapsto \nabla^{\varphi}_X s$$

is called a **covariant derivative operator in E along φ** if the following four conditions are satisfied for any $X, Y \in \mathfrak{X}(M)$, $s, r \in \Gamma_{\varphi}(E)$, and $f \in C^{\infty}(M)$:

(i) $\nabla^{\varphi}_{X+Y} s = \nabla^{\varphi}_X s + \nabla^{\varphi}_Y s$,

(ii) $\nabla^{\varphi}_{fX} s = f\nabla^{\varphi}_X s$,

(iii) $\nabla^{\varphi}_X (s+r) = \nabla^{\varphi}_X s + \nabla^{\varphi}_X r$,

(iv) $\nabla^{\varphi}_X (fs) = X(f)s + f\nabla^{\varphi}_X s$.

We call $\nabla^{\varphi}_X s$ the **covariant derivative of s with respect to X**. If $M = N$ and $\varphi = \mathrm{id}$ then we write ∇ instead of ∇^{id} and call ∇ a **covariant derivative operator on E**.

By property (ii) the operator

$$\mathfrak{X}(M) \to \Gamma_{\varphi}(E), \qquad X \mapsto \nabla^{\varphi}_X s$$

is $C^\infty(M)$-linear, and hence by Theorem 20.20 it is also a point operator. We can therefore

$$\nabla^\varphi_\xi s := (\nabla^\varphi_X s)(p), \tag{31.7}$$

where X is any vector field on M such that $X(p) = \xi$.

Note that the left-hand side of (31.7) does not have reference to the point p, since this is implicitly contained in the fact that $\xi \in T_pM$.

Note however that property (iv) implies that $s \mapsto \nabla^\varphi_X s$ is *not* $C^\infty(M)$-linear, and thus $s \mapsto \nabla^\varphi_X s$ is *not* a point operator. Therefore value of $\nabla^\varphi_X s$ at p depends on s not just through $s(p)$. In fact, as we will later prove, $(\nabla^\varphi_X s)(p)$ depends on s only through the values of $s(\gamma(t))$ for t small, where γ is any smooth curve such that $\gamma(0) = p$.

REMARK 31.7. Take $E = TM$ and $\varphi = \mathrm{id}$. Then both $Y \mapsto \nabla_X Y$ and $Y \mapsto \mathcal{L}_X Y$ are operators $\mathfrak{X}(M) \to \mathfrak{X}(M)$. The Lie derivative is *not* a point operator (cf. Remark 22.23), whereas ∇_X is. In this sense ∇_X is more "useful" than $\mathcal{L}_X$ – the disadvantage is that ∇_X depends on the choice of ∇, whereas the Lie derivative is canonical.

Here then is the main result that links connections and covariant derivative operators. Just as with connections and parallel transport operators, the proof is quite involved, and we split it into two stages: one direction is Theorem 31.8 below, and the other is Theorem 32.1 in the next lecture.

THEOREM 31.8. *Let $\pi\colon E \to N$ be a vector bundle and let Δ be a connection on E with connection map K. If $\varphi\colon M \to N$ is any smooth map then*

$$(\nabla^\varphi_X s)(p) := K\big(Ds(p)X(p)\big) \tag{31.8}$$

defines a covariant derivative operator in E along φ. This covariant derivative operator has the property that a section $s \in \Gamma_\varphi(E)$ is parallel if and only if $\nabla^\varphi_X s = 0$ for all $X \in \mathfrak{X}(M)$. Moreover the chain rule holds: if $\psi\colon L \to M$ is a smooth map then for all $q \in L$ and $\xi \in T_qL$,

$$\nabla^{\varphi\circ\psi}_\xi(s \circ \psi) = \nabla^\varphi_{D\psi(q)\xi} s. \tag{31.9}$$

REMARK 31.9. If $\psi\colon L \to M$ is actually a diffeomorphism then (31.9) can be written as

$$\nabla^{\varphi\circ\psi}_Y(s \circ \psi) = (\nabla^\varphi_{\psi_* Y} s) \circ \psi, \qquad Y \in \mathfrak{X}(L),\ s \in \Gamma_\varphi(E). \tag{31.10}$$

This only makes sense for ψ a diffeomorphism, as otherwise $\psi_* Y$ is not defined!

Proof. The formula (31.8) certainly defines an element of $\Gamma_\varphi(E)$. We show that ∇^φ really is a covariant derivative operator along φ in four steps.

1. In this step we show that a section $s \in \Gamma_\varphi(E)$ is parallel with respect to Δ if and only if $\nabla^\varphi_X s = 0$ for every vector field $X \in \mathfrak{X}(M)$. This is clear, since $\Delta = \ker K$ and by Definition 29.2 a section s is parallel (or horizontal – cf. Remark 30.3) if and only if $Ds(TM) \subset \Delta$.

2. Let us now verify (31.9). Note $s \circ \psi \in \Gamma_{\varphi\circ\psi}(E)$. Fix $q \in L$ and

$\xi \in T_q L$. Then

$$\begin{aligned}\nabla_\xi^{\varphi\circ\psi}(s\circ\psi) &= K\big(D(s\circ\psi)(q)\xi\big)\\ &= K\Big(Ds(\psi(q))\circ D\psi(q)\xi\Big)\\ &= \nabla^\varphi_{D\psi(q)\xi}s.\end{aligned}$$

3. Let $s, r \in \Gamma_\varphi(E)$ and $f \in C^\infty(M)$. Fix $p \in M$, $\xi \in T_pM$. In this step we show that

$$\nabla_\xi^\varphi(s+r)(p) = (\nabla_\xi^\varphi s)(p) + (\nabla_\xi^\varphi r)(p). \tag{31.11}$$

Let γ be a curve in M with $\gamma(0) = p$ and $\dot\gamma(0) = \xi$. Then with A as in (31.2) we have from (31.3) that

$$\begin{aligned}Ds(p)\xi \mathbin{\pmb{+}} Dr(p)\xi &= DA(s(p), r(p))\big(Ds(p)\xi, Dr(p)\xi\big)\\ &= \frac{d}{dt}\Big|_{t=0} s(\gamma(t)) + r(\gamma(t))\\ &= \frac{d}{dt}\Big|_{t=0}(s+r)(\gamma(t))\\ &= D(s+r)(p)\xi.\end{aligned}$$

Since K is a vector bundle morphism along π_{TM} by Theorem 31.5, we obtain

$$\begin{aligned}\nabla_\xi^\varphi(s+r) &= K\big(D(s+r)(p)\xi\big)\\ &= K\big(Ds(p)\xi \mathbin{\pmb{+}} Dr(p)\xi\big)\\ &= K(Ds(p)\xi) + K(Dr(p)\xi)\\ &= \nabla_\xi^\varphi s + \nabla_\xi^\varphi r.\end{aligned}$$

This proves (31.11).

4. Let $s \in \Gamma_\varphi(E)$ and $f \in C^\infty(M)$. Fix $p \in M$ and $\xi \in T_pM$. In this step we prove that

$$\nabla_\xi^\varphi(fs) = \xi(f)s + f(p)\nabla_\xi^\varphi s. \tag{31.12}$$

Let $\mu\colon \mathbb{R}\times E \to E$ be the scalar multiplication $(c, v) \mapsto \mu_c(v) = cv$. Then for $c \neq 0$, $b \in \mathbb{R}$ and $\zeta \in T_vE$,

$$D\mu(c,v)\left(b\frac{\partial}{\partial t}\Big|_c, \zeta\right) = D\mu_c(v)\zeta + \mathcal{J}_{cv}(bv). \tag{31.13}$$

The section $p \mapsto f(p)s(p)$ can be written as the composition $\mu\circ(f,s)$, and hence using (31.13) with $\zeta = Ds(p)\xi$ we compute

$$\begin{aligned}D(fs)(p)\xi &= D\big(\mu\circ(f,s)\big)(p)\xi\\ &= D\mu(f(p), s(p))\circ\big(Df(p)\xi, Ds(p)\xi\big)\\ &= D\mu_{f(p)}(s(p))Ds(p)\xi + \mathcal{J}_{f(p)s(p)}((Df(p)\xi)s(p))\\ &= D\mu_{f(p)}(s(p))Ds(p)\xi + \mathcal{J}_{f(p)s(p)}(\xi(f)s(p)).\end{aligned}$$

Now by definition

$$D\mu_{f(p)}(s(p))Ds(p) = f(p) \bullet Ds(p).$$

Thus applying K to both sides and using Theorem 31.5 we obtain

$$K\big(D(fs)(p)\xi\big) = f(p)K(Ds(p)\xi) + \xi(f)s(p),$$

which gives (31.12). This completes the proof. ∎

Corollary 31.10. *Let $\pi\colon E \to N$ be a vector bundle with connection Δ, and let $\varphi\colon M \to N$ be smooth. If $s, r \in \Gamma_\varphi(E)$ are horizontal then so is $cs + r$ for any $c \in \mathbb{R}$. Thus the horizontal sections form a vector subspace of $\Gamma_\varphi(E)$.*

Proof. For any vector field X on M,

$$\nabla^\varphi_X(c\,s + r) = c\nabla^\varphi_X s + \nabla^\varphi_X r = 0.$$

∎

LECTURE 32

Holonomy

We begin this lecture by completing the various chain of equivalences and proving that a covariant derivative operator uniquely determines a connection. We then introduce the notion of *holonomy*, which will give us a way to measure how "non-trivial" a connection is.

THEOREM 32.1. *Let ∇ be a covariant derivative operator on a vector bundle $\pi\colon E \to M$. Then there exists a connection Δ on E such that if $s \in \Gamma(E)$ and $\xi \in T_pM$ then $Ds(p)\xi \in \Delta_{s(p)}$ if and only if $\nabla_\xi s = 0$.*

Proof. Given $v \in E_p$, we define Δ_v to be the set of all vectors in T_vE of the form

$$Ds(p)\xi - \mathcal{J}_v(\nabla_\xi s),$$

for s a section of E such that $s(p) = v$. We then set $\Delta = \bigsqcup_{v\in E} \Delta_v$. In contrast to the proof of Theorem 30.1, this time it is clear that Δ_v is a linear subspace of T_vE. Moreover $D\pi(v)|_{\Delta_v}\colon \Delta_v \to T_{\pi(v)}M$ is a linear isomorphism by construction, since

$$\begin{aligned} D\pi(s(p))\big(Ds(p)\xi - \mathcal{J}_v(\nabla_\xi s)\big) &= D(\underbrace{\pi \circ s}_{=\mathrm{id}})(p)\xi - D\pi(s(p))(\underbrace{\mathcal{J}_v(\nabla_\xi s)}_{\in VE}) \\ &= \xi - 0 \\ &= \xi. \end{aligned}$$

The proof that Δ really is a vector subbundle goes along exactly the same lines as the proof of Step 3 of Theorem 30.1: If x is a chart on M with local coordinates then $dx \circ D\pi$ is a vector bundle chart on Δ that can be extended to a vector bundle chart on TE. Thus Δ is a preconnection. Finally if $c \in \mathbb{R}$ then

$$D(cs)(p)\xi - \mathcal{J}_{cv}(\nabla_\xi(cs)) = D\mu_c(p)\Big(Ds(p)\xi - \mathcal{J}_v(\nabla_\xi s)\Big),$$

and hence Δ is a connection. ■

As a corollary, we deduce that a covariant derivative operator ∇^φ is uniquely determined by what it does to $\varphi = \mathrm{id}$.

COROLLARY 32.2. *Suppose $\nabla = \nabla^{\mathrm{id}}$ is a covariant derivative operator in $\pi\colon E \to N$. Then ∇ induces a covariant derivative operator ∇^φ for any smooth map $\varphi\colon M \to N$.*

Proof. Theorem 31.8 tells us that if we start with a connection Δ, the associated covariant derivative operator satisfies the chain rule (31.10). Thus for $p \in M$ and $\xi \in T_pM$ one has

$$\nabla^\varphi_\xi(s \circ \varphi) \coloneqq \nabla_{D\varphi(p)\xi}s.$$

Similarly sections of the form $\tilde{s} \circ \varphi$ locally generate $\Gamma_\varphi(E)$ over $C^\infty(M)$, this shows ∇^φ is uniquely determined by ∇. Moreover Theorem 32.1 tells us that all such covariant derivative operators $\nabla = \nabla^{\mathrm{id}}$ come from connections. ■

Remark 32.3. Here is another way to view Corollary 32.2. Suppose $\nabla = \nabla^{\mathrm{id}}$ is a covariant derivative operator in $\pi\colon E \to N$. Let Δ denote the connection on $E \to N$ corresponding to ∇ given to us by Theorem 32.1. Then Δ induces a connection $\varphi^*\Delta$ on $\varphi^*E \to M$ by Proposition 29.1, and hence also a covariant derivative operator on $\mathfrak{X}(M) \times \Gamma(\varphi^*E) \to \Gamma(\varphi^*E)$ by Theorem 31.8. The desired covariant derivative operator ∇^{φ} is then obtained using Lemma 29.5.

Next, we finally make rigorous the discussion from the beginning of Lecture 28 when we initially motivated the definition of a connection. This requires a couple of preliminary results, starting with following result, whose proof is on Problem Sheet M.

Proposition 32.4. *Let $\pi\colon E \to M$ be a vector bundle of rank n with connection Δ. Fix $p \in M$, and let $\psi_p\colon U_p \to \mathcal{O}_p$ be a ray parametrisation at p. For $\xi \in T_pM$ write $\gamma_{p,\xi}(t) \coloneqq \psi_p(t\xi)$, as in (29.5). Fix a basis $\{v_1, \dots, v_n\}$ of E_p. There exists a local frame $\{e_1, \dots, e_n\}$ on U_p such that $e_i(p) = v_i$ and such that for all $\xi \in T_pM$, $e_i \circ \gamma_{p,\xi}$ is parallel along $\gamma_{p,\xi}$.*

The next result is an easy corollary of Proposition 32.4, and whose proof is also on Problem Sheet M.

Corollary 32.5. *Let $\pi\colon E \to M$ be a vector bundle of rank n with connection Δ. Fix $p \in M$ and let $\{v_1, \dots, v_n\}$ be a basis of E_p. Let $\gamma\colon (-\varepsilon, \varepsilon) \to M$ be a smooth curve with $\gamma(a) = p$ and $\dot\gamma(t) \neq 0$ for all $t \in (-\varepsilon, \varepsilon)$. Then there exists a local frame $\{e_1, \dots, e_n\}$ of E over an open set U containing p such that $e_i(p) = v_i$ and such that $e_i \circ \gamma$ is parallel along γ for each $i = 1, \dots, n$.*

Up to shrinking ε, this can always be achieved provided $\dot\gamma(0) \neq 0$.

We call $\{e_1, \dots, e_n\}$ a **parallel local frame along** γ. If $\rho \in \Gamma_\gamma(E)$ is any section along γ then we can write

$$\rho(t) = f^i(t)\, e_i(\gamma(t))$$

for some smooth functions $f^i(t)$. We claim:

Lemma 32.6. *Let $\pi\colon E \to M$ be a vector bundle of rank n with connection Δ. Let γ be a curve in M with $\gamma(0) = p$, and let $\{e_1, \dots, e_n\}$ be a parallel local frame along γ. Fix $\rho \in \Gamma_\gamma(E)$ and write $\rho(t) = f^i(t)\, e_i(\gamma(t))$ as above. Then ρ is parallel along γ if and only if each f^i is a constant function.*

Proof. Set $v = \rho(0)$. Then ρ is parallel if and only if $\rho = \mathbb{p}_{\gamma;v}$. If $v_i \coloneqq e_i(\gamma(0))$ then we can write $c = a^i\, v_i$ for constants a^i, and then by Axiom (i) of parallel transport,

$$\mathbb{p}_{\gamma;v}(t) = a^i\, \mathbb{p}_{\gamma;v_i}(t) = a^i e_i(\gamma(t)).$$

Thus $\rho = \mathbb{p}_{\gamma;v}$ if and only if $f^i(t) \equiv a^i$. ∎

Proposition 32.7. *Let $\pi\colon E \to N$ be a vector bundle with connection ∇. Let $\varphi\colon M \to N$ be a smooth map. Let $\gamma\colon [0,1] \to M$ be a smooth curve and abbreviate by*

$$\mathbb{P}_t\colon E_{\varphi(\gamma(0))} \to E_{\varphi(\gamma(t))}$$

the parallel transport along the curve $r \mapsto \varphi(\gamma(r))$ for $0 \leq r \leq t$. Then if $s \in \Gamma_\varphi(E)$ one has

$$\nabla^{\varphi}_{\dot\gamma(0)} s = \mathcal{J}^{-1}_{s(\gamma(0))}\left(\frac{d}{dt}\Big|_{t=0} \mathbb{P}_t^{-1}(s(\gamma(t)))\right) \tag{32.1}$$

Proof. Let $\{e_i\}$ be a parallel local frame along $\varphi \circ \gamma$. We can write $s \circ \gamma = f^i(e_i \circ \varphi \circ \gamma)$ for smooth functions f^i. Then

$$\begin{aligned}\mathbb{P}_t^{-1}(s(\gamma(t)) &= \mathbb{P}_t^{-1}\big(f^i(t)e_i(\varphi(\gamma(t)))\big)\\ &= f^i(t)e_i(\varphi(\gamma(0))).\end{aligned} \tag{32.2}$$

Let T denote the vector field $\frac{\partial}{\partial t}$ on $[0,1]$. Then we have by the chain rule (31.10) that

$$\begin{aligned}\nabla^{\varphi}_{\dot\gamma(0)} s &= \nabla^{\gamma}_{T(0)}(s\circ\gamma)\\ &= \nabla^{\gamma}_{T(0)}\big(f^i(e_i\circ\varphi\circ\gamma)\big)\\ &= (f^i)'(0)e_i(\varphi(\gamma(0)))\\ &= \mathcal{J}^{-1}_{s(\gamma(0))}\left(\frac{d}{dt}\Big|_{t=0} \mathbb{P}_t^{-1}(s(\gamma(t)))\right)\end{aligned}$$

where the penultimate line used property (iv) of a connection and the final line used (32.2). ∎

Remark 32.8. The equation (32.1) shows how parallel transport allows us to make sense of (28.2). Indeed, if $\mathbb{P}$ is the trivial parallel transport system from Example 29.13 then this defines exactly what we called "the trivial connection" in Definition 28.1.

Remark 32.9. The proof of Proposition 32.7 used that we already knew that the parallel transport system $\mathbb{P}$ determined a covariant derivative operator ∇ – we merely had to identify it. However a minor modification of the argument would allow us to *define* ∇ via (32.1). This would allow us to go directly from a parallel transport system to a covariant derivative operator and bypass connections entirely. Many introductory treatments of Differential Geometry do this. We will see one concrete advantage of why having the connection definition on hand is useful next lecture (Theorem 33.9).

Summary

Here is how to go back and forth between the definitions:

(i) **If you have a connection Δ and you want...**

(a) *a parallel transport system*, then for $\gamma\colon [a,b] \to M$ and a vector $v \in E_{\gamma(0}$, set $\mathbb{p}_{\gamma;v}$ to be the unique horizontal section $\rho \in \Gamma_\gamma(E)$ with $\rho(0) = v$.

(b) *a covariant derivative operator*, then set $\nabla_X s := K(Ds(X))$, where K is the connection map of Δ.

(ii) **If you have a parallel transport system $\mathbb{P}$ and you want...**

(a) *a connection*, then set $\Delta_v \subset T_v E$ to be the set of all tangent vectors $\dot{\mathbb{p}}_{\gamma;v}(0)$ of parallel lifts of curves γ starting at $\pi(v)$.

(b) *a covariant derivative operator*, then to define $\nabla_\xi s$ take any smooth curve γ such that $\dot\gamma(0) = \xi$ and use parallel transport $\mathbb{P}_t^{-1}$ to shift all the vectors $s(\gamma(t))$ into the same vector space $E_{\gamma(0)}$. Then (up to suppressing the $\mathcal{J}$ maps), simply differentiate as normal:

$$\nabla_\xi s := \frac{d}{dt}\Big|_{t=0} \mathbb{P}_t^{-1}(s(\gamma(t))).$$

(iii) **If you have a covariant derivative operator ∇ and you want...**

(a) *a connection*, then set $\Delta_v \subset T_v E$ to be the set of all tangent vectors of the form $Ds(\pi(v))\xi \; - \; \mathcal{J}_v(\nabla_\xi s)$, where s is any section such that $s(\pi(v)) = v$ and $\xi \in T_{\pi(v)} M$.

(b) *a parallel transport system*, then for $\gamma\colon [a,b] \to M$ and a vector $v \in E_{\gamma(0}$, set $\mathbb{p}_{\gamma;v}$ to be the unique section $\rho \in \Gamma_\gamma(E)$ such that $\rho(0) = v$ and $\nabla_{\dot\gamma}\rho = 0$.

With all that being said, we now introduce the arguably somewhat contradictory:

Important convention: Since connections, parallel transport systems and covariant derivative operators are really three different ways of expressing the same concept, we will abuse language and refer to all of them as a "connection" – the notation will make it clear which one we mean $(\Delta, \mathbb{P}, \nabla)$. In fact, since we will typically use the covariant derivative viewpoint more often than the other two, our generic notation for a connection will become ∇.

We now move onto studying the *holonomy* of a connection. In the following, we will have cause to work with **piecewise smooth** curves. By definition a piecewise smooth curve $\gamma\colon [a,b] \to M$ in a manifold M is a continuous map γ such that there exist finitely many points $a_0 = a < a_1 < \ldots < a_r = b$ such that $\gamma|_{[a_i,a_{i+1}]}\colon [a_i,a_{i+1}] \to M$ is smooth for each $i = 0,\ldots r-1$ (thinking of $[a_i,a_{i+1}]$ as a one-dimensional manifold with boundary). The simplest way to manufacture such a curve is simply to glue two smooth curves together:

EXAMPLE 32.10. Suppose $\gamma\colon [a,b] \to M$ and $\delta\colon [b,c] \to M$ are two smooth curves with $\gamma(b) = \delta(b)$. Then the **concatenation** of γ and δ is the piecewise smooth curve $\gamma * \delta\colon [a,c] \to M$ defined by

$$(\gamma * \delta)(t) := \begin{cases} \gamma(t), & a \le t \le b, \\ \delta(t), & b \le t \le c. \end{cases}$$

DEFINITION 32.11. Let $\pi\colon E \to M$ be a vector bundle with connection ∇. Suppose $\gamma\colon [a,b] \to M$ and $\delta\colon [b,c] \to M$ are two smooth curves with $\gamma(b) = \delta(b)$. We extend Axiom (ii) of Definition 29.11 and define the **parallel transport along the piecewise smooth curve** $\gamma * \delta$ to be the linear isomorphism

$$\mathbb{P}_{\gamma*\delta}\colon E_{\gamma(a)} \to E_{\delta(c)}, \qquad \mathbb{P}_{\gamma*\delta} := \mathbb{P}_\delta \circ \mathbb{P}_\gamma.$$

The same definition works for any piecewise smooth curve; as the composition of finitely many linear isomorphisms, it is again a linear isomorphism.

REMARK 32.12. More generally, suppose $\gamma\colon [a,b] \to M$ and $\delta\colon [b_1,c] \to M$ are two smooth curves with $\gamma(b) = \delta(b_1)$ but $b \ne b_1$. Then we cannot directly concatenate γ and δ, and thus we cannot directly define $\mathbb{P}_{\gamma*\delta}$. But this is easily rectified by reparametrising. Indeed, we can choose a diffeomorphism $h\colon [a,b_1] \to [a,b]$ such that $h(a) = a$ and $h(b) = b_1$ and replace γ with $\gamma \circ h$. Then $(\gamma \circ h) * \delta$ is defined. Alternatively, we could reparametrise δ. This reparametrisation will have no effect on the parallel transport thanks to Axiom (iii) from Definition 29.11. From now on we will often suppress the reparametrisation, and speak of the concatentated curve $\gamma * \delta$ and the parallel transport $\mathbb{P}_{\gamma*\delta}$ whenever γ and δ are two curves such that γ ends (in M) where δ begins.

REMARK 32.13. It follows from Axiom (iii) that parallel transport along piecewise smooth curves is associative:

$$\mathbb{P}_{\gamma*(\delta*\varepsilon)} = \mathbb{P}_{(\gamma*\delta)*\varepsilon}$$

for three curves γ, δ, ε such that γ ends where δ begins, and δ ends where ε begins.

Since the inverse of $\mathbb{P}_\gamma$ is $\mathbb{P}_{\gamma^-}$, where γ^- is the reverse path – this is part of Axiom (i), it follows that if we fix a basepoint we get a group.

DEFINITION 32.14. Let $\pi\colon E \to M$ be a vector bundle with connection ∇. Fix $p \in M$. The **holonomy group of** ∇ at p is the subgroup $\mathrm{Hol}^\nabla(p) \subset \mathrm{GL}(E_p)$ consisting of all parallel transport maps

$\mathbb{P}_\gamma \colon E_p \to E_p$ where γ is a piecewise smooth loop at p. We always consider $\mathrm{Hol}^\nabla(p)$ as carrying the subspace topology inherited from $\mathrm{GL}(E_p)$.

If the base manifold M is connected then the holonomy group $\mathrm{Hol}^\nabla(p)$ is – up to isomorphism – independent of p.

Recall a manifold is connected if and only if it is path connected, Proposition 1.37.

LEMMA 32.15. *Let $\pi \colon E \to M$ be a vector bundle with connection ∇. Assume that M is connected. Fix $p, q \in M$ and let γ denote a piecewise smooth curve from p to q. Then the map*

$$\mathrm{Hol}^\nabla(p) \to \mathrm{Hol}^\nabla(q), \qquad \mathbb{P}_\delta \mapsto \mathbb{P}_{\gamma^- * \delta * \gamma} \tag{32.3}$$

is an isomorphism.

Proof. The map (32.3) is a group homomorphism by associativity of parallel transport, since parallel transport around $\gamma^- * (\delta * \varepsilon) * \gamma$ is the same as parallel transport around $\big(\gamma^- * \delta * \gamma\big) * \big(\gamma^- * \varepsilon * \gamma\big)$. Moreover it is an isomorphism as the inverse homomorphism is given by $\mathbb{P}_\delta \mapsto \mathbb{P}_{\gamma * \delta * \gamma^-}$. ■

Suppose E has rank n. It is often useful to think of the holonomy group $\mathrm{Hol}^\nabla(p)$ as a subgroup of $\mathrm{GL}(n)$ rather than $\mathrm{GL}(E_p)$. This can be done, provided we only work up to conjugation. Recall the frame bundle $\mathrm{Fr}(E)$ from Definition 17.24. Fix $p \in M$ and $A \in \mathrm{Fr}(E_p)$; thus $A \colon \mathbb{R}^n \to E_p$ is a linear isomorphism. Then

$$\mathrm{Hol}^\nabla(p; A) := \Big\{ A^{-1} \circ \mathbb{P}_\gamma \circ A \mid \mathbb{P}_\gamma \in \mathrm{Hol}^\nabla(p) \Big\}$$

is a subgroup of $\mathrm{GL}(n)$. If $B \in \mathrm{Fr}(E_p)$ is another frame then the subgroup $\mathrm{Hol}^\nabla(p; B)$ is not equal to $\mathrm{Hol}^\nabla(p; A)$, but it is conjugate to it:

$$\mathrm{Hol}^\nabla(p; B) = \Big\{ TST^{-1} \mid S \in \mathrm{Hol}^\nabla(p; A) \Big\}$$

where $T := B^{-1} \circ A \in \mathrm{GL}(n)$. Moreover if M is connected then Lemma 32.15 shows that if p and q are two points in M and $A \in \mathrm{Fr}(E_p)$ and $B \in \mathrm{Fr}(E_q)$, then the subgroups $\mathrm{Hol}^\nabla(p; A)$ and $\mathrm{Hol}^\nabla(q; B)$ are also conjugate in $\mathrm{GL}(n)$. This proves:

COROLLARY 32.16. *Let $\pi \colon E \to M$ be a vector bundle of rank n over a connected manifold M with connection ∇. Then for all $p \in M$, the holonomy group $\mathrm{Hol}^\nabla(p)$ can be regarded as a subgroup of $\mathrm{GL}(n)$, defined up to conjugation, and in this sense it is independent of p.*

We will explore applications of holonomy over the next few lectures.

LECTURE 33

Curvature

Our first use of holonomy will be to define what it means for a connection on a connected manifold to be trivial.

DEFINITION 33.1. Let $\pi\colon E \to M$ be a vector bundle over a connected manifold, and let ∇ be a connection on E. We say that ∇ is a **trivial connection** if Hol^∇ is the trivial group.

This definition is consistent with Definition 28.1 and Example 29.13.

PROPOSITION 33.2. *Let $\pi\colon E \to M$ be a vector bundle over a connected manifold, and let ∇ be a connection on E. Then ∇ is trivial if and only if E is a trivial vector bundle and the parallel transport system $\mathbb{P}$ is the trivial parallel transport system from Example 29.13.*

Proof. Suppose Hol^∇ is the trivial group, and fix $p \in M$. Define $\varepsilon\colon E \to E_p$ by

$$\varepsilon(v) := \mathbb{P}_{\gamma;v}(1)$$

where γ is a smooth path in M from $\pi(v)$ to p. Then ε is well-defined because $\mathrm{Hol}^\nabla(p)$ is trivial, and ε is smooth by Axiom (iv) of Definition 29.11. By Axiom (i) it follows that ε is a parallel vector bundle chart on E, and thus E is the trivial bundle and $\mathbb{P}$ is the trivial parallel transport system.

i.e. the frame (e_i) corresponding to ε is a parallel frame.

Conversely, if E is the trivial bundle and $\mathbb{P}$ is the trivial parallel transport system, then if γ is a path in M then any parallel section ρ along γ is of the form $\rho = s \circ \gamma$, where s is a global parallel section of E. Thus if $\gamma\colon [0,1] \to M$ is a loop then for any parallel ρ along γ,

$$\rho(1) = s(\gamma(1)) = s(\gamma(0)) = \rho(0).$$

This shows $\mathrm{Hol}^\nabla(\gamma(0))$ is the trivial group. ∎

It is often convenient to restrict to *contractible* loops.

DEFINITION 33.3. Let $\pi\colon E \to M$ be a vector bundle over a connected manifold, and let ∇ be a connection on E. Fix $p \in M$. The **restricted holonomy group** $\mathrm{Hol}_0^\nabla(p)$ is the subgroup of $\mathrm{Hol}^\nabla(p)$ consisting of all parallel transports around contractible (i.e. null-homotopic) piecewise smooth loops at p.

Let $\pi_1(M,p)$ denote the fundamental group of M at p. The Whitney Approximation Theorem 7.13 tells us that any class $[\gamma] \in \pi_1(M,p)$ can be represented by a smooth map γ.

PROPOSITION 33.4. *The restricted holonomy group $\mathrm{Hol}_0^\nabla(p)$ is a path-connected normal subgroup of $\mathrm{Hol}^\nabla(p)$, and there exists a surjective group homomorphism*

$$\pi_1(M,p) \to \mathrm{Hol}^\nabla(p) / \mathrm{Hol}_0^\nabla(p). \tag{33.1}$$

Proof. Suppose $\gamma\colon [0,1] \to M$ is a contractible piecewise smooth loop based at p. Thus there exists a continuous map $H\colon [0,1]\times[0,1] \to M$ such that $H(0,t) = \gamma(t)$, $H(1,t)$ is the constant loop $c_p(t) := p$ and such that $H(s,\cdot)$ is a piecewise smooth loop based at p for each $s \in [0,1]$ (using Theorem 7.17 again). Then $s \mapsto \mathbb{P}_{H(s,\cdot)}$ is a path in $\mathrm{Hol}_0^{\nabla}(p)$ from $\mathbb{P}_\gamma$ to the $\mathbb{P}_{c_p}$. Thus $\mathrm{Hol}_0^{\nabla}(p)$ is path-connected.

Next, if δ and γ are any two loops at p such that γ is nullhomotopic, then the concatenation $\delta^- * \gamma * \delta$ is also nullhomotopic. Thus if $\mathbb{P}_\gamma \in \mathrm{Hol}_0^{\nabla}(p)$ and $\mathbb{P}_\delta \in \mathrm{Hol}^{\nabla}(p)$ then $\mathbb{P}_\delta \circ \mathbb{P}_\gamma \circ \mathbb{P}_{\delta^-} = \mathbb{P}_{\delta^- * \gamma * \delta}$ belongs to $\mathrm{Hol}_0^{\nabla}(p)$. This shows that $\mathrm{Hol}_0^{\nabla}(p)$ is normal in $\mathrm{Hol}^{\nabla}(p)$.

Finally, the desired homomorphism (33.1) sends $[\gamma]$ to the equivalence class of $\mathbb{P}_\gamma$ in the quotient for γ a smooth representative of $[\gamma]$. This is a well-defined surjective group homomorphism by the argument above. ∎

Corollary 32.16 tells us that the holonomy groups are subgroups of the matrix Lie group $\mathrm{GL}(n)$. In fact, much more is true: they are *Lie* subgroups.

Theorem 33.5. *Let $\pi\colon E \to M$ be a vector bundle over a connected manifold, and let ∇ be a connection on E. Then $\mathrm{Hol}^{\nabla}(p)$ is a Lie group, and $\mathrm{Hol}_0^{\nabla}(p)$ is the connected component containing the identity.*

The proof of Theorem 33.5 goes beyond the scope of this course. However a sketch is presented in the bonus section below.

We now explore what it means to say that a connection Δ forms an *integrable* distribution in the sense of Definition 14.9. This will lead us naturally to the concept of the *curvature* of a connection, which roughly speaking measures how far the connection is from being integrable.

Definition 33.6. Let $\pi\colon E \to M$ be a vector bundle with connection ∇. We say that ∇ is a **flat connection** if the corresponding distribution Δ of E is integrable. The pair (E, ∇) is referred to as a **flat vector bundle**.

Trivial connections are always flat. To see this, let us first give an alternative criterion for a connection to be trivial.

Lemma 33.7. *Let $\pi\colon E \to M$ be a vector bundle with connection ∇. Then ∇ is the trivial connection if and only if for every point $v \in E$ there exist a global parallel section $s \in \Gamma(E)$ such that $s(\pi(v)) = v$.*

Proof. This is just a rephrasing of the last part of Proposition 33.2. It is clear that the trivial connection on the trivial vector bundle has this property. Meanwhile if such a section exists through every point then the argument in the last paragraph of the proof of Proposition 33.2 shows that the holonomy groups are trivial, whence Proposition 33.2 itself then shows that ∇ is the trivial connection. ∎

Why is this relevant? If $s \in \Gamma(E)$ is a global parallel section then

$s(M) \subset E$ is an embedded submanifold of E (Lemma 20.8) with

$$D\imath_{s(p)}\big(T_{s(p)}s(M)\big) = \Delta_{s(p)}, \qquad \forall\, p \in M.$$

Thus $s(M)$ is an integral manifold for the distribution Δ passing through $s(p)$ in the sense of Definition 14.6. Therefore by Lemma 14.11 we have:

COROLLARY 33.8. *The trivial connection is flat.*

The converse is true locally. This uses the hard direction of the Frobenius Theorem.

THEOREM 33.9. *Let $\pi\colon E \to M$ be a vector bundle over a connected manifold, and suppose ∇ is a flat connection on E. Then ∇ is a locally trivial connection and $\mathrm{Hol}_0^{\nabla}(p)$ is the trivial group for all $p \in M$.*

Here by a "locally trivial connection" we meant that every point $p \in M$ has a neighbourhood U such that the restriction of ∇ to the trivial subbundle $\pi^{-1}(U) \to U$ of E is the trivial connection.

Proof. We prove the result in two steps.

1. In this step we show that ∇ is locally trivial. By the Global Frobenius Theorem 15.4, Δ induces a foliation of E. Let L be a leaf of the foliation, i.e. a maximal connected integral manifold of the distribution Δ corresponding to ∇. We claim that $\pi|_L\colon L \to M$ is surjective. Indeed, given $v \in L$ and $p \in M$, let $\gamma\colon [0,1] \to M$ be a smooth curve such that $\gamma(0) = \pi(v)$ and $\gamma(1) = p$. The section $\mathbb{p}_{\gamma;v}$ is horizontal and thus has image contained in L. Since $\pi(\mathbb{p}_{\gamma;v}(1)) = p$, this shows that $\pi|_L$ is surjective.

Since π is a submersion, the Inverse Function Theorem 5.10 tells us that $\pi|_L$ is a local diffeomorphism from L to M. Let $U \subset M$ be a connected and simply connected open subset over which E is trivial. Then the intersection $L \cap \pi^{-1}(U)$ is a disjoint union of connected embedded submanifolds of L such that for each component L_k, $\pi|_{L_k}\colon L_k \to U$ is a diffeomorphism. Thus $s_k \coloneqq \pi|_{L_k}^{-1}\colon U \to L_k$ is a section of the vector subbundle $\pi^{-1}(U) \to U$. Since L_k is an integral submanifold of $\Delta|_{\pi^{-1}(U)}$, s_k is a parallel section. Thus for every point of $L \cap \pi^{-1}(U)$ there is a parallel section of $\pi^{-1}(U)$. By Lemma 33.7, the restriction of ∇ to $\pi^{-1}(U)$ is the trivial connection.

Note we have actually shown something slightly stronger than local triviality: namely that ∇ is trivial over *any* connected and simply connected set. We will use this fact in the next step.

2. In this step we show that the restricted holonomy groups are always trivial. Fix a point $p \in M$, and let $\gamma\colon [0,1] \to M$ be a contractible piecewise smooth loop at p. Then as in the proof of Proposition 33.4, there exists a continuous map $H\colon [0,1] \times [0,1] \to M$ such that $H(0,t) = \gamma(t)$, $H(1,t)$ is the constant loop $c_p(t) \coloneqq p$ and such that $H(s,\cdot)$ is a piecewise smooth contractible loop based at p for each $s \in [0,1]$. Fix $v \in E_p$, and let L be the maximal integral manifold of ∇ passing through v. Then as in the previous step, each section $\mathbb{p}_{H(s,\cdot);v}$ has image contained in L. Consider the map

$$\tilde{H}\colon [0,1] \times [0,1] \to L, \qquad \tilde{H}(s,t) \coloneqq \mathbb{p}_{H(s,\cdot);v}(t). \tag{33.2}$$

This map is a lift of H to L in the sense that

$$\pi(\tilde{H}(s,t)) = H(s,t).$$

Since $H(s,1)$ is independent of s, so is $\tilde{H}(s,1)$. Thus

See the bonus section below for more details here.

$$\begin{aligned}\mathbb{P}_{\gamma;v}(1) &= \tilde{H}(0,1)\\ &= \tilde{H}(1,1)\\ &= \mathbb{P}_{c_p;v}(1)\\ &= v.\end{aligned}$$

Thus parallel transport around γ is trivial. Since γ was arbitrary, it follows that $\mathrm{Hol}_0^\nabla(p)$ is the trivial group. This completes the proof. ■

Corollary 33.10. *Let $\pi\colon E \to M$ be a vector bundle over a connected and simply connected manifold M, and let ∇ be a connection on E. Then ∇ is flat if and only if E is the trivial bundle and ∇ is the trivial connection.*

Proof. If M is simply connected then $\mathrm{Hol}^\nabla(p) = \mathrm{Hol}_0^\nabla(p)$ for all $p \in M$. Thus the claim is immediate from Proposition 33.2 and Theorem 33.9. ■

A connection is flat if and only if the vector space $\Gamma(\Delta)$ of horizontal vector fields is a Lie subalgebra of the space $\mathfrak{X}(E) = \Gamma(TE)$ of all vector fields on E. The *curvature* of a connection gives a quantitative way to measure how far a given connection is from being flat.

Definition 33.11. Let $\pi\colon E \to M$ be a vector bundle with connection ∇ and connection map $K\colon TE \to E$. The **curvature tensor** R^∇ of ∇ is defined as follows. Fix vector fields $X, Y \in \mathfrak{X}(M)$ and $v \in E$. Let $\overline{X}$ and $\overline{Y}$ denote the horizontal lifts of X and Y to E (cf. Definition 28.9) and set

$$R^\nabla(X,Y)(v) := -K\big([\overline{X},\overline{Y}](v)\big). \tag{33.3}$$

That is, we take the vertical component of the tangent vector $[\overline{X},\overline{Y}](v) \in T_vE$, which therefore belongs to V_pE, and then project it to E_p via the map $\mathrm{pr}_2\colon VE \to E$.

The minus sign on the right-hand side of (33.3) may look a little unnatural, and indeed some authors define it with the other sign. Our preference for this sign convention will become clear next lecture when we give an alternative way of expressing the curvature (see Theorem 35.10).

Remark 33.12. The meaning of the word “curvature” will become apparent when we study Riemannian Geometry in the second half of the course. We will see that the curvature of a (Riemannian) manifold does indeed correspond to what you would naively guess it does. For example, the sphere S^n with its standard Euclidean metric is “positively” curved.

Since $K(\zeta) = \widetilde{\mathrm{pr}}_2(\zeta^\mathsf{v})$ and $\mathrm{pr}_2\colon VE \to E$ is an vector bundle isomorphism along $\pi\colon E \to M$, we have:

Corollary 33.13. *Let $\pi\colon E \to M$ be a vector bundle with connection ∇. Then ∇ is flat if and only if the curvature R^{∇} is identically zero.*

If s is a section of E then the correspondence

$$p \mapsto R^{\nabla}(X,Y)(s(p))$$

defines another section of E, since it satisfies the section property and is smooth (being the composition of smooth maps). We write this section as $R^{\nabla}(X,Y)(s)$. Thus we can think of R^{∇} as defining a map

$$R^{\nabla}\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\times\Gamma(E)\to\Gamma(E).$$

The main result of this lecture proves that this map is a point operator in all three variables.

Theorem 33.14. *Let $\pi\colon E \to M$ be a vector bundle with connection ∇. Then R^{∇} is $C^\infty(M)$-linear in all three variables, and antisymmetric in the first two variables. Thus R^{∇} can be thought of as a section of the bundle* $\operatorname{Hom}\big(\bigwedge^2(TM),\operatorname{Hom}(E,E)\big)=\bigwedge^2(T^*M)\otimes E\otimes E^*$.

This means that for any fixed $p\in M$ we can unambiguously define

$$R^{\nabla}(\xi,\zeta)\colon E_p\to E_p,\qquad \xi,\zeta\in T_pM \tag{33.4}$$

by picking any vector fields X,Y such that $X(p)=\xi$ and $Y(p)=\zeta$ and setting

$$R^{\nabla}(\xi,\zeta)(v) := R^{\nabla}(X,Y)(v)$$

Proof. We prove the result in two steps.

1. By part (iii) of Lemma 28.10, if X,Y,Z are three vector fields on M and $s\in\Gamma(E)$ we have

$$\begin{aligned}[\overline{X+Y},\overline{Z}](s(p))^{\mathsf{v}} &= [\overline{X}+\overline{Y},\overline{Z}](s(p))^{\mathsf{v}}\\ &= [\overline{X},\overline{Z}](s(p))^{\mathsf{v}}+[\overline{Y},\overline{Z}](s(p))^{\mathsf{v}}.\end{aligned}$$

Since $\mathrm{pr}_2\colon VE\to E$ is a vector bundle morphism along π, this shows that for any section $s\in\Gamma(E)$, we have

$$R^{\nabla}(X+Y,Z)(s)=R^{\nabla}(X,Z)(s)+R^{\nabla}(Y,Z)(s).$$

Next, since the Lie bracket is anti-symmetric we certainly have

$$R^{\nabla}(X,Y)(s)=-R^{\nabla}(Y,X)(s).$$

Now suppose $f\in C^\infty(M)$. Then by part (ii) of Lemma 28.10 and Problem D.5, we have

$$\begin{aligned}[\overline{fX},\overline{Y}](s(p))^{\mathsf{v}} &= [(f\circ\pi)\overline{X},\overline{Y}](s(p))^{\mathsf{v}}\\ &= (f\circ\pi)(s(p))\,[\overline{X},\overline{Y}](s(p))^{\mathsf{v}}-\overline{Y}(f\circ\pi)(s(p))\,\overline{X}(s(p))^{\mathsf{v}}\\ &= (f\circ\pi)(s(p))\,[\overline{X},\overline{Y}](s(p))^{\mathsf{v}}\end{aligned}$$

since $\overline{X}(p)^{\mathsf{v}}=0$ by definition of a horizontal lift. Thus

$$R^{\nabla}(fX,Y)(s)=fR^{\nabla}(X,Y)(s).$$

We have thus show that for a given section s, the map $R^\nabla(\cdot,\cdot)(s)$ is alternating and bilinear over $C^\infty(M)$. Thus it defines a section of the bundle $\mathrm{Hom}(\bigwedge^2(TM), \mathrm{Hom}(E,E))$.

cf. Theorem 36.10.

2. It remains to show that R^∇ is $C^\infty(M)$-linear in the third argument, ie. that

$$R^\nabla(X,Y)(fs) = fR^\nabla(X,Y)(s).$$

This is a bit trickier. Since we already know R^∇ is a point operator in the first two variables, it is sufficient to show that for fixed $p \in M$ and $\xi, \zeta \in T_pM$, the map $R^\nabla(\xi,\zeta)\colon E_p \to E_p$ from (33.4) is $\mathbb{R}$-linear.

Let $\mu_c\colon E \to E$ denote scalar multiplication. Any horizontal lift is μ_c-invariant:

$$\overline{X}(cp) = \overline{X}(\mu_c(p)) = D\mu_c(p)\overline{X}(p).$$

For $c \neq 0$, μ_c is a diffeomorphism, and thus we can write this as $(\mu_c)_*(\overline{X}) = \overline{X}$. Proposition 8.19 therefore tells us that for $c \neq 0$ we have

$$(\mu_c)_*[\overline{X},\overline{Y}] = [\overline{X},\overline{Y}].$$

Now we use Theorem 31.5 to obtain

i.e. the second commutative diagram in (31.6).

$$K\big([\overline{X},\overline{Y}]\big) = K\big((\mu_c)_*[\overline{X},\overline{Y}]\big) = \mu_c \circ K\big([\overline{X},\overline{Y}]\big).$$

This shows that $R^\nabla(\xi,\zeta)\colon E_p \to E_p$ is a homogeneous map, i.e. $R^\nabla(\xi,\zeta)(cv) = cR^\nabla(\xi,\zeta)(v)$ for $c \neq 0$. Since $R^\nabla(\xi,\zeta)$ is differentiable at $0_p \in E_p$, Problem M.1 implies that $R^\nabla(\xi,\zeta)$ it a linear map. This completes the proof. ∎

Bonus Material for Lecture 33

In this bonus section we first sketch of the proof of Theorem 33.5, and then clear up a detail in the proof of Theorem 33.9 that was glossed over.

Proof of Theorem 33.5. A difficult theorem, proved independently by Kuranishi and Yamabe, says that any path connected subgroup of a Lie group is itself a Lie group. Applying this to $\mathrm{Hol}_0^\nabla(p) \subset \mathrm{GL}(n)$ shows that $\mathrm{Hol}_0^\nabla(p)$ is a Lie group. Since M is connected and second countable, its fundamental group is countable. Thus $\mathrm{Hol}^\nabla(p)/\mathrm{Hol}_0^\nabla(p)$ is countable by Proposition 33.4. This implies that $\mathrm{Hol}^\nabla(p)$ is also a Lie group with $\mathrm{Hol}_0^\nabla(p)$ the connected component containing the identity. ∎

Yamabe's proof is very short, and quite easy to understand. See here.

Now let us return to the proof of Theorem 33.9; more specifically, to the map $\tilde{H}$ from (33.2). In the proof of Step 2 of Theorem 33.9, we claimed that the fact that $H(s,1)$ was independent of s implied that $\tilde{H}(s,1)$ was also independent of s. Why is this true?

The answer is hiding a small amount of algebraic topology. In-

Hence the relegation to the bonus section.

deed, the argument in Step 1 actually shows that $\pi|_L \colon L \to M$ is a **covering space**. Covering spaces enjoy the **unique homotopy lifting property**. One way to phrase this is as follows: if $\pi \colon Y \to X$ is a covering space and $\gamma, \delta \colon [0,1] \to X$ are two paths in X which are homotopic with fixed endpoints, then if $p \in Y$ is any point in Y such that $\pi(p) = \gamma(0)$ then there are unique lifts $\tilde{\gamma}, \tilde{\delta}$ of γ and δ that $\tilde{\gamma}(0) = \tilde{\delta}(0) = p$, and *moreover* these lifts also satisfy $\tilde{\gamma}(1) = \tilde{\delta}(1)$.

LECTURE 34

The Holonomy Algebra

In this lecture we define the *holonomy algebra* of a connection. We first recall how to see the endomorphism bundle of a vector bundle as a Lie algebra bundle.

We begin at the level of linear algebra. Let V be a vector space. The vector spaces $\mathrm{End}(V)$ and $\mathfrak{gl}(V)$ are canonically isomorphic. Explicitly, the isomorphism $\mathrm{End}(V) \to \mathfrak{gl}(V)$ is given by differentiation at 0:

$$A \in \mathrm{End}(V) \qquad \Rightarrow \qquad DA(0) \in \mathfrak{gl}(V). \tag{34.1}$$

Nevertheless, as algebras they are different: $\mathrm{End}(V)$ admits the structure of an associative algebra under composition

$$(A, B) \mapsto A \circ B, \qquad A, B \in \mathrm{End}(V). \tag{34.2}$$

Meanwhile $\mathfrak{gl}(V)$ admits the structure of a Lie algebra under commutation

$$(A, B) \mapsto [A, B] = A \circ B - B \circ A, \qquad A, B \in \mathfrak{gl}(V). \tag{34.3}$$

Now let us investigate what this means in terms of bundles. Let $\pi \colon E \to M$ be a vector bundle, and let $\mathrm{End}(E) \to M$ denote the associated endomorphism bundle. This bundle admits the structure of algebra bundle (Definition 19.29) in two ways: firstly, via fibrewise composition (34.2), and secondly via fibrewise commutation (34.3).

CONVENTION. To help distinguish the two, we denote by $\mathfrak{gl}(E)$ the bundle $\mathrm{End}(E)$, thought of as a Lie algebra bundle under the Lie bracket (34.3). Meanwhile the algebra structure on $\mathrm{End}(E)$ should always be understood as coming from composition (34.2).

Thus both $\mathrm{End}(E)$ and $\mathfrak{gl}(E)$ have the same underlying vector bundle structure, but as algebra bundles they are different. Our default choice of notation remains $\mathrm{End}(E)$ – we use the notation $\mathfrak{gl}(E)$ only when it is important to emphasise the Lie algebra structure.

DEFINITION 34.1. Let $\pi \colon E \to M$ be a vector bundle and let ∇ be a connection on E. We define the **holonomy algebra at** $p \in M$, written $\mathfrak{hol}^{\nabla}(p)$, to be the Lie algebra of $\mathrm{Hol}^{\nabla}(p)$. Since $\mathrm{Hol}^{\nabla}(p)$ is a Lie subgroup of $\mathrm{GL}(E_p)$ by Theorem 33.5, it follows that $\mathfrak{hol}^{\nabla}(p)$ is a Lie subalgebra of $\mathfrak{gl}(E_p)$, with Lie bracket given by matrix commutation (cf. Proposition 11.9):

$$[A, B] := A \circ B - B \circ A, \qquad A, B \in \mathfrak{hol}^{\nabla}(p).$$

We then define

$$\mathfrak{hol}^{\nabla} := \bigsqcup_{p \in M} \mathfrak{hol}^{\nabla}(p).$$

We call $\mathfrak{hol}^{\nabla}$ the **holonomy algebra** of ∇.

The holonomy algebra is itself a vector bundle over M. In fact, it is a Lie algebra subbundle of $\mathfrak{gl}(E)$. Before proving this, we need another definition.

Definition 34.2. Let $\pi\colon E \to M$ be a vector bundle and let ∇ be a connection on E. Suppose $F \subset E$ is a vector subbundle of E. We say that the connection ∇ is **reducible** to F if F is invariant under parallel transport in the sense that if $\gamma\colon [a,b] \to M$ is a smooth curve then $\mathbb{P}_\gamma(F_{\gamma(a)}) \subseteq F_{\gamma(b)}$.

Alternatively, in terms of distributions: a connection Δ is reducible to F if $\Delta_v \subset T_v F$ for all $v \in F$.

On Problem Sheet M you will show that if ∇ is reducible to F then ∇ induces a connection on F. In fact, the hypothesis that F is a vector subbundle of E is superfluous, as the next lemma shows.

Lemma 34.3. *Let $\pi\colon E \to M$ be a vector bundle over a connected manifold and let ∇ be a connection on E. Assume $F \subset E$ is a subset invariant under parallel transport with the property that there exists $p \in M$ such that $F \cap E_p$ is a linear subspace of E_p. Then F is a vector subbundle of E, and ∇ is reducible to F.*

Proof. Since $\mathbb{P}_\gamma\colon E_{\gamma(a)} \to E_{\gamma(b)}$ is a linear isomorphism for any smooth curve $\gamma\colon [a,b] \to M$, it follows that if $F \cap E_p$ is a linear subspace of E_p for *some* point $p \in M$ then $F \cap E_q$ is a linear subspace of E_q for *every* point $q \in M$. Vector subbundle charts on F can be obtained by taking the restriction of the vector bundle charts on E built from Proposition 32.4. ∎

Recall a vector bundle chart is the same thing as a local frame, cf. Lemma 20.6.

Next, by part (ii) of Problem M.4, ∇ induces a connection ∇^{End} on $\mathrm{End}(E)$. Since $\mathrm{End}(E)$ and $\mathfrak{gl}(E)$ are the same vector bundle, we can also regard ∇^{End} as a connection $\nabla^{\mathfrak{gl}}$ on $\mathfrak{gl}(E)$. We denote the associated parallel transport systems by $\mathbb{P}^{\mathrm{End}}$ and $\mathbb{P}^{\mathfrak{gl}}$.

Proposition 34.4. *Let $\pi\colon E \to M$ be a vector bundle, and let ∇ denote a connection on M. Then $\mathbb{P}^{\mathrm{End}}$ respects the composition algebra structure on $\mathrm{End}(E)$ and $\mathbb{P}^{\mathfrak{gl}}$ respects the Lie algebra structure on $\mathfrak{gl}(E)$.*

Proof. We begin by identified what parallel transport with respect to ∇^{End} in the bundle $\mathrm{End}(E)$ looks like. Suppose $p \in M$ and $\gamma\colon [0,1] \to M$ is a smooth curve with $\gamma(0) = p$. Abbreviate by $\mathbb{P}_t\colon E_p \to E_{\gamma(t)}$ parallel transport along the curve $r \mapsto \gamma(r)$ for $0 \le r \le t$ with respect to ∇ and similarly by

$$\mathbb{P}_t^{\mathrm{End}}\colon \mathrm{End}(E_{\gamma(0)}) \to \mathrm{End}(E_{\gamma(t)}).$$

the parallel transport with respect to ∇^{End}. Suppose $C \in \Gamma_\gamma(\mathrm{End}(E))$ is a section along γ, i.e.

$$C(t)\colon E_{\gamma(t)} \to E_{\gamma(t)}$$

is a linear map for each $t \in [0,1]$. It follows from Problem M.3, Problem M.4 and Proposition 32.7 that a section C is parallel along γ with respect to ∇^{End} if and only if for every section $\rho \in \Gamma_\gamma(E)$ which is parallel along γ with respect to ∇, the section $C(\rho) \in \Gamma_\gamma(E)$ defined by $t \mapsto C(t)\rho(t)$ is also parallel with respect to γ. This means that for $A \in \mathrm{End}(E_{\gamma(0)})$ we have

$$\mathbb{P}_t^{\mathrm{End}}(A) = \mathbb{P}_t \circ A \circ \mathbb{P}_t^{-1}. \tag{34.4}$$

From this equation it is immediate that

$$\mathbb{P}_t^{\mathrm{End}}(A \circ B) = \mathbb{P}_t^{\mathrm{End}}(A) \circ \mathbb{P}_t^{\mathrm{End}}(B),$$

which shows that $\mathbb{P}^{\mathrm{End}}$ respects the algebra structure on $\mathrm{End}(E)$. Next, by (34.1) the parallel transport $\mathbb{P}_t^{\mathfrak{gl}}$ on $\mathfrak{gl}(E)$ is given by differentiating $\mathbb{P}_t^{\mathrm{End}}$ at id:

$$\mathbb{P}_t^{\mathfrak{gl}} := D\mathbb{P}_t^{\mathrm{End}}(\mathrm{id}) \tag{34.5}$$

Since $\mathbb{P}_t^{\mathrm{End}}$ is a linear map, Problem B.3 tells us that its differential is given by

$$D\mathbb{P}_t^{\mathrm{End}}(\mathrm{id})(A) = \mathcal{J}_0^{-1} \circ \mathbb{P}_t^{\mathrm{End}}(A) \circ \mathcal{J}_0$$

Thus from (34.4) we see that

$$\mathbb{P}_t^{\mathfrak{gl}}([A,B]) = \left[\mathbb{P}_t^{\mathfrak{gl}}(A), \mathbb{P}_t^{\mathfrak{gl}}(B)\right],$$

which shows that $\mathbb{P}^{\mathfrak{gl}}$ respects the algebra structure on $\mathfrak{gl}(E)$. ■

We are now ready to prove that the holonomy algebra bundle is a Lie subbundle of $\mathfrak{gl}(E)$.

THEOREM 34.5. *Let $\pi\colon E \to M$ be a vector bundle over a connected manifold and let ∇ be a connection on E. Then $\mathfrak{hol}^\nabla$ is a Lie algebra subbundle of $\mathfrak{gl}(E)$. Moreover the induced connection ∇^{End} on $\mathfrak{gl}(E)$ is reducible to $\mathfrak{hol}^\nabla$.*

Proof. Suppose $p \in M$ and $\gamma\colon [0,1] \to M$ is a smooth curve with $\gamma(0) = p$, and let $\mathbb{P}_t^{\mathrm{End}}$ be as in the proof of Proposition 34.4. Comparing (34.4) and Lemma 32.15, we see that the isomorphism $\mathrm{Hol}^\nabla(p) \cong \mathrm{Hol}^\nabla(\gamma(t))$ is exactly given by $\mathbb{P}_t^{\mathrm{End}}$:

$$\mathbb{P}_t^{\mathrm{End}}\colon \mathrm{Hol}^\nabla(p) \xrightarrow{\sim} \mathrm{Hol}^\nabla(\gamma(t)).$$

If we differentiate $\mathbb{P}_t^{\mathrm{End}}$ at $\mathrm{id} \in \mathrm{Hol}^\nabla(p)$, we get a linear map:

$$D\mathbb{P}_t^{\mathrm{End}}(\mathrm{id})\colon \mathfrak{hol}^\nabla(p) \to \mathfrak{hol}^\nabla(\gamma(t)).$$

Since $\mathbb{P}_t^{\mathrm{End}}$ is itself linear map, this is equivalent to the the statement that $\mathbb{P}_t^{\mathrm{End}}$ defines a map

$$\mathbb{P}_t^{\mathrm{End}}\colon \mathfrak{hol}^\nabla(p) \to \mathfrak{hol}^\nabla(\gamma(t)).$$

By (34.5) this exactly the assertion that $\mathfrak{hol}^\nabla$ is invariant under parallel transport $\mathbb{P}^{\mathfrak{gl}}$. Lemma 34.3 then implies that $\mathfrak{hol}^\nabla$ is a vector subbundle of $\mathfrak{gl}(E)$ and that the connection $\nabla^{\mathfrak{gl}}$ is reducible to $\mathfrak{hol}^\nabla$.

It remains to show that $\mathfrak{hol}^\nabla$ is actually a Lie algebra subbundle. For this we apply Proposition 32.4 to the vector bundle $\mathfrak{gl}(E)$ equipped with the connection $\nabla^{\mathfrak{gl}}$. Let ε denote the vector bundle chart on $\mathfrak{gl}(E)$ corresponding to the local frame (e_i). This bundle chart has the property that if C is a parallel sections along a curve of the form $\gamma = \gamma_{p,\xi}$ then

$$\varepsilon_{\gamma(t)}(C(t)) = \varepsilon_p(C(0)). \tag{34.6}$$

Thus if C, D are parallel then applying this to C, D and $[C, D]$ (which is also parallel by Proposition 34.4) we obtain

$$\begin{aligned}
\varepsilon_{\gamma(t)}\big([C, D](t)\big) &= \varepsilon_p\big([C, D](0)\big) \\
&= \varepsilon_p\big([C(0), D(0)]\big) \\
&= \big[\varepsilon_p(C(0)), \varepsilon_p(D(0))\big] \\
&= \big[\varepsilon_{\gamma(t)}(C(t)), \varepsilon_{\gamma(t)}(D(t))\big]
\end{aligned}$$

Thus the vector bundle charts on $\mathfrak{gl}(E)$ constructed in this way all preserve the Lie bracket, and hence may be taken as Lie algebra bundle charts on $\mathfrak{gl}(E)$. Moreover these charts restrict to Lie algebra charts on $\mathfrak{hol}^\nabla$, since the latter is invariant under parallel transport by the first part of the proof. ■

LECTURE 35

Reinterpreting Curvature

This lecture is devoted to giving two additional viewpoints on the curvature – one is geometric in nature (Proposition 35.3) and the other is useful for computations (Definition 35.7). Along the way we state the famous *Ambrose-Singer Holonomy Theorem*, whose proof will follow later in the course.

We begin with the following technical lemma, which is a souped-up version of Problem E.5.

LEMMA 35.1. *Let M be a smooth manifold and let X, Y be vector fields on M with local flows Φ_t and Ψ_t respectively. Fix $p \in M$ and consider the curve*

If Φ and Ψ commute the curve γ is constant.

$$\gamma \colon [0, \varepsilon) \to M, \qquad \gamma(t) := \Psi_{-\sqrt{t}} \circ \Phi_{-\sqrt{t}} \circ \Psi_{\sqrt{t}} \circ \Phi_{\sqrt{t}}(p),$$

which is well-defined for small enough ε. If $f \in C^\infty(U)$ is a smooth function on a neighbourhood U of p then

$$[X, Y](f)(p) = \lim_{t \to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t}.$$

Proof. Let $\delta(t) := \gamma(t^2)$. Then we claim that

(i) $(f \circ \delta)'(0) = 0$,

(ii) $(f \circ \delta)''(0) = 2[X, Y](f)(p)$.

This implies

$$\begin{aligned} [X, Y](f)(p) &= \frac{1}{2}(f \circ \delta)''(0) \\ &= \lim_{t \to 0} \frac{f(\delta(t)) - f(\delta(0))}{t^2} \\ &= \lim_{t \to 0} \frac{f(\delta(\sqrt{t})) - f(\delta(0))}{t} \\ &= \lim_{t \to 0} \frac{(f(\gamma(t)) - f(\gamma(0))}{t}. \end{aligned}$$

To prove (i) and (ii), consider the rectangles

$$\begin{aligned} A(s, t) &:= \Psi_s \circ \Phi_t(p) \\ B(s, t) &:= \Phi_{-s} \circ \Psi_t \circ \Phi_t(p) \\ C(s, t) &:= \Psi_{-s} \circ \Phi_{-t} \circ \Psi_t \circ \Phi_t(p). \end{aligned}$$

Then $\delta(t) = C(t, t)$ and $C(0, t) = B(t, t)$ and $B(0, t) = A(t, t)$. Abbreviate

$$\partial_s(f \circ C)(0, 0) := D(f \circ C)(0, 0) \left[\frac{\partial}{\partial s}\Big|_{s=0}, 0 \right]$$

and similarly for the other partial derivatives. Then by the chain rule

$$\begin{aligned}(f\circ\delta)'(0) &= \partial_s(f\circ C)(0,0)+\partial_t(f\circ C)(0,0)\\ &= \partial_s(f\circ C)(0,0)+\partial_s(f\circ B)(0,0)+\partial_t(f\circ B)(0,0)\\ &= \partial_s(f\circ C)(0,0)+\partial_s(f\circ B)(0,0)+\partial_s(f\circ A)(0,0)+\partial_t(f\circ A)(0,0)\\ &= -Y(f)(p)-X(f)(p)+Y(f)(p)+X(f)(p)\\ &= 0.\end{aligned}$$

This proves (i). To prove (ii) we start from

$$(f\circ\delta)''(0)=\partial_{ss}(f\circ C)(0,0)+2\partial_{ts}(f\circ C)(0,0)+\partial_{tt}(f\circ C)(0,0). \quad (35.1)$$

Since $\partial_s(f\circ C)=-Y(f)\circ C$, the first term on the right-hand side of (35.1) is equal to

$$\partial_{ss}(f\circ C)(0,0)=\partial_s(-Y(f)\circ C)(0,0)=Y(Y(f))(p).$$

Similarly since

$$\partial_s(f\circ A)=Y(f)\circ A, \qquad \partial_s(f\circ\Phi_s)=-X(f)\circ B,$$

and

$$\partial_t(f\circ A)(0,t)=X(f)\circ A(0,t),$$

we obtain

$$2\partial_{ts}(f\circ C)(0,0)=-2Y(Y(f)(p))$$

and

$$\partial_{tt}(f\circ C)(0,0)=Y(Y(f))(p)+2[X,Y](f)(p).$$

Substituting these into (35.1) gives

$$(f\circ\delta)''(0)=2Y(Y(f))(p)-2Y(Y(f))(p)+2[X,Y](f)(p),$$

which proves (ii). ∎

Remark 35.2. The curve γ from the statement of Lemma 35.1 is typically not differentiable (not even right differentiable) at 0. Thus strictly speaking, the tangent vector $\dot\gamma(0)$ is not defined. However if we formally define a tangent vector $\dot\gamma(0)$ by declaring that

$$\dot\gamma(0)(f)\overset{\text{def}}{=}\lim_{t\to 0}\frac{f(\gamma(t))-f(\gamma(0))}{t}$$

then $\dot\gamma(0)$ is a well-defined element of T_pM. In this sense the conclusion of Lemma 35.1 can be restated as

$$[X,Y](p)=\dot\gamma(0).$$

We will use this convention without comment in the future.

Suppose now $\xi,\zeta\in T_pM$ are two fixed tangent vectors. Choose vector fields X, Y such that $X(p)=\xi$ and $Y(p)=\zeta$. We may without loss of generality assume that $[X,Y]=0$ on a neighbourhood of p. Let Φ_t and Ψ_t denote the local flows of X and Y. Since $[X,Y]=0$ near p, by either Lemma 35.1 above (or Problem E.5), for sufficiently small $t>0$ the curve η_t obtained by concatenating the four curves:

(i) $s \mapsto \Phi_s(p)$ for $0 \leq s \leq \sqrt{t}$,

(ii) $s \mapsto \Psi_s \circ \Phi_{\sqrt{t}}(p)$ for $0 \leq s \leq \sqrt{t}$,

(iii) $s \mapsto \Phi_{-s} \circ \Psi_{\sqrt{t}} \circ \Phi_{\sqrt{t}}(p)$ for $0 \leq s \leq \sqrt{t}$,

(iv) $s \mapsto \Psi_{-s} \circ \Phi_{-\sqrt{t}} \circ \Psi_{\sqrt{t}} \circ \Phi_{\sqrt{t}}(p)$ for $0 \leq s \leq \sqrt{t}$,

is a piecewise smooth *loop* based at p. See Figure 35.1.

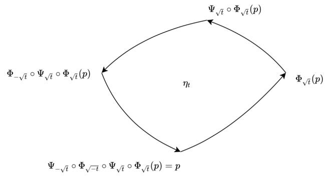


Figure 35.1: The piecewise smooth loop η_t

Fix $v \in E_p$ and consider the curve in E given by

$$t \mapsto \mathbb{P}_{\eta_t}(v) = \mathbb{p}_{\eta_t;v}(1).$$

This is smooth by Axiom (iv). Since η_t is a loop, its image is contained in E_p, and hence its tangent vector at $t = 0$ is vertical:

$$\frac{d}{dt}\Big|_{t=0} \mathbb{P}_{\eta_t}(v) \in V_v E. \tag{35.2}$$

Do not confuse (35.2) with the tangent vector $\dot{\mathbb{p}}_{\eta_t;v}(0)$!

is a vertical tangent vector in $T_v E$. The following statement can be thought of as a geometric interpretation of the curvature.

Proposition 35.3. *The curvature operator $R^\nabla(\xi, \zeta) \colon E_p \to E_p$ is given by*

$$R^\nabla(\xi,\zeta)(v) = -\mathcal{J}_v^{-1}\left(\frac{d}{dt}\Big|_{t=0} \mathbb{P}_{\eta_t}(v)\right).$$

In words: The curvature at v is the time derivative of parallel transporting v around shorter and shorter loops η_t.

Proof. Let $\overline{\Phi}_t$ and $\overline{\Psi}_t$ denote the local flows of the horizontal lifts $\overline{X}$ and $\overline{Y}$. Then by Problem E.4, we have

$$\pi \circ \overline{\Phi}_t = \Phi_t \circ \pi, \qquad \pi \circ \overline{\Psi}_t = \Psi_t \circ \pi,$$

and by definition of the horizontal lift, for all $v \in E$ sufficiently close to E_p, we have

$$\overline{\Phi}_t(v) = \mathbb{p}_{\delta;v}(t), \qquad \text{where } \delta(t) := \Phi_t(p).$$

and similarly

$$\overline{\Psi}_t(v) = \mathbb{p}_{\varepsilon;v}(t), \qquad \text{where } \varepsilon(t) := \Psi_t(p).$$

Thus for $v \in E_p$ and $t > 0$ sufficiently small, one has

$$\mathbb{P}_{\eta_t}(v) = \overline{\Psi}_{-\sqrt{t}} \circ \overline{\Phi}_{-\sqrt{t}} \circ \overline{\Psi}_{\sqrt{t}} \circ \overline{\Phi}_{\sqrt{t}}(v).$$

By Lemma 35.1 we have

$$[\overline{X}, \overline{Y}](v) = \frac{d}{dt}\Big|_{t=0} \mathbb{P}_{\eta_t}(v).$$

Since $\frac{d}{dt}\big|_{t=0}\mathbb{P}_{\eta_t}(v)$ is vertical, by (31.1) we have

$$K\left(\frac{d}{dt}\Big|_{t=0} \mathbb{P}_{\eta_t}(v)\right) = \mathcal{J}_v^{-1}\left(\frac{d}{dt}\Big|_{t=0} \mathbb{P}_{\eta_t}(v)\right).$$

Thus

$$R^\nabla(\xi,\zeta)(v) = -\mathcal{J}_v^{-1}\left(\frac{d}{dt}\Big|_{t=0} \mathbb{P}_{\eta_t}(v)\right)$$

as desired. ■

Remark 35.4. The preceding proof gives another way to see that R^∇ is a point operator in the third variable, which bypasses the use of Theorem 31.5: Define a curve $\ell(t)$ in $\mathrm{GL}(E_p)$ by

$$\ell(t)v := \mathbb{P}_{\eta_t}(v)$$

for small $t > 0$. Thus $\dot{\ell}(0) \in T_{\mathrm{id}}\,\mathrm{GL}(E_p) = \mathfrak{gl}(E_p)$. Then Proposition 35.3 tells us that

$$R^\nabla(\xi, \zeta) = -\underline{\dot{\ell}}(0) \in \mathrm{End}(E_p)$$

is a linear operator.

We now investigate how curvature affects the holonomy algebra. We first have:

Corollary 35.5. *Let $\pi\colon E \to M$ be a vector bundle and suppose ∇ is a connection on E. Then for all $x \in M$ and $\xi, \zeta \in T_pM$, the linear operator $R^\nabla(\xi, \zeta) \in \mathrm{End}(E_p)$ actually belongs to $\underline{\mathfrak{hol}}^\nabla(p)$.*

Proof. This is immediate from Proposition 35.3. ∎

Corollary 35.5 actually shows us rather more: namely, how the holonomy algebra $\underline{\mathfrak{hol}}^\nabla(p)$ is influenced by the curvature across the entire manifold. Indeed, if γ is a smooth path in M from q to p and $\xi, \zeta \in T_pM$, then the operator

$$\mathbb{P}_\gamma^{\mathrm{End}}(R^\nabla(\xi, \zeta)) = \mathbb{P}_\gamma \circ R^\nabla(\xi, \zeta) \circ \mathbb{P}_\gamma^{-1}$$

also belongs to $\underline{\mathfrak{hol}}^\nabla(p)$. The next theorem, which is one of the cornerstones of the subject, tells us this is all there is.

Theorem 35.6 (The Ambrose–Singer Holonomy Theorem). *Let $\pi\colon E \to M$ be a vector bundle over a connected manifold M and let ∇ be a connection on E. Then for any $p \in M$, the holonomy algebra $\underline{\mathfrak{hol}}^\nabla(p)$ at p is the vector subspace of $\mathrm{End}(E_p)$ spanned by all the elements of the form*

$$\mathbb{P}_\gamma \circ R^\nabla(\xi, \zeta) \circ \mathbb{P}_\gamma^{-1}, \qquad q \in M,\ \xi, \zeta \in T_qM$$

where γ is a piecewise smooth path in M from q to p.

In Lecture 42 we will prove a version of the Ambrose–Singer Holonomy Theorem for principal bundles. Theorem 35.6 is a corollary of this more general principal bundle version, as you will prove on Problem Sheet O.

Instead, now we work towards deriving a more convenient formula for R^∇. As with our approach to covariant derivatives, it will be useful to formulate this in the more general setting of sections along a map.

Definition 35.7. Let $\pi\colon E \to N$ denote a vector bundle with connection ∇, and let $\varphi\colon M \to N$ denote a smooth map. Define for $X, Y \in \mathfrak{X}(M)$ and $s \in \Gamma_\varphi(E)$

$$R^\nabla_\varphi(X, Y)(s) = \nabla^\varphi_X \nabla^\varphi_Y s - \nabla^\varphi_Y \nabla^\varphi_X s - \nabla^\varphi_{[X,Y]} s. \tag{35.3}$$

We will prove next lecture that for $\varphi = \mathrm{id}$ we have

$$R^{\nabla}_{\mathrm{id}} = R^{\nabla}.$$

One can interpret (35.3) as

$$R^{\nabla}_{\varphi}(X,Y) = [\nabla^{\varphi}_X, \nabla^{\varphi}_Y] - \nabla^{\varphi}_{[X,Y]}.$$

The first term measures the failure of ∇^{φ}_X and ∇^{φ}_Y to commute, and the second term is subtracted to make the following result true.

Proposition 35.8. *The operator R^{∇}_{φ} is $C^\infty(M)$-linear in all three variables, and antisymmetric in the first two variables.*

Proof. We prove only that $R^{\nabla}_{\varphi}(fX,Y)(s) = fR^{\nabla}_{\varphi}(X,Y)(s)$; the remaining computations are similar and left as an exercise. By Problem D.5 we have $[fX,Y] = f[X,Y] - Y(f)X$ and hence

$$\begin{aligned}
R^{\nabla}_{\varphi}(fX,Y)(s) &= \nabla^{\varphi}_{fX}\nabla^{\varphi}_{Y}s - \nabla^{\varphi}_{Y}\nabla^{\varphi}_{fX}s - \nabla^{\varphi}_{[fX,Y]}s \\
&= f\nabla^{\varphi}_{X}\nabla^{\varphi}_{Y}s - \nabla^{\varphi}_{Y}(f\nabla^{\varphi}_{X}s) - \nabla^{\varphi}_{f[X,Y]}(s) + \nabla^{\varphi}_{Y(f)X}s \\
&= f\left(\nabla^{\varphi}_{X}\nabla^{\varphi}_{Y}s - \nabla^{\varphi}_{Y}\nabla^{\varphi}_{X}s - \nabla^{\varphi}_{[X,Y]}s\right) - Y(f)\nabla^{\varphi}_{X}s + Y(f)\nabla^{\varphi}_{X}s \\
&= fR^{\nabla}_{\varphi}(X,Y)(s).
\end{aligned}$$

This completes the proof. ∎

Since R^{∇}_{φ} is $C^\infty(M)$-linear is all variables, it is a point operator in all three variables by Theorem 20.20, and hence $R^{\nabla}_{\varphi}(\xi,\zeta)(v)$ is well defined for any $\xi,\zeta \in T_pM$ and $v \in E_{\varphi(p)}$.

Proposition 35.9. *Let $\pi\colon E \to N$ denote a vector bundle with connection ∇, and let $\varphi\colon M \to N$ denote a smooth map. Then for all $p \in M$, $\xi,\zeta \in T_pM$ and $v \in E_{\varphi(p)}$ we have*

$$R^{\nabla}_{\varphi}(\xi,\zeta)(p) = R^{\nabla}_{\mathrm{id}}\big(D\varphi(p)\xi, D\varphi(p)\zeta\big)(v). \tag{35.4}$$

In particular, if $\varphi\colon M \to N$ is a diffeomorphism then for all $X,Y \in \mathfrak{X}(M)$ and $s \in \Gamma_\varphi(E)$ we have

$$R^{\nabla}_{\mathrm{id}}(\varphi_*X, \varphi_*Y)(s) = R^{\nabla}_{\varphi}(X,Y)(s).$$

Proof. Assume $X,Y \in \mathfrak{X}(M)$ are φ-related to vector fields $Z,W \in \mathfrak{X}(N)$, and assume $s \in \Gamma_\varphi(E)$ has the property that $s(p) = \tilde{s}(\varphi(p))$ for some section $\tilde{s}$ of E and all $x \in M$. Then by repeatedly applying the chain rule for covariant derivatives (31.9) we have

$$\begin{aligned}
\nabla^{\varphi}_{X}\nabla^{\varphi}_{Y}s &= \nabla^{\varphi}_{X}\nabla^{\varphi}_{Y}(\tilde{s}\circ\varphi) \\
&= \nabla^{\varphi}_{X}\nabla^{\varphi}_{W\circ\varphi}\tilde{s} \\
&= \nabla^{\varphi}_{X}((\nabla_W\tilde{s})\circ\varphi) \\
&= \nabla^{\varphi}_{Z\circ\varphi}\nabla_W\tilde{s} \\
&= (\nabla_Z\nabla_W\tilde{s})\circ\varphi.
\end{aligned}$$

Similarly $\nabla^{\varphi}_{Y}\nabla^{\varphi}_{X}s = (\nabla_W\nabla_Z\tilde{s})\circ\varphi$. Moreover by Problem D.6 we have $\nabla^{\varphi}_{[X,Y]}s = (\nabla_{[Z,W]}\tilde{s})\circ\varphi$, and hence

$$R^{\nabla}_{\varphi}(X,Y)(s) = \Big(R^{\nabla}_{\mathrm{id}}(Z,W)(\tilde{s})\Big)\circ\varphi. \tag{35.5}$$

Since both sides of (35.4) are point operators, the general case follows from this special case. For sections, this is easy: sections of the form $\tilde{s} \circ \varphi$ locally generate $\Gamma_\varphi(E)$ over $C^\infty(M)$. For vector fields, the argument is a bit subtler. In general for a given $\xi \in T_pM$ it is not possible (even locally) to find vector fields X, Z on M and N respectively which are φ related and satisfy $X(p) = \xi$. Nevertheless this *is* the case if either φ is an embedding (when $\dim M \leq \dim N$) or a submersion (when $\dim M \geq \dim N$). This however suffices, since every map φ can be written as a composition of an embedding and a submersion $\varphi = \psi \circ \iota$, where $\iota\colon M \hookrightarrow M \times N$ is the map $p \mapsto (p, q_0)$ (for some fixed $q_0 \in N$) and $\psi\colon M \times N \to N$ is the map $(p, q) \mapsto \varphi(p)$. Thus arguing as in (35.5) gives us

This argument is due to Alessandro Pigati.

$$\begin{aligned} R^\nabla_\varphi(\xi,\zeta)(v) &= R^\nabla_\psi\big(D\iota(p)\xi, D\iota(p)\zeta\big)(v) \\ &= R^\nabla_{\mathrm{id}}\big(D\varphi(p)\xi, D\varphi(p)\zeta\big)(v). \qquad \blacksquare \end{aligned}$$

We conclude today's lecture by completing our discussion of curvature and proving that for $\varphi = \mathrm{id}$ the operator R^∇_{id} from Definition 35.7 agrees with the curvature R^∇.

THEOREM 35.10. *Let $\pi\colon E \to M$ denote a vector bundle with connection ∇. Then $R^\nabla_{\mathrm{id}} = R^\nabla$.*

Proof. Consider $\mathbb{R}^2$ with coordinates (s,t). Let S, T denote the vector fields $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ on $\mathbb{R}^2$ respectively. Now fix $p \in M$, $\xi, \zeta \in T_pM$ and $v \in E_p$. Choose a smooth map $\gamma\colon (-\varepsilon,\varepsilon) \times (-\varepsilon,\varepsilon) \to M$ such that $\gamma(0,0) = p$ and

We break our usual convention that the coordinates on $\mathbb{R}^2$ are (u^1, u^2) here so as to simplify the notation in this proof.

$$D\gamma(0,0)S(0,0) = \xi, \qquad D\gamma(0,0)T(0,0) = \zeta.$$

Now define a section $\rho \in \Gamma_\gamma(E)$ such that $\rho(0,0) = v$ and such that:

(i) ρ is parallel along the curve $t \mapsto \gamma(0,t)$,

(ii) ρ is parallel along the curve $s \mapsto \gamma(s,t)$ for all $t \in (-\varepsilon,\varepsilon)$.

Such a section exists and is unique by Proposition 29.9. To see this first apply Proposition 29.9 along to curve to $t \mapsto \gamma(0,t)$ so that (i) is satisfied. Then define ρ along each curve $s \mapsto \gamma(s,t)$ again via Proposition 29.9. The fact that the resulting section ρ is smooth in both s and t is due to the fact that integral curves depend smoothly on initial conditions (see the proof of Proposition 29.9 and Theorem 9.1). Using (ii) and the fact that $[S,T] = 0$ we obtain

$$R^\nabla_\gamma(S(0,0), T(0,0))(v) = (\nabla^\gamma_S \nabla^\gamma_T \rho)(0,0).$$

Let $\mathbb{P}_s\colon E_p \to E_{\gamma(s,0)}$ denote parallel transport along $r \mapsto \gamma(r,0)$ for $0 \leq r \leq s$ and let $\mathbb{P}_{s,t}\colon E_{\gamma(s,0)} \to E_{\gamma(s,t)}$ denote parallel transport along $r \mapsto \gamma(s,r)$ for $0 \leq r \leq t$. Then by Proposition 32.7 we have

$$(\nabla^\gamma_T \rho)(s,0) = \frac{d}{dt}\Big|_{t=0} \mathbb{P}^{-1}_{s,t}(\rho(s,t))$$

and thus

$$R^\nabla_\gamma(S(0,0), T(0,0))(p) = \frac{d^2}{dsdt}\Big|_{(s,t)=(0,0)} \mathbb{P}^{-1}_s \mathbb{P}^{-1}_{s,t}(\rho(s,t))$$

Thus by the definition of the derivative as a limit, the right-hand side is equal to

$$\lim_{s,t\to 0} \frac{\mathbb{P}_s^{-1}\mathbb{P}_{s,t}^{-1}(\rho(s,t)) - \mathbb{P}_s^{-1}\mathbb{P}_{s,0}^{-1}(\rho(s,0)) - \mathbb{P}_0^{-1}\mathbb{P}_{0,t}^{-1}(\rho(0,t)) + \mathbb{P}_0^{-1}\mathbb{P}_{0,0}^{-1}(\rho(0,0))}{st}$$

Since $\rho(s,0) = \mathbb{P}_s(v)$ by assumption (ii), $\rho(0,t) = \mathbb{P}_{0,t}(v)$ by assumption (i) and $\mathbb{P}_{s,0} = \mathrm{id}$ by definition we can simplify this to

$$\lim_{s,t\to 0} \frac{\mathbb{P}_s^{-1}\mathbb{P}_{s,t}^{-1}(\rho(s,t)) - v}{st}$$

Now take $s = t$ to obtain

$$R_\gamma^\nabla(S(0,0),T(0,0))(p) = \lim_{t\to 0} \frac{\mathbb{P}_t^{-1}\mathbb{P}_{t,t}^{-1}(\rho(t,t)) - v}{t^2}.$$

Finally set $r = \sqrt{t}$ and observe that the $\mathbb{P}_t^{-1}\mathbb{P}_{t,t}^{-1}(\rho(t,t))$ is exactly the parallel transport of v along the inverse of the loop η_r used in Proposition 35.3. Thus we obtain

The inverse is consistent with the minus sign in our original Definition 33.11 of R^∇.

$$R_\gamma^\nabla(S(0,0),T(0,0))(v) = R^\nabla(\xi,\zeta)(v).$$

Finally by Proposition 35.9 we have

$$R_\gamma^\nabla(S(0,0),T(0,0))(p) = R_{\mathrm{id}}^\nabla(\xi,\zeta)(v).$$

This completes the proof. ■

LECTURE 36

Exterior Covariant Differentials

In this lecture we will push our treatment of differential forms a little further and allow them to take values in an arbitrary vector space, or later, a vector bundle. This additional formalism will grant us yet another viewpoint on connections of vector bundles: as a graded derivation d^∇ on the space of bundle-valued forms. In contrast to the normal exterior differential d, one does not necessarily have $d \circ d = 0$. In fact, $d^\nabla \circ d^\nabla = R^\nabla$. Thus the curvature can be throught of as the obstruction to $(\Omega^\bullet(M, E), d^\nabla)$ forming a chain complex.

As usual, we start at the level of linear algebra. If V is a vector space, we have studied extensively the exterior wedge $\bigwedge^k V^*$, and its identification with the space $\mathrm{Alt}_k(V)$ of alternating multilinear maps

$$A \colon \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R}.$$

Now suppose W is another vector space. In Definition 19.23 we actually originally introduced the space $\mathrm{Alt}_k(V, W)$ of alternating multilinear maps

$$A \colon \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to W.$$

Moreover Lemma 19.25 and Corollary 19.3 show that

$$\begin{aligned} \mathrm{Alt}_k(V, W) &\cong \mathrm{Hom}\big(\textstyle\bigwedge^k V, W\big) \\ &\cong \big(\textstyle\bigwedge^k V\big)^* \otimes W \\ &\cong \textstyle\bigwedge^k V^* \otimes W. \end{aligned}$$

This gives:

Lemma 36.1. *Let V and W be two vector spaces. For $k \geq 0$ there is a canonical isomorphism between $\mathrm{Alt}_k(V, W)$ and $\bigwedge^k V^* \otimes W$.*

We now generalise this idea. If E is a vector bundle over M and V is a vector space, we denote by $E \otimes V$ the bundle over M whose fibre is $(E \otimes V)_p := E_p \otimes V$ (equivalently, this is the bundle obtained by tensoring E with the trivial bundle $M \times V \to M$).

Definition 36.2. Let M be a smooth manifold and let V be a vector space. A **differential k-form on M with values in V** (also called a **vector-valued form**) is a section of the bundle $\bigwedge^k T^*M \otimes V \to M$. We denote the space of sections by

$$\Omega^k(M, V) := \Gamma\Big(\textstyle\bigwedge^k T^*M \otimes V\Big).$$

This is not as scary as it looks (and reduces to the normal definition if $V = \mathbb{R}$). For instance, a V-valued one-form ω associates to every $p \in M$ a linear map $\omega_p \colon T_pM \to V$. Thus if we feed ω_p a tangent vector ξ we get an element of V, rather than an element of $\mathbb{R}$. If X is a vector field on M and $f\colon M \to V$ is a smooth function then

$$X(f)\colon M \to V, \qquad X(f)(p) := \mathcal{J}^{-1}_{f(p)}\big(Df(p)X(p)\big)$$

is another smooth function. Thus the analogue of Proposition 8.2 holds for vector-valued functions as well. In fact, almost all of our earlier work on differential forms goes through without any changes (just insert V in appropriate places). Instead of regurgitating unnecessary details, we content ourselves with merely stating a few pertinent results.

THEOREM 36.3 (The Vector-valued Differential Form Criterion). *Let M be a smooth manifold and let $U \subset M$ be a non-empty open set, and let V be a vector space. Then there is a canonical identification between $\Omega^k(U,V)$ and alternating $C^\infty(M)$-multilinear functions*

$$\underbrace{\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U)}_{k \text{ copies}} \to C^\infty(U,V).$$

Warning: Unfortunately we tend to use the same letters U, V, W etc to denote both open sets in manifolds and arbitrary vector spaces. This means expressions like $\Omega^k(U,V)$ can be somewhat confusing...

Assume now that V has dimension n, and let $(e_1,\dots,e_n)$ be a basis. If $\omega \in \Omega^k(M,V)$ and $p \in M$, then for any tangent vectors $\xi_1,\dots,\xi_k \in T_pM$, we can write $\omega_p(\xi_1,\dots,\xi_k)$ as a linear combination of the e_i. If we denote the coefficient of e_i by $\omega^i_p(\xi_1,\dots,\xi_k)$, we can thus write

$$\omega_p(\xi_1,\dots,\xi_k) = \omega^i_p(\xi_1,\dots,\xi_k)\, e_i.$$

Since ω_p is an alternating multilinear map, so is each ω^i_p. It follows that ω^i is a normal differential k-form on M, and we can write

$$\omega = \omega^i \otimes e_i.$$

This is of course, consistent with thinking of ω as a section of the tensored bundle $\bigwedge^k T^*M \otimes V$. This allows us to extend the exterior differential to a graded derivation $d\colon \Omega^k(M,V) \to \Omega^{k+1}(M,V)$ by declaring that

$$d\big(\omega^i \otimes e_i\big) := d\omega^i \otimes e_i.$$

Exercise: Why is this independent of the choice of basis of V?

The wedge product requires a little more thought to define, since this requires us to multiply vectors together. This isn't possible in an arbitrary vector space (only in algebras, cf. Definition 19.18). Thus in general we need to specify a bilinear map. This works as follows: suppose V, V_1, V_2 and W are four vector spaces, and assume we are given a bilinear map $\beta\colon V_1 \times V_2 \to W$ (equivalently, a linear map $V_1 \otimes V_2 \to W$, cf. Lemma 19.2). Then motivated by Lemma 22.18, we make the following definition:

DEFINITION 36.4. Let $\omega \in \mathrm{Alt}_h(V,V_1)$ and $\theta \in \mathrm{Alt}_k(V,V_2)$. We define $\omega \wedge_\beta \theta \in \mathrm{Alt}_{h+k}(V,W)$ by

$$(\omega\wedge_\beta\theta)(\xi_1,\dots,\xi_{h+k}) = \frac{1}{h!k!}\sum_{\varrho\in\mathfrak{S}_{h+k}} \mathrm{sgn}(\varrho)\beta\left(\omega\big(\xi_{\varrho(1)},\dots,\xi_{\varrho(h)}\big),\theta\big(\xi_{\varrho(h+1)},\dots,\xi_{\varrho(h+k)}\big)\right).$$

Equivalently we can think of $\wedge_\beta$ as defining a map

$$\left(\textstyle\bigwedge^h V^* \otimes V_1\right) \times \left(\textstyle\bigwedge^k V^* \otimes V_2\right) \to \left(\textstyle\bigwedge^{h+k} V^* \otimes W\right)$$

If for instance $W = V_1 = V_2$ is an algebra (and thus there is natural algebra multiplication $W \otimes W \to W$) we can regard the wedge product as a map

$$\left(\bigwedge^h V^* \otimes W\right) \times \left(\bigwedge^k V^* \otimes W\right) \to \left(\bigwedge^{h+k} V^* \otimes W\right),$$

and in this case we typically omit reference of the map β. Moreover if we have no convenient map β, we can always take $W = V_1 \otimes V_2$ and have β be induced from the identity map $V_1 \times V_2 \to V_1 \times V_2$. Thus there is always a wedge product

$$\left(\bigwedge^h V^* \otimes V_1\right) \times \left(\bigwedge^k V^* \otimes V_2\right) \to \left(\bigwedge^{h+k} V^* \otimes V_1 \otimes V_2\right),$$

Now let us apply this to manifolds: if $\beta\colon V_1 \times V_2 \to W$ is a bilinear map then we obtain a map

$$\Omega^h(M, V_1) \times \Omega^k(M, V_2) \xrightarrow{\wedge_\beta} \Omega^{h+k}(M, W)$$

by applying the above construction pointwise:

$$(\omega \wedge_\beta \theta)_p = \omega_p \wedge_\beta \theta_p, \qquad p \in M.$$

Let (e_i) be a basis of V_1 and (e_j) be a basis of V_2. If we write $\omega = \omega^i \otimes e_i$ and $\theta = \theta^j \otimes e'_j$ then from the definition it follows that

$$\omega \wedge_\beta \theta = \omega^i \wedge \theta^j \beta(e_i, e'_j).$$

If (f_l) is a basis of W then we can write $\beta(e_i, e'_j) = a^l_{ij} f_l$ for real numbers a^l_{ij}, and thus

$$\omega \wedge_\beta \theta = a^l_{ij}\, \omega^i \wedge \theta^j\, f_l,$$

which also proves that $\omega \wedge_\beta \theta$ is smooth (if you were worried). The following result, whose proof is on Problem Sheet O, shows that the exterior differential on vector-valued forms is still skew-commutative.

Proposition 36.5. *Let M be a smooth manifold, and let V_1, V_2 and W be vector spaces. Let $\omega \in \Omega^h(M, V_1)$ and let $\theta \in \Omega^k(M, V_2)$, and let $\beta\colon V_1 \times V_2 \to W$ be a bilinear map. Then*

$$d(\omega \wedge_\beta \theta) = d\omega \wedge_\beta \theta + (-1)^h \omega \wedge_\beta d\theta.$$

Let us give an example of how this is useful.

Example 36.6. Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ is in particular a vector space, and the Lie bracket $(v, w) \mapsto [v, w]$ is a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Suppose M is a manifold. Given $\omega \in \Omega^k(M, \mathfrak{g})$ and $\theta \in \Omega^s(M, \mathfrak{g})$, we typically use the notation

$$[\omega, \theta] := \omega \wedge_{\beta = [\cdot,\cdot]} \theta.$$

We claim that this wedge product satisfies the following version of skew-commutativity:

$$[\omega, \theta] = (-1)^{hk+1}[\theta, \omega]. \qquad (36.1)$$

To see this, let (e_i) be a basis for $\mathfrak{g}$. Write $\omega = \omega^i \otimes e_i$ and $\theta = \theta^j \otimes e_j$. Then

$$\begin{aligned}[\omega, \theta] &= \omega^i \wedge \theta^j [e_i, e_j] \\ &= (-1)^{hk+1} \theta^j \wedge \omega_i [e_j, e_i] \\ &= (-1)^{hk+1} [\theta, \omega],\end{aligned}$$

where the $(-1)^{hk}$ came from swapping $\omega^i \wedge \theta^j$ to $\theta^j \wedge \omega^i$ and the other -1 came from $[e_i, e_j] = -[e_j, e_i]$. In particular, this shows that if $h = sk = 1$ then $[\omega, \theta]$ is *symmetric* in ω and θ. (This should surprise you, since the normal Lie bracket is anti-symmetric). In particular, it is **not** (!) necessarily true that $[\omega, \omega] = 0$ for $\omega \in \Omega^1(M, \mathfrak{g})$. This will be important in Lecture 41 (see Theorem 41.6 in particular).

The analogue of Theorem 23.13 holds for d.

THEOREM 36.7. *Let M be a smooth manifold, $\omega \in \Omega^k(M, W)$ and $X_0, \dots X_k \in \mathfrak{X}(M)$. Then:*

$$\begin{aligned}d\omega(X_0, \dots, X_k) = &\sum_{i=0}^{k} (-1)^i X_i\big(\omega(X_0, \dots, \widehat{X}_i, \dots, X_k)\big) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k).\end{aligned}$$

Proof. The proof is identical to the proof of Theorem 23.13 – the only difference is that both sides are functions $M \to V$ rather than functions $M \to \mathbb{R}$. ∎

Similarly the proof of Lemma 23.4 goes through without any changes to give:

LEMMA 36.8. *Let $\varphi \colon M \to N$ be a smooth map and let $\omega \in \Omega(N, V)$. Then*

$$\varphi^*(d\omega) = d(\varphi^*\omega).$$

Let us now take this one step further and look at differential forms with values in a vector bundle, rather than just a vector space.

DEFINITION 36.9. Let M be a smooth manifold and $\pi \colon E \to M$ a vector bundle over M. A **differential k form with values in** E (or a **bundle-valued form**) is a section of the bundle $\bigwedge^k T^*M \otimes E$. As usual, we denote by $\Omega^k(M, E)$ the space of such sections.

Thus an element $\omega \in \Omega^k(M, E)$ defines for each $p \in M$ an alternating multilinear map

$$\omega_p \colon \underbrace{T_pM \times \cdots \times T_pM}_{k \text{ copies}} \to E_p.$$

Again, this may seem confusing, but in reality is no more complicated than the case of a vector-valued form; the only difference is that the target vector space E_p now also depends on p. If (e_i) is a local frame for E over an open set U then any element $\omega \in \Omega^k(U, E)$ can be written as a sum

Warning: Do not confuse $\Omega^k(U, E)$ and $\Gamma(U, \bigwedge^k E)$!

$$\omega = \omega^i \otimes e_i$$

where ω^i is a normal differential k-form on U.

Theorem 36.10 (The Bundle-valued Differential Form Criterion). *Let* $\pi\colon E \to M$ *be a vector bundle and suppose* $U \subset M$ *is a non-empty open set. There is a natural* $C^\infty(U)$*-module isomorphism between* $\Omega^k(U, E)$ *and alternating* $C^\infty(U)$*-multilinear functions*

$$\underbrace{\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U)}_{k \text{ copies}} \to \Gamma(U, E).$$

This follows in the same way as Theorem 22.15 and Theorem 36.3, but since this is arguably the hardest of these sort of results, let us recap the details.

Proof. Take $U = M$ for simplicity. We use the Hom-Gamma Theorem 20.25:

$$\begin{aligned}\Omega^k(M, E) &= \Gamma\Big(M, \textstyle\bigwedge^k T^*M \otimes E\Big) \\ &= \Gamma\Big(\mathrm{Hom}\big(\textstyle\bigwedge^k TM, E\big)\Big) \\ &= \mathrm{Hom}\Big(\Gamma\big(\textstyle\bigwedge^k TM\big), \Gamma(E)\Big).\end{aligned}$$

Now the argument used in the proof of Theorem 21.5 (which was proved as Problem I.4) shows that this latter space can be identified with alternating $C^\infty(M)$-multilinear functions

$$\underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k \text{ copies}} \to \Gamma(E).$$

Alternatively, one could use the alternating version of Proposition 21.14 and Proposition 21.15 to show that

$$\mathrm{Hom}\Big(\Gamma\big(\textstyle\bigwedge^k TM\big), \Gamma(E)\Big) \cong \mathrm{Alt}_k\big(\mathfrak{X}(M), \Gamma(E)\big)$$

(this is more efficient, but harder, since $\mathfrak{X}(M)$ and $\Gamma(E)$ are infinite-dimensional vector spaces). ∎

Thus we can think of an element of $\Omega^k(M, E)$ as a alternating map that eats vector fields and produces a section of E:

$$\omega(X_1, \dots, X_k) \in \Gamma(E).$$

Here is are two examples:

Examples 36.11.

(i) Let $\varphi\colon M \to N$ be a smooth map. Then $D\varphi$ can be thought of as an element of $\Omega^1(M, \varphi^*(TN))$.

(ii) Let ∇ be a connection on E. Then by Theorem 33.14 the curvature R^∇ is an element of $\Omega^2(M, \mathrm{End}(E))$.

We now develop a version of the wedge product for bundle-valued forms. Rather than work in maximal generality, we will give the relevant definitions only for the case we are interested in. Let us say that a **decomposable** element of $\Omega^k(U, E)$ is an element of the form $\alpha = \omega \otimes s$ where $\omega \in \Omega^k(U)$ and $s \in \Gamma(U, E)$.

Definition 36.12. The space $\Omega(M,E)$ is a sheaf of $\Omega(M)$**-bimodules** in the sense that there are wedge products

$$\wedge\colon \Omega(M)\times\Omega(M,E)\to\Omega(M,E)$$

and

$$\wedge\colon \Omega(M,E)\times\Omega(M)\to\Omega(M,E)$$

which restrict to define maps

$$\Omega^h(M)\times\Omega^k(M,E)\to\Omega^{h+k}(M,E),\qquad \Omega^k(M,E)\times\Omega^k(M)\to\Omega^{h+k}(M,E)$$

and are compatible in the sense that

$$(\omega\wedge\alpha)\wedge\theta=\omega\wedge(\alpha\wedge\theta),\qquad \alpha\in\Omega(M,E),\ \omega,\theta\in\Omega(M).$$

Explicitly, these wedge product are defined on decomposable elements $\alpha=\omega\otimes s$ as follows:

$$(\omega\otimes s)\wedge\theta\coloneqq(\omega\wedge\theta)\otimes s.$$

where wedge product on the right-hand side is normal wedge product, and similarly

$$\theta\wedge(\omega\otimes s)\coloneqq(\theta\wedge\omega)\otimes s,$$

and then extended by linearity. Just as the wedge product reduces to multiplication for 0-forms, we define

$$\omega\wedge s=s\wedge\omega\coloneqq\omega\otimes s,\qquad \omega\in\Omega(M),\ s\in\Omega^0(M,E).\tag{36.2}$$

It follows from the definition that the wedge product is again graded commutative in the sense that

$$\omega\wedge\alpha=(-1)^{hk}\alpha\wedge\omega,\qquad \omega\in\Omega^h_M,\ \alpha\in\Omega^k(M,E).\tag{36.3}$$

With all the necessary formalism developed, we can proceed to heart of today's lecture. The starting point is the following observation:

Lemma 36.13. *A connection ∇ on E is equivalent to an $\mathbb{R}$-linear local operator $\nabla\colon\Omega^0(M,E)\to\Omega^1(M,E)$ which satisfies the Leibniz rule*

$$\nabla(fs)=df\otimes s+f\,\nabla s.$$

Proof. The axioms of a covariant derivative (Definition 31.6) tell us that we get an $\mathbb{R}$-linear map

$$\nabla\colon\Omega^0(M,E)\to\Omega^1(M,E)$$

satisfying the Leibniz rule. The proof that ∇ is a local operator is a standard argument using a bump function: if $s|_U\equiv 0$ and $p\in U$, choose an open set $V\subset U$ containing p and a bump function χ such that $\chi|_V\equiv 1$ and $\operatorname{supp}(\chi)\subset U$. Then $\chi s\equiv 0$ and $d\chi_p=0$ as χ is constant on a neigbourhood of p, leading to:

$$\begin{aligned}0&=\nabla(\chi s)(p)\\&=d\chi_p\otimes s(p)+\chi(p)(\nabla s)(p)\\&=(\nabla s)(p).\end{aligned}$$

∎

So far all we have done is added notational complexity. Recall that the exterior differential $d\colon \Omega(M) \to \Omega(M)$ is a graded derivation of degree 1 (Definition 22.24), which extends the operation $f \mapsto df$ to higher differential forms. We now play the same game with connections.

We can now formulate our main result.

THEOREM 36.14. *Let $\pi\colon E \to M$ be a vector bundle with connection ∇. There exists a unique local operator*

$$d^{\nabla}\colon \Omega(M,E) \to \Omega(M,E)$$

of degree 1, i.e. that d^{∇} restricts to define local operators

$$d^{\nabla}\colon \Omega^k(M,E) \to \Omega^{r+1}(M,E)$$

such that:

(i) *d^{∇} is a graded derivation with respect to the wedge products from Definition 36.12, i.e. for $\omega \in \Omega^k(M)$ and $\alpha \in \Omega^k(M,E)$ we have*

$$\begin{aligned} d^{\nabla}(\omega \wedge \alpha) &= d\omega \wedge \alpha + (-1)^k \omega \wedge d^{\nabla}\alpha. \\ d^{\nabla}(\alpha \wedge \omega) &= d^{\nabla}\alpha \wedge \omega + (-1)^k \alpha \wedge d\omega. \end{aligned} \tag{36.4}$$

(ii) *d^{∇} is equal to ∇ on $\Omega^0(M,E)$: $d^{\nabla}s = \nabla s$ for $s \in \Omega^0(M,E)$.*

We call d^{∇} the **exterior covariant differential** *associated to the connection ∇ and refer to $d^{\nabla}\alpha$ as the exterior covariant differential of α.*

The proof of Theorem 36.14 is very similar to the proof of Theorem 23.1. Indeed, if one takes E to be the trivial bundle $M \times \mathbb{R} \to \mathbb{R}$ and ∇ to be the trivial connection then $d^{\nabla} = d$ and the proof is of Theorem 36.14 reduces exactly to that of Theorem 23.1. The general case is only notationally different, and we leave it to the interested reader as an exercise.

We then have the following analogue of Theorem 23.13, which uses the Bundle-Valued Differential Form Criterion (Theorem 36.10) to make sense of its statement.

THEOREM 36.15. *Let $\pi\colon E \to M$ be a vector bundle with connection ∇. Let $\alpha \in \Omega^k(M,E)$ and let $X_0, \dots X_k \in \mathfrak{X}(M)$. Then:*

$$\begin{aligned} d^{\nabla}\alpha(X_0,\dots,X_k) = &\sum_{i=0}^{k} (-1)^i \nabla_{X_i}\big(\alpha(X_0,\dots,\widehat{X}_i,\dots,X_k)\big) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \alpha([X_i,X_j],X_0,\dots\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_k). \end{aligned}$$

The proof is by induction on k, and proceeds in exactly the same was as Theorem 23.13. Similarly we have the following version of Lemma 23.4:

LEMMA 36.16. *Let $\pi\colon E \to N$ be a vector bundle with connection ∇. Let $\varphi\colon M \to N$ be a smooth map and let $\alpha \in \Omega(N,E)$. Then*

$$\varphi^*(d^{\nabla}\alpha) = d^{\nabla}(\varphi^*\alpha).$$

that is, φ^* commutes with the exterior covariant differentials.

On Problem Sheet N you will show:

Proposition 36.17. *Let* $\pi\colon E \to M$ *be an algebra bundle and let be a connection on* ∇ *such that the algebra multiplication* $\beta\colon E \times E \to E$ *is parallel in the sense that if* γ *is a curve in* M *and* ρ_1, ρ_2 *are two parallel sections then* $\beta(\rho_1, \rho_2)$ *is also parallel along* γ*. Then* d^∇ *satisfies the product rule*

$$d^\nabla\big(\beta(\alpha_1, \alpha_2)\big) = \beta\big(d^\nabla \alpha_1, \alpha_2\big) + (-1)^k \beta\big(\alpha_1, d^\nabla \alpha_2\big),$$

for $\alpha_1 \in \Omega^k(M, E)$ *and* $\alpha_2 \in \Omega(M, E)$*.*

Proposition 34.4 shows that the bundles $\mathrm{End}(E)$ and $\mathfrak{gl}(E)$ satisfy the hypotheses of this proposition.

Unlike the exterior differential however, the exterior covariant differential does *not* necessarily square to zero. For this we need a bit more formalism.

Recall a decomposable element of $\Omega^k(M, \mathrm{End}(E))$ is of the form $\omega \otimes \Phi$ where $\omega \in \Omega^k(M)$ and $\Phi \in \Gamma(\mathrm{End}(E))$. We can therefore extend Definition 36.12 and define (even) more wedge products.

Definition 36.18. The vector space $\Omega(M, \mathrm{End}(E))$ are $C^\infty(M)$-algebras under the multiplication

$$\Omega(M, \mathrm{End}(E)) \times \Omega(M, \mathrm{End}(E)) \to \Omega(M, \mathrm{End}(E)).$$

given by

$$(\omega \otimes \Phi) \wedge (\theta \otimes \Psi) := (\omega \wedge \theta) \otimes (\Phi \circ \Psi),$$

Moreover $\Omega(M, E)$ is also a left $\Omega(M, \mathrm{End}(E))$-module via the wedge product

$$\wedge\colon \Omega(M, \mathrm{End}(E)) \times \Omega(M, E) \to \Omega(M, E)$$

defined on decomposable elements by

$$(\omega \otimes \Phi) \wedge (\theta \otimes s) := (\omega \wedge \theta) \otimes \Phi(s).$$

This wedge products restrict to a map

$$\Omega^h(M, \mathrm{End}(E)) \times \Omega^k(M, E) \to \Omega^{h+k}(M, E).$$

This wedge product makes $\Omega(M, E)$ into a $\Omega(M, \mathrm{End}(E))$-$\Omega(M)$ bimodule, in the sense that for $\Theta \in \Omega(M, \mathrm{End}(E))$, $\alpha \in \Omega(M, E)$ and $\omega \in \Omega(M)$ one has

If the algebraic terminology is unfamiliar, just ignore it. Only (36.5) is important.

$$(A \wedge \alpha) \wedge \omega = A \wedge (\alpha \wedge \omega). \tag{36.5}$$

We are now ready to prove the main result of today's lecture: the curvature R^∇ is the obstruction to $(\Omega(M, E), d^\nabla)$ being a chain complex.

Theorem 36.19. *Let* $\pi\colon E \to M$ *be a vector bundle with connection* ∇*. For all* $\alpha \in \Omega^k(M, E)$ *one has*

$$d^\nabla \circ d^\nabla(\alpha) = R^\nabla \wedge \alpha.$$

Thus $d^\nabla \circ d^\nabla = 0$ *if and only if* ∇ *is flat.*

Proof. We first prove the result in the special case $k = 0$, so that $\alpha = s$ is just a section of E. Let $X, Y \in \mathfrak{X}(M)$. Then using Theorem 36.15 and Theorem 35.10 we compute:

$$\begin{aligned} d^\nabla \circ d^\nabla(s)(X,Y) &= \nabla_X(\nabla s(Y)) - \nabla_Y(\nabla s(X)) - \nabla s([X,Y]) \\ &= \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \\ &= R^\nabla(X,Y)(s), \end{aligned}$$

and hence

$$d^\nabla \circ d^\nabla s = R^\nabla \wedge s. \tag{36.6}$$

For the general case it suffices to take $\alpha = \omega \otimes s$ to be a decomposable element. Then we compute

$$\begin{aligned} d^\nabla \circ d^\nabla \alpha &\overset{(36.2)}{=} d^\nabla \circ d^\nabla(\omega \wedge s) \\ &\overset{(36.4)}{=} d^\nabla \left(d\omega \wedge s + (-1)^k \omega \wedge d^\nabla s \right) \\ &= d(d\omega) \wedge s + (-1)^{k+1} d\omega \wedge d^\nabla(s) + (-1)^k d\omega \wedge d^\nabla s + (-1)^{2k} \omega \wedge \left(d^\nabla \circ d^\nabla s \right) \\ &\overset{(36.6)}{=} \omega \wedge (R^\nabla \wedge s) \\ &\overset{(36.3)}{=} (R^\nabla \wedge s) \wedge \omega \\ &\overset{(36.5)}{=} R^\nabla \wedge (s \wedge \omega) \\ &\overset{(36.3)}{=} R^\nabla \wedge (\omega \wedge s) \\ &\overset{(36.2)}{=} R^\nabla \wedge \alpha. \end{aligned}$$

This completes the proof. ■

We conclude this lecture by stating and proving the *Bianchi identity*. As we will see next lecture, this identity is the starting point for using connections to study de Rham cohomology of a manifold via *characteristic classes*.

We denote by

$$d^{\nabla^{\mathrm{End}}} : \Omega(M, \mathrm{End}(E)) \to \Omega(M, \mathrm{End}(E))$$

the exterior covariant differential associated to the connection ∇^{End} on $\mathrm{End}(E)$. On Problem Sheet N you will prove:

Proposition 36.20. *For $\Theta \in \Omega^k(M, \mathrm{End}(E))$ and $\alpha \in \Omega(M, E)$ one has*

$$d^\nabla(\Theta \wedge \alpha) = d^{\nabla^{\mathrm{End}}} \Theta \wedge \alpha + (-1)^k \Theta \wedge d^\nabla \alpha.$$

By part (ii) of Examples 36.11 the curvature R^∇ of ∇ is an element of $\Omega^2(M, \mathrm{End}(E))$, and hence $d^{\nabla^{\mathrm{End}}}(R^\nabla) \in \Omega^3(M, \mathrm{End}(E))$. In fact, this element is always zero.

Theorem 36.21 (The Bianchi Identity). *Let $\pi \colon E \to M$ be a vector bundle with connection ∇. Then*

$$d^{\nabla^{\mathrm{End}}}(R^\nabla) = 0.$$

Proof. Let $\alpha \in \Omega(M, E)$. We compute $(d^{\nabla})^3(\alpha) \coloneqq d^{\nabla} \circ d^{\nabla} \circ d^{\nabla}\alpha$ in two ways. Firstly, by Theorem 36.19 we have

$$(d^{\nabla})^3(\alpha) = (d^{\nabla})^2\big(d^{\nabla}\alpha\big) = R^{\nabla} \wedge d^{\nabla}\alpha. \tag{36.7}$$

Alternatively, using Proposition 36.20 in addition to Theorem 36.19 we have

$$\begin{aligned}
(d^{\nabla})^3(\alpha) &= d^{\nabla}\big((d^{\nabla})^2(\alpha)\big) \\
&= d^{\nabla}(R^{\nabla} \wedge \alpha) \\
&= d^{\nabla^{\mathrm{End}}}(R^{\nabla}) \wedge \alpha + (-1)^2 R^{\nabla} \wedge d^{\nabla}\alpha \\
&= d^{\nabla^{\mathrm{End}}}(R^{\nabla}) \wedge \alpha + R^{\nabla} \wedge d^{\nabla}\alpha.
\end{aligned}$$

Comparing this with (36.7) tells us that

$$R^{\nabla} \wedge d^{\nabla}\alpha = d^{\nabla^{\mathrm{End}}}(R^{\nabla}) \wedge \alpha + R^{\nabla} \wedge d^{\nabla}\alpha,$$

and hence

$$d^{\nabla^{\mathrm{End}}}(R^{\nabla}) \wedge \alpha = 0, \qquad \forall\, \alpha \in \Omega(M, E).$$

This implies that $d^{\nabla^{\mathrm{End}}}(R^{\nabla}) = 0$, and thus completes the proof. ■

LECTURE 37

Riemannian Vector Bundles

Next lecture we will introduce *characteristic classes* of a vector bundle. In this lecture we motivate their construction by considering a simple – and ultimately, useless – example. Along the way we introduce *Riemannian metrics* on vector bundles, whose study we will return to in Lecture 43.

Suppose $\pi\colon E \to M$ is a vector bundle and η is a section of the dual bundle E^*. If $\alpha \in \Omega^k(M, E)$ is an E-valued differential k-form on M, then we can feed α to η to obtain a normal differential k-form $\eta(\alpha) \in \Omega^k(M)$. Explicitly, if $\alpha = \omega \otimes s$ is decomposable then $\eta(\alpha) := \eta(s)\,\omega$.

LEMMA 37.1. *Suppose $\pi\colon E \to M$ is a vector bundle with connection ∇. Suppose $\eta \in \Gamma(E^*)$ is a section of the dual bundle which is parallel with respect to the induced connection ∇^*. Then for any $\alpha \in \Omega^k(M, E)$, we have*

$$d(\eta(\alpha)) = \eta\big(d^\nabla \alpha\big)$$

as elements of $\Omega^{k+1}(M)$.

Proof. If $X_0, \dots, X_k$ are vector fields on M then by Theorem 23.13

$$\begin{aligned} d(\eta(\alpha))(X_0, \dots, X_k) = &\sum_{i=0}^{k} (-1)^i X_i\big(\eta(\alpha)(X_0, \dots, \widehat{X}_i, \dots, X_k)\big) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \eta(\alpha)([X_i, X_j], X_0, \dots \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k). \end{aligned}$$

Using the definition of the induced connection ∇^*, we have

i.e. part (ii) of Problem M.3.

$$\begin{aligned} X_i\big(\eta(\alpha)(X_0, \dots, \widehat{X}_i, \dots, X_k)\big) = &(\nabla^*_{X_i}\eta)\big(\alpha(X_0, \dots, \widehat{X}_i, \dots, X_k)\big) \\ &+ \eta\big(\nabla_{X_i}\alpha(X_0, \dots, \widehat{X}_i, \dots, X_k)\big), \end{aligned}$$

and thus by Theorem 36.15 we have

$$\begin{aligned} d(\eta(\alpha))(X_0, \dots, X_k) = &\sum_{i=0}^{k} (-1)^i (\nabla^*_{X_i}\eta)\big(\alpha(X_0, \dots, \widehat{X}_i, \dots, X_k)\big) \\ &+ \eta\big(d^\nabla \alpha(X_0, \dots, X_k)\big). \end{aligned}$$

If η is parallel then $\nabla^*\eta = 0$, and thus the result follows. ■

Now consider the trace operator

$$\mathtt{tr}\colon \mathrm{Mat}(n) \to \mathbb{R}$$

that sends a matrix to its trace. We will show that $\mathtt{tr}$ induces a parallel section of the dual bundle to the homomorphism bundle. Recall the frame bundle $\mathrm{Fr}(E)$ associated to E from Definition 17.24. An element of the fibre $\mathrm{Fr}(E_p)$ is a linear isomorphism $\ell\colon \mathbb{R}^n \to E_p$.

Proposition 37.2. *Let $\pi\colon E \to M$ be a vector bundle of rank n with connection ∇. There is a well-defined section η of the bundle $(\mathrm{End}(E))^*$ given by the trace:*

$$\eta_p(A) := \mathsf{tr}(\ell_0^{-1} \circ A \circ \ell_0), \qquad A \in \mathrm{End}(E_p), \tag{37.1}$$

where $\ell_0 \in \mathrm{Fr}(E_p)$ is any fixed element. Moreover this section η is parallel with respect to the dual connection on $\mathrm{End}(E)^*$ induced by the connection ∇^{End} on $\mathrm{End}(E)$.

Proof. To prove that η is well defined we observe that if $\ell_1\colon \mathbb{R}^n \to E_p$ was another element of $\mathrm{Fr}(E_p)$ then $\ell := \ell_0^{-1}\ell_1 \in \mathrm{GL}(n)$, and

$$\begin{aligned} \mathsf{tr}(\ell_1^{-1} \circ A \circ \ell_1) &= \mathsf{tr}(\ell^{-1}\ell_0^{-1} \circ A \circ \ell_0\ell) \\ &= \mathsf{tr}(\ell_0^{-1} \circ A \circ \ell_0), \end{aligned}$$

since the trace of a matrix is invariant under conjugation by an invertible matrix.

To prove that η is parallel with respect to the dual connection on $(\mathrm{End}(E))^*$ induced by ∇^{End}, by part (i) of Problem M.3 we need to show that η is constant along parallel sections of $\mathrm{End}(E)$ with respect to ∇^{End}. Fix $p \in M$ and let $\gamma\colon [0,1] \to M$ be a curve with $\gamma(0) = p$ and $\dot\gamma(t) \neq 0$. Let (e_i) be a local frame of E over an open set U containing p which is parallel along γ (Corollary 32.5), and let $\varepsilon\colon \pi^{-1}(U) \to \mathbb{R}^n$ denote the associated vector bundle chart. Suppose $C \in \Gamma_\gamma(\mathrm{End}(E))$ is parallel with respect to ∇^{End}. Then just as in (34.6), the curve $\varepsilon_{\gamma(t)}(C(t))$ is constant in t. Unravelling the definitions then gives

$$\eta_{\gamma(t)}(C(t)) = \mathsf{tr}\big(\varepsilon_p \circ C(0) \circ \varepsilon_p^{-1}\big),$$

which thus is constant as required. ∎

From now on by an abuse of notation we will denote the section η defined in (37.1) also by tr. What have we gained from this construction?

Corollary 37.3. *Let $\pi\colon E \to M$ be a vector bundle with connection ∇. Then the differential 2-form $\mathsf{tr}(R^\nabla)$ is closed, and hence defines a de Rham cohomology class $[\mathsf{tr}(R^\nabla)] \in H^2_{\mathrm{dR}}(M)$.*

Proof. We apply Lemma 37.1 applied with "E" equal to $\mathrm{End}(E)$ and "∇" equal to ∇^{End}. Then using also the Bianchi Identity (Theorem 36.21) we have

$$d(\mathsf{tr}(R^\nabla)) = \mathsf{tr}\big(d^{\nabla^{\mathrm{End}}}(R^\nabla)\big) = 0.$$

Thus $\mathsf{tr}(R^\nabla)$ is closed, as required. ∎

What is more surprising is that the cohomology class $[\mathsf{tr}(R^\nabla)]$ is actually independent of the choice of connection ∇.

Proposition 37.4. *Let $\pi\colon E \to M$ denote a vector bundle and let ∇_0 and ∇_1 denote two connections on E. Then as elements of $H^2_{\mathrm{dR}}(M)$, we have*

$$[\mathsf{tr}(R^{\nabla_0})] = [\mathsf{tr}(R^{\nabla_1})].$$

Proof. Let $\mathrm{pr}_1 \colon M \times [0,1] \to M$ denote the first projection, and consider the pullback bundle $\mathrm{pr}_1^\star E$ over $M \times [0,1]$. Let $\overline{\nabla}_i$ denote the pullback connection $\mathrm{pr}_1^\star \nabla_i$. If $\mathrm{pr}_2 \colon M \times [0,1] \to [0,1]$ is the second projection, then

An alternative approach to this proof can be found in Problem N.1.

$$\nabla := (1 - \mathrm{pr}_2)\overline{\nabla}_0 + \mathrm{pr}_2 \overline{\nabla}_1$$

is a connection on $\mathrm{pr}_1^\star E$. If $\iota_t \colon M \to M \times [0,1]$ is the map $\iota_t(p) := (p,t)$ then

$$\iota_t^\star \nabla = (1-t)\nabla_0 + t\nabla_1,$$

and thus in particular

$$\iota_0^\star \nabla = \nabla_0, \qquad \iota_1^\star \nabla = \nabla_1.$$

If R^∇ denotes the curvature of ∇ and R^{∇_i} denotes the curvature of ∇_i then using Proposition 35.9 and Theorem 35.10 we obtain

$$\mathsf{tr}(R^{\nabla_0}) = \iota_0^\star\big(\mathsf{tr}(R^\nabla)\big), \qquad \mathsf{tr}(R^{\nabla_1}) = \iota_1^\star\big(\mathsf{tr}(R^\nabla)\big).$$

By Proposition 27.4 we obtain

$$[\mathsf{tr}(R^{\nabla_0})] = \iota_0^\star[\mathsf{tr}(R^\nabla)] = \iota_1^\star[\mathsf{tr}(R^\nabla)] = [\mathsf{tr}(R^{\nabla_1})].$$

This completes the proof. ■

We have thus shown that the trace of the curvature of a connection gives rise to a de Rham cohomology class in the base manifold that depends only on the vector bundle. Amusingly however, this cohomology class is not particularly interesting.

PROPOSITION 37.5. *Let $\pi \colon E \to M$ denote a vector bundle. Then* $[\mathsf{tr}(R^\nabla)] = 0$.

One should view Proposition 37.5 as a very special case of the construction of characteristic classes of vector bundles. The key idea behind their construction is that we can play the same game with any "invariant polynomial", rather than just the trace. More adventurous choices of polynomial will lead to cohomology classes that are not necessarily zero, and thus give rise to algebraic invariants of the vector bundle. We will explore this further next lecture.

The proof of Proposition 37.5 is not particularly hard, but it requires us to introduce another concept, that of a *Riemannian metric*. After connections, this is the second most important idea of the entire course.

DEFINITION 37.6. Let $\pi \colon E \to M$ be a vector bundle. A **Riemannian metric** on E (often shorted to just "a metric on E") is a section $g \in \Gamma(E^* \otimes E^*)$ with the property that for all $p \in M$, the element $g_p \in E_p^* \otimes E_p^* \cong (E \otimes E)^*|_p$ is an inner product on the vector space E_p. We call the pair (E, g) a **Riemannian vector bundle**.

For historical reasons, "g" is the standard symbol for a metric. This is slightly unfortunate, as the symbol g is already somewhat overloaded (eg. it is also the default letter for an element in a Lie group.) Nevertheless, we stick to the established convention.

In the special case $E = TM$, we say that g is a **Riemannian metric on** M and refer to the pair (M, g) as a **Riemannian manifold**. The field of *Riemannian geometry* is the study of Riemannian metrics on manifold.

> **Warning:** Do not confuse a Riemannian metric with a normal metric in the sense of point-set topology. They are *not* the same thing! We will eventually prove that if (M, g) is a Riemannian manifold then the Riemannian metric g induces an actual metric d_g on M, which moreover induces the given topology on M.

We will often use the notation

$$\langle u, v\rangle := g_p(u, v)$$

to emphasise that g_p is an inner product. Often we will omit the subscript p and just write $\langle u, v\rangle$, and sometimes we will refer to the entire metric by $\langle\cdot,\cdot\rangle$. Similarly we abbreviate by

$$\|v\|_p := \sqrt{g_p(v, v)}$$

the associated norm on E_p, again sometimes omitting the subscript p.

DEFINITION 37.7. Let $\pi_1 \colon E \to M$ and $\pi_2 \colon F \to N$ be two vector bundles equipped with Riemmanian metrics g_i for $i = 1, 2$. Suppose $\varphi \colon M \to N$ is a smooth map and $\Phi \colon E \to F$ is a vector bundle morphism along φ:

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M & \xrightarrow[\varphi]{} & N \end{array}$$

We say that Φ is an **isometric vector bundle morphism** if

$$g_{1|p}(u, v) = g_{2|\varphi(p)}(\Phi(u), \Phi(v)), \qquad \forall\, p \in M,\ u, v \in E_p.$$

As with connections, every vector bundle admits a Riemannian metric.

PROPOSITION 37.8. *Every vector bundle $\pi \colon E \to M$ admits a Riemannian metric.*

Proof. This is a standard partition of unity argument. Suppose E has rank n. Let $\{U_a \mid a \in A\}$ be an open cover of M such that there exist a vector bundle chart $\varepsilon_a \colon \pi^{-1}(U_a) \to \mathbb{R}^n$ for each $a \in A$. Let $\langle\cdot,\cdot\rangle$ denote the standard Euclidean inner product on $\mathbb{R}^n$, and define for $p \in U_a$

$$g_{a|p}(u, v) := \langle \varepsilon_{a|p}(u), \varepsilon_{a|p}(v)\rangle$$

Then g_a is a Riemannian metric on the trivial bundle $\pi^{-1}(U_a) \to U_a$. To globalise this, let $\{\kappa_a \mid a \in A\}$ denote a partition of unity subordinate to $\{U_a \mid a \in A\}$ and extend the local section $\kappa_a g_a$ of $E^* \otimes E^*$ to be defined on all of M by setting it to be zero outside of U_a. Then define

$$g := \sum_{a \in A} \kappa_a g_a \in \Gamma(E^* \otimes E^*).$$

This is a Riemannian metric on E as the sum is finite at every point. ∎

We also have:

LEMMA 37.9. *Let $\pi\colon E \to M$ be a vector bundle and suppose g is a Riemannian metric on E. Then around any point $p \in M$ there exists a local frame (e_i) for E which is orthonormal with respect to g.*

Proof. Apply the Gram-Schmidt process to an arbitrary local frame. ∎

A corollary of this is that we can always reduce the structure group of a vector bundle to the orthogonal group.

cf. Remark 16.10.

COROLLARY 37.10. *If $\pi\colon E \to M$ is a vector bundle of rank n then the structure group G of E may be reduced to $\mathrm{O}(n) \subset \mathrm{GL}(n)$.*

This was originally Problem G.5.

Proof. Lemma 37.9 furnishes the necessary vector bundle charts. ∎

Now we relate connections to metrics.

DEFINITION 37.11. Let (E, g) be a Riemannian vector bundle. A connection ∇ on E is said to be **compatible** with g if g is a parallel section with respect to the induced connection on $E^* \otimes E^*$.

If g is understood we simply say that ∇ is a **metric connection**.

PROPOSITION 37.12. *Let $(E, g = \langle \cdot, \cdot \rangle)$ be a Riemannian vector bundle over M. A connection ∇ on E is a metric connection if and only if*

The equation (37.2) is known as the **Ricci Identity**.

$$X\langle r, s\rangle = \langle \nabla_X r, s\rangle + \langle r, \nabla_X s\rangle, \qquad \forall\, X \in \mathfrak{X}(M),\ r, s \in \Gamma(E). \quad (37.2)$$

Proof. Denote the induced connection on $E^* \otimes E^*$ (also) by ∇. By Problem M.4 we have for $X \in \mathfrak{X}(M)$ and $r, s \in \Gamma(E)$ that

$$(\nabla_X g)(r, s) = X\langle r, s\rangle - \langle \nabla_X r, s\rangle - \langle r, \nabla_X s\rangle.$$

Thus $\nabla_X g = 0$ if and only if (37.2) holds. ∎

The Ricci Identity also holds for the pullback of a metric connection.

COROLLARY 37.13. *Let $\pi\colon E \to N$ be a vector bundle with Riemannian metric $g = \langle \cdot, \cdot \rangle$ and let ∇ denote a connection on E which is compatible with respect to g. Suppose $\varphi\colon M \to N$ is a smooth map. Then the pullback connection satisfies the Ricci identity too: for every $X \in \mathfrak{X}(M)$ and $r, s \in \Gamma_\varphi(E)$ one has*

$$X\langle r, s\rangle = \langle \nabla^\varphi_X r, s\rangle + \langle r, \nabla^\varphi_X s\rangle,$$

where both sides are smooth functions on M.

Proof. Apply the chain rule (31.9) for covariant derivative operators. ∎

On Problem Sheet N you will prove that if ∇ is a Riemannian connection then $\mathrm{Hol}^\nabla(p) \subset \mathrm{O}(E_p, g_p) \subset \mathrm{GL}(E_p)$, for every $x \in M$, where $\mathrm{O}(E_p, g_p)$ denotes the orthogonal transformations with respect to the inner product g_p. We denote by $\mathfrak{o}(E_p, g_p) \subseteq \mathfrak{gl}(E_p)$ its Lie algebra and by

$$\mathfrak{o}(g) = \bigsqcup_{p \in M} \mathfrak{o}(E_p, g_p).$$

This is an algebra subbundle of $\mathfrak{gl}(E)$, which we call the **orthogonal algebra bundle**.

Exercise: Check this!

Proposition 37.14. *Let $(E, g = \langle \cdot, \cdot \rangle)$ be a Riemannian vector bundle over M. Then metric connections exist.*

Proof. The argument is again via a partition of unity. Suppose E has rank n. Let $\{U_a \mid a \in A\}$ be an open cover of M such that there exist a orthonormal frame (e_i^a) for E over U_a. Define a covariant derivative operator on the trivial bundle $\pi^{-1}(U_a) \to U_a$ by

$$\nabla_X^a s := \sum_{i=1}^k X \langle e_i^a, s \rangle e_i^a.$$

Now let $\{\kappa_a \mid a \in A\}$ denote a partition of unity subordinate to $\{U_a \mid a \in A\}$ and extend the local section $\kappa_a \nabla^a$ to be defined on all of M by setting it to be zero outside of U_a. Then define

$$\nabla := \sum_{a \in A} \kappa_a \nabla^a.$$

This is a covariant derivative operator on M. Moreover we claim that ∇ is Riemannian: indeed if $X \in \mathfrak{X}(M)$ and $r, s \in \Gamma(E)$ then

If the associated distribution to ∇^a is Δ^a then the distribution Δ constructed in Step 2 of Theorem 28.6 has its associated covariant derivative equal to ∇.

$$\begin{aligned} X\langle r, s \rangle &= \sum_{a \in A} \sum_{i=1}^n \kappa_a X \big(\langle r, e_i^a \rangle \langle e_i^a, s \rangle \big) \\ &= \sum_{a \in A} \kappa_a \Big(\langle \nabla_X^a r, s \rangle + \langle r, \nabla_X^a s \rangle \Big) \\ &= \langle \nabla_X r, s \rangle + \langle r, \nabla_X s \rangle, \end{aligned}$$

where as usual the interchange of summation signs is justified as the sum is locally finite. ■

We now prove that the curvature tensor of a metric connection is skew-symmetric.

Proposition 37.15. *Let (E, g) be a Riemannian vector bundle over M, and let ∇ be a metric connection. Then for all $X, Y \in \mathfrak{X}(M)$ and $r, s \in \Gamma(E)$, one has*

$$\langle R^\nabla(X,Y)(r), s \rangle + \langle r, R^\nabla(X,Y)(s) \rangle = 0.$$

Proof. It is sufficient to prove the result in the case $[X, Y] = 0$ since R^∇ is a point operator. Let $s \in \Gamma(E)$. Then by Theorem 35.10 and the Ricci Identity (37.2), we have

In fact, by Problem D.4 it suffices to prove the result in the (even more) special case where $X = \frac{\partial}{\partial x^i}$ and $Y = \frac{\partial}{\partial x^j}$.

$$\begin{aligned} \langle R^\nabla(X,Y)(s), s \rangle &= \langle \nabla_X \nabla_Y s, s \rangle - \langle \nabla_Y \nabla_X s, s \rangle \\ &= X \langle \nabla_Y s, s \rangle - \langle \nabla_Y s, \nabla_X s \rangle - Y \langle \nabla_X s, s \rangle + \langle \nabla_X s, \nabla_Y s \rangle \\ &= \frac{1}{2} \left(XY \langle s, s \rangle - YX \langle s, s \rangle \right) \\ &= \frac{1}{2} [X, Y] \langle s, s \rangle \\ &= 0. \end{aligned}$$

This completes the proof. ■

Corollary 37.16. *Let (E, g) be a Riemannian vector bundle over M, and let ∇ be a metric connection. Then for all $X, Y \in \mathfrak{X}(M)$, the curvature $R^\nabla(X, Y)$ belongs to the orthogonal algebra bundle $\mathfrak{o}(g)$.*

Proof. Proposition 37.15 shows us that $R^\nabla(\xi, \zeta) \in \mathfrak{o}(E_p, g_p)$ for all $p \in M$ and $\xi, \zeta \in T_pM$. ■

We conclude this lecture by using Corollary 37.16 to prove Proposition 37.5.

Proof of Proposition 37.5. It suffices to find a single connection for which $[\mathsf{tr}(R^\nabla)] = 0$. Let g denote any Riemannian metric on E and let ∇ denote any metric connection. Then Proposition 37.15 shows that $R^\nabla(X, Y)$ is skew-symmetric and hence has trace zero. ■

LECTURE 38

Characteristic Classes

In this lecture we construct the characteristic classes of a vector bundle in generality. The formalism is a little daunting, so you are urged to keep in mind the example of the trace function from the last lecture. We begin as usual at the level of linear algebra. Let $\mathbb{R}[\mathsf{X}_1, \dots, \mathsf{X}_n]$ denote the $\mathbb{R}$-algebra of polynomials in n indeterminates X_i. A polynomial $\mathsf{p} \in \mathbb{R}[\mathsf{X}_1, \dots, \mathsf{X}_n]$ is said to be **homogeneous of degree** k if we can write

$$\mathsf{p}(\mathsf{X}_1, \dots, \mathsf{X}_n) = \sum c_{i_1 \cdots i_k} \mathsf{X}_{i_1} \cdots \mathsf{X}_{i_k}.$$

where the sum is over all n^k tuples $(i_1, \dots, i_k)$ such that $1 \le i_j \le n$ for each i_j. We may without loss of generality always assume that the coefficients $c_{i_1 \cdots i_k}$ are *symmetric* in the indices $i_1, \dots, i_k$.

Warning: Do not confuse this with requiring the polynomial p itself to be a *symmetric polynomial*!

Definition 38.1. Let V be a vector space of dimension n. A **homogeneous polynomial of degree** k on V is a map

$$\mathsf{q} \colon V \to \mathbb{R}$$

such that for every basis (e^i) of the dual space V^*, there exists a unique homogeneous $\mathsf{p} \in \mathbb{R}[\mathsf{X}_1, \dots, \mathsf{X}_n]$ such that

$$\mathsf{q}(v) = \mathsf{p}(e^1, \dots, e^k)(v) = \sum c_{i_1 \cdots i_k} e^{i_1}(v) \cdots e^{i_k}(v). \tag{38.1}$$

This notation "q" clashes with our previous use of q in Lecture 25 to denote a singular chain (Definition 25.13). Luckily, there is no overlap in the mathematical content, and hence there should be no danger of confusion.

It is easy to see that this property is independent of the choice of basis in the sense that we could replace "for every basis" with "there exists a basis". Moreover any polynomial is obviously smooth.

Definition 38.2. Let V be a vector space. We let $\mathscr{P}_k(V)$ denote the set of all homogeneous polynomials of degree k, and $\mathscr{P}(V) = \bigoplus_{k \ge 0} \mathscr{P}_k(V)$. Then $\mathscr{P}(V)$ is an algebra under the usual pointwise product of functions.

Definition 38.3. Let V be a vector space, and suppose $\mathsf{q} \in \mathscr{P}_k(V)$. The **polarisation** of q is the tensor $\operatorname{polar}(\mathsf{q}) \in T^{0,k}(V) \cong \operatorname{Mult}_{k,0}(V)$ (cf. Proposition 19.8) defined by

$$\operatorname{polar}(\mathsf{q}) = \sum c_{i_1 \cdots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k}.$$

where (e^i) is some basis of V^* and the coefficients $c_{i_1 \cdots i_k}$ are determined by (38.1).

As above, it is easy to see that definition of $\operatorname{polar}(\mathsf{q})$ does not depend on the choice of basis (e^i) of V^*. Since we assumed that the original coefficients $c_{i_1 \cdots i_k}$ were symmetric in the indices i_j, the tensor $\operatorname{polar}(\mathsf{q})$ is actually a *symmetric* tensor in the following sense.

Definition 38.4. Let V be a vector space. A tensor $S \in T^{0,k}V$ is said to be **symmetric** if we can write

$$S = \sum c_{i_1 \cdots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k}$$

for some basis (e^i) such that the coefficients $c_{i_1 \cdots i_k}$ are symmetric in the indices $i_1, \dots, i_k$.

In fact, any such symmetric tensor is the polarisation of a homogeneous polynomial, i.e. the map

$$\text{polar} \colon \mathscr{P}(V) \to \{\text{symmetric tensors}\}.$$

is a bijection. An explicit inverse is given by

$$\text{polar}^{-1}(S)(v) := S(v, \dots, v).$$

Definition 38.5. Let G be a Lie group with Lie algebra $\mathfrak{g}$. A homogeneous polynomial $\mathsf{q} \colon \mathfrak{g} \to \mathbb{R}$ is said to be **invariant** if

$$\mathsf{q}(\mathrm{Ad}_g(\xi)) = \mathsf{q}(\xi), \qquad \forall\, g \in G,\ \xi \in \mathfrak{g},$$

where $\mathrm{Ad}_g \colon \mathfrak{g} \to \mathfrak{g}$ was defined in Definition 13.16. We denote by $P_{\mathrm{inv}}(\mathfrak{g}) \subset P(\mathfrak{g})$ the subalgebra of all invariant polynomials.

In this lecture we are concerned only with characteristic classes on vector bundles, and hence it is enough to restrict to the case $G = \mathrm{GL}(n)$. In this case the adjoint action on $\mathfrak{gl}(n)$ is given by conjugation:

On Problem Sheet O you are asked to develop the theory of characteristic classes for an arbitrary principal G-bundle. This theory requires the general form of Definition 38.5.

$$\mathrm{Ad}_A \colon \mathfrak{gl}(n) \to \mathfrak{gl}(n), \qquad \mathrm{Ad}_A(B) = ABA^{-1}.$$

and thus a polynomial $\mathsf{q} \colon \mathfrak{gl}(n) \to \mathbb{R}$ is invariant if

$$\mathsf{q}(ABA^{-1}) = \mathsf{q}(B), \qquad \forall\, A \in \mathrm{GL}(n),\ B \in \mathfrak{gl}(n).$$

We abbreviate

$$\mathscr{P}_{\mathrm{inv}}(n) := \mathscr{P}_{\mathrm{inv}}(\mathfrak{gl}(n)).$$

The following lemma is elementary linear algebra.

Lemma 38.6. *The coefficients q_k of the characteristic polynomial*

$$\det(tI + A) = \sum_{k=0}^{n} \mathsf{q}_k(A) t^{n-k}, \qquad A \in \mathfrak{gl}(n),\ t \in \mathbb{R} \tag{38.2}$$

are invariant polynomials of degree k on $\mathfrak{gl}(n)$. In particular, the trace and determinant are invariant polynomials:

$$1 = \mathsf{q}_0(A), \qquad \mathsf{tr} A = \mathsf{q}_1(A), \qquad \det A = \mathsf{q}_k(A)$$

In fact, polynomials of this form generate the entire algebra $\mathscr{P}_{\mathrm{inv}}(n)$. This is a classical result which goes by the somewhat flamboyant name:

Theorem 38.7 (Fundamental Theorem on Symmetric Polynomials). *The space $\mathscr{P}_{\mathrm{inv}}(n)$ is generated as an $\mathbb{R}$-algebra by the coefficients q_i of the characteristic polynomial (38.2).*

Theorem 38.7 is not hard to prove. However it has nothing to do with Differential Geometry and therefore we will omit it.

Definition 38.8. Let G be a Lie group with Lie algebra $\mathfrak{g}$. We say a symmetric tensor $S \in S^{0,k}\mathfrak{g}$ is **invariant** if

$$S(\mathrm{Ad}_g(\xi_1), \dots, \mathrm{Ad}_g(\xi_k)) = S(\xi_1, \dots, \xi_k)$$

for all $g \in G$ and $\xi_i \in \mathfrak{g}$.

It follows readily from the definition that a symmetric tensor $S = \mathrm{polar}(\mathsf{q})$ is invariant if and only if q is an invariant polynomial on $\mathfrak{g}$.

Let us now proceed to vector bundles. The following lemma is the analogue of Proposition 37.2 in this new more complicated setting.

Lemma 38.9. *Let $\pi\colon E \to M$ be a vector bundle of rank n with connection ∇, and let $\mathsf{q} \in \mathscr{P}_{\mathrm{inv}}(n)$ denote an invariant polynomial of degree k. Then q induces a parallel section Q of the dual bundle $\mathrm{End}^k(E)^*$.*

Proof. Let $S = \mathrm{polar}(\mathsf{q})$ denote the polarisation of q. Fix $p \in M$ and choose $\ell_0 \in \mathrm{Fr}(E_p)$. We define

$$\mathsf{Q}_p(A_1 \otimes \dots \otimes A_k) \coloneqq S\big(\ell_0^{-1} \circ A_1 \circ \ell_0, \dots, \ell_0^{-1} \circ A_k \circ \ell_0\big)$$

for $A_i \in \mathfrak{gl}(E_p)$. This definition is independent of the choice of F, since if ℓ_1 was another element of $\mathrm{Fr}(E_p)$ then $\ell \coloneqq \ell_0^{-1}\ell_1 \in \mathrm{GL}(n)$, and

$$\begin{aligned} S\big(\ell_1^{-1} \circ A_1 \circ \ell_1, \dots, \ell_1^{-1} \circ A_k \circ \ell_1\big) &= S\big(\ell^{-1}\ell_0^{-1} \circ A_1 \circ \ell_0\ell, \dots, \ell^{-1}\ell_0^{-1} \circ A_k \circ \ell_0\ell\big) \\ &= S\big(\,\mathrm{Ad}_\ell(\ell_0^{-1} \circ A_1 \circ \ell_0), \dots, \mathrm{Ad}_\ell(\ell_0^{-1} \circ A_k \circ \ell_0)\big) \\ &= S\big(\ell_0^{-1} \circ A_1 \circ \ell_0, \dots, \ell_0^{-1} \circ A_k \circ \ell_0\big) \end{aligned}$$

since S is invariant. The proof that Q is parallel is identical to the proof of Proposition 37.2. ■

Let us now consider differential forms with values in $\mathrm{End}^k(E)$. The tensor product gives us another way to multiply such forms together:

Definition 38.10. Let $\Theta \in \Omega^h(M, \mathrm{End}^k(E))$ and $\Upsilon \in \Omega^l(M, \mathrm{End}^j(E))$. We define an element $\Theta \otimes \Upsilon \in \Omega^{h+l}(M, \mathrm{End}^{k+r}(E))$ by wedging together the $\Omega(M)$ factors and tensoring the $\mathrm{End}(E)$ factors. Thus for decomposable elements

$$\Theta = \omega \otimes \big(\Phi_1 \otimes \dots \otimes \Phi_k\big), \qquad \Upsilon = \theta \otimes (\Psi_1 \otimes \dots \otimes \Psi_r),$$

where $\omega \in \Omega^h(M)$, $\theta \in \Omega^l(M)$ and $\Theta_i, \Psi_j \in \Gamma(\mathrm{End}(E))$, one has

$$\Theta \otimes \Upsilon \coloneqq (\omega \wedge \theta) \otimes \big(\Phi_1 \otimes \dots \otimes \Phi_k \otimes \Psi_1 \otimes \dots \otimes \Psi_r).$$

Just as in the discussion before Lemma 37.1, if we are given a section η of the dual bundle $\mathrm{End}^k(E)^*$ and a bundle-valued differential form $\Theta \in \Omega(M, \mathrm{End}^k(E))$, we can feed Θ to η to obtain a normal differential form $\eta(\Theta)$ of the same degree. We then have the following generalisation of Lemma 37.1.

LEMMA 38.11. *Suppose $\pi\colon E \to M$ is a vector bundle of rank n with connection ∇, and suppose $\mathsf{q} \in P_{\mathrm{inv}}(n)$ is an invariant polynomial of degree k. Let Q denote the induced parallel section of the dual bundle $\mathrm{End}^k(E)^*$. Suppose $\Theta_i \in \Omega^{h_i}(M, \mathrm{End}(E))$ for $i = 1, \dots, k$. Then denoting by ∇^{End} the induced connection on $\mathrm{End}(E)$, we have*

$$d\big(\mathsf{Q}(\Theta_1 \otimes \cdots \otimes \Theta_k)\big) = \mathsf{Q}\left(\sum_{j=1}^{k} (-1)^{h_1+\cdots+h_{j-1}} \Theta_1 \otimes \cdots \otimes d^{\nabla^{\mathrm{End}}} \Theta_j \otimes \cdots \otimes \Theta_k\right).$$

Proof. Use Proposition 36.17 together with the argument from the proof of Lemma 37.1. ■

The wedge product on differential forms is also well defined on the level of de Rham cohomology.

DEFINITION 38.12. Let M be a smooth manifold. If $[\omega] \in H^h_{\mathrm{dR}}(M)$ and $[\theta] \in H^k_{\mathrm{dR}}(M)$ are two de Rham cohomology classes represented by closed forms ω and θ respectively, then we define

$$[\omega] \wedge [\theta] := [\omega \wedge \theta] \in H^{h+k}_{\mathrm{dR}}(M).$$

This is well defined as (a) the $(h + k)$-form $\omega \wedge \theta$ is closed, since $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^k \omega \wedge d\theta = 0$, and (b) $[\omega \wedge \theta]$ is independent of the choice of representatives ω and θ, since if ω_1 and θ_1 were two more representatives (meaning that $\omega - \omega_1$ and $\theta - \theta_1$ are both exact) then the same formula shows that $\omega \wedge \theta - \omega_1 \wedge \theta_1$ is exact.

We are now ready to state and prove our main result.

THEOREM 38.13 (The Chern-Weil Theorem). *Let $\pi\colon E \to M$ denote a vector bundle of rank n and let $\mathsf{q} \in \mathscr{P}_{\mathrm{inv}}(n)$ have degree k. Then:*

(i) *If ∇ is a connection on E and Q is the induced parallel section of $\mathrm{End}^k(E)^*$ from Lemma 38.9 then the $2k$-form*

$$\mathsf{q}(\nabla) := \mathsf{Q}\Big(\underbrace{R^\nabla \otimes \cdots \otimes R^\nabla}_{k \text{ copies}}\Big)$$

is closed.

(ii) *The cohomology class $[\mathsf{q}(\nabla)] \in H^{2k}_{\mathrm{dR}}(M)$ is independent of ∇.*

(iii) *The map*

$$\mathsf{CW}_E\colon \mathscr{P}_{\mathrm{inv}}(n) \to H_{\mathrm{dR}}(M)$$

given by

$$\mathsf{CW}_E(\mathsf{q}) := [\mathsf{q}(\nabla)]$$

is an algebra homomorphism.

Here "CW" stands for "Chern-Weil" (don't confuse this with a CW complex in algebraic topology!)

Proof. The proof of (i) is the same as Corollary 37.3, and uses the Bianchi Identity (Theorem 36.21) and Lemma 38.11:

$$\begin{aligned} d\big(\mathsf{Q}(R^\nabla \otimes \cdots \otimes R^\nabla)\big) &= \mathsf{Q}\Big(\sum_i (-1)^{2+\cdots+2} R^\nabla \otimes \cdots \otimes \underbrace{d^{\nabla^{\mathrm{End}}}(R^\nabla)}_{=0} \otimes \cdots \otimes R^\nabla\Big) \\ &= \mathsf{Q}\left(R^\nabla \otimes \cdots \otimes 0 \otimes \cdots \otimes R^\nabla\right) \\ &= 0. \end{aligned}$$

The proof of (ii) is identical to that of Proposition 37.4: using the notation from that proof one has

$$\mathsf{q}(\nabla_0) = \iota_0^*\big(\mathsf{q}(\nabla)\big), \qquad \mathsf{q}(\nabla_1) = \iota_1^*\big(\mathsf{q}(\nabla)\big),$$

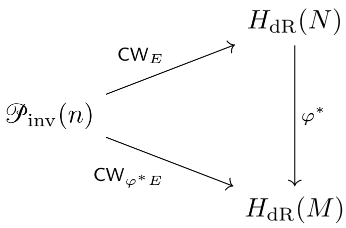
and hence by Proposition 27.4 again

$$[\mathsf{q}(\nabla_0)] = \iota_0^*[\mathsf{q}(\nabla)] = \iota_1^*[\mathsf{q}(\nabla)] = [\mathsf{q}(\nabla_1)].$$

The proof of (iii) is on Problem Sheet N. ■

We now prove that CW behaves nicely with respect to pullbacks.

Proposition 38.14. *Let $\pi\colon E \to N$ denote a vector bundle of rank n and let $\varphi\colon M \to N$ denote a smooth map. Then the following diagram commutes:*



Proof. This follows from the equality $R^{\varphi^*\nabla} = \varphi^* R^{\nabla}$, which in turn follows from Proposition 35.9. ■

Definition 38.15. Let $\pi\colon E \to M$ be a vector bundle of rank n. We call an element $\mathsf{CW}_E(\mathsf{q}) \in H_{\mathrm{dR}}(M)$ a **characteristic class** of E. The map $\mathsf{CW}_E\colon \mathscr{P}_{\mathrm{inv}}(n) \to H_{\mathrm{dR}}(M)$ is called the **Chern-Weil homomorphism**.

It follows from Proposition 38.14 that isomorphic vector bundles have the same characteristic classes. Turning this on its head, if E and F are any two vector bundles, then in order to show that E and F are *not* isomorphic, it suffices to find a single characteristic class which takes different values on E and F.

The following generalisation of Proposition 37.5 is on Problem Sheet N.

Proposition 38.16. *If $\mathsf{q} \in \mathscr{P}_{\mathrm{inv}}(n)$ is an invariant homogeneous polynomial of odd degree $2k+1$ then $\mathsf{CW}_E(\mathsf{q}) = 0$ for any vector bundle of rank n.*

Combining this with Theorem 38.7 tells us that the ring of characteristic classes on M is generated by the coefficients q_{2k} of even degree of the characteristic polynomial (38.2). This motivates the following definition.

Definition 38.17. Let $\pi\colon E \to M$ be a vector bundle. We define the **kth Pontryagin class** of E to be

$$p_k(E) := \left[\mathsf{q}_{2k}\left(\frac{i}{2\pi}\nabla\right)\right] \in H^{4k}_{\mathrm{dR}}(M).$$

The factor of $\frac{i}{2\pi}$ is not too important, it is just there to make certain other formulae prettier. Note $p_k(E) = 0$ if $k > \lfloor \frac{\dim M}{4} \rfloor$. It is also formally useful to define $p_0(E) \coloneqq 1$, where $1 \in H^0_{\mathrm{dR}}(M) = \mathbb{R}$ is the cohomology class containing the constant function 1.

Moreover since q_{2k} is homogeneous of degree $2k$, the factor of i disappears when fed to q_{2k}

PROPOSITION 38.18 (The Whitney Product Formula). *If E and F are vector bundles over M then*

$$p_k(E \oplus F) = \sum_{i=0}^{k} p_i(E) \wedge p_{k-i}(F).$$

For those of you who are familiar with Algebraic Topology: the statement would be more complicated if one worked with (singular) cohomology with coefficients in $\mathbb{Z}$, since then one would need to worry about 2-torsion elements.

We conclude this lecture with a sample application.

PROPOSITION 38.19. *Suppose M is a compact manifold of dimension m which can be embedded in $\mathbb{R}^{m+1}$. Then $p_k(TM) = 0$ for $k > 0$.*

Proof. If M embeds in $\mathbb{R}^{m+1}$ then the normal bundle $\mathrm{Norm}(M)$ from Definition 7.7 is a one-dimensional vector bundle and hence has $p_k(\mathrm{Norm}(M)) = 0$ for $k > 0$. We have a vector bundle isomorphism:

$$T\mathbb{R}^{m+1}|_M \cong TM \oplus \mathrm{Norm}(M).$$

Proposition 38.14 applied to the embedding $M \hookrightarrow \mathbb{R}^{m+1}$ tells us that $p_k(T\mathbb{R}^{m+1}|_M) = 0$ for $k > 0$. Thus the Whitney Product Formula (Proposition 38.18) implies that $p_k(TM) = 0$ for $k > 0$. This completes the proof. ■

Of course, the usefulness of Proposition 38.19 depends on our ability to compute the Pontryagin classes! But this is a topic best suited for a course on Algebraic Topology. We just state here one result.

COROLLARY 38.20. *$\mathbb{C}P^2$ does not embed in $\mathbb{R}^5$.*

Proof. We can think of $\mathbb{C}P^2$ as a compact manifold of (real) dimension four. One can show that the class $p_1(T\mathbb{C}P^2) \in H^4_{\mathrm{dR}}(\mathbb{C}P^2)$ is of the form $3c^2$, where $c \in H^2_{\mathrm{dR}}(\mathbb{C}P^2)$ is a generator (and thus in particular is non-zero). ■

LECTURE 39

Connections on Principal Bundles

In this lecture we return to principal bundles. We define connections on principal bundles, and relate them to connections on associated vector bundles. We begin with some generalities concerning differential forms on principal bundles and their associated bundles.

Let $\pi\colon P \to M$ be a principal G-bundle, and suppose σ is a representation of G on a vector space V. Abbreviate by $E = P \times_G V$ the associated vector bundle over M (Corollary 18.5). Using the formalism from Lecture 36, we are interested in the relationship between

In Theorem 18.3 we restricted to faithful representations. This however was not used anywhere in the proof (cf. Remark 18.2). Throughout our treatment of connections on principal bundles (Lectures 39–42), it is convenient not to impose this restriction.

$$\Omega^k(P,V), \qquad \text{i.e. } V \text{ vector-valued differential forms on } P,$$
$$\Omega^k(M,E), \qquad \text{i.e. } E \text{ bundle-valued differential forms on } M.$$

We construct a subspace $\Omega^k_G(P,V) \subset \Omega^k(P,V)$ consisting of *horizontal equivariant* forms. Then we will prove that $\Omega^k_G(P,V) \cong \Omega^k(M,E)$.

We begin with the following very general definition.

DEFINITION 39.1. A differential form $\omega \in \Omega^k(P,V)$ is said to be **horizontal** if ω vanishes whenever any of its variables is a vertical vector. If $k = 0$, we declare all forms to be horizontal.

Let τ denote the principal bundle action of G on P. The next definition is the differential form version of Definition 12.16.

DEFINITION 39.2. Let $\omega \in \Omega^k(P,V)$ denote an V-valued form. We say that ω is **(σ,τ)-equivariant** if

The reason for the g^{-1} is that σ is a left-action and τ is a right action.

$$\tau_g^*\omega = \sigma_{g^{-1}}\omega, \qquad \forall\, g \in G.$$

Explicitly, this means that for any $u \in P$ and $\zeta_1, \dots, \zeta_k \in T_uP$,

$$(\tau_g^*\omega)_u(\zeta_1, \dots, \zeta_k) = \sigma_{g^{-1}}(\omega_u(\zeta_1, \dots, \zeta_k)).$$

When there is no danger of confusion, we will simply "equivariant" rather than (σ,τ)-equivariant.

We set:

$$\Omega^k_G(P,V) \coloneqq \left\{\omega \in \Omega^k(P,V) \mid \omega \text{ is horizontal and equivariant}\right\}.$$

Let us now explain how an element $\omega \in \Omega^k_G(P,V)$ gives rise to an element $\breve{\omega} \in \Omega^k(M,E)$. Fix $p \in M$ and $\xi_1, \dots, \xi_k \in T_pM$. Choose any point $u \in P_p$, and choose any vectors $\zeta_i \in T_uP$ such that $D\pi(u)\zeta_i = \xi_i$ (such vectors exist as π is a submersion). Then $\omega_u(\zeta_1, \dots, \zeta_k)$ belongs to V. As in the proof of Theorem 18.3, we denote by $\psi_u\colon V \to E_{\pi(u)}$ the map $v \mapsto [u,v]$. This map is a linear isomorphism (cf. the paragraph before Corollary 18.5). We apply ψ_u to $\omega_u(\zeta_1, \dots, \zeta_k)$ to get an element of E_p, which we denote by $\breve{\omega}_p(\xi_1, \dots, \xi_k)$:

Recall an element of E is an equivalence class $[u,v]$, where the equivalence relation is $(\tau_g(u), v) \sim (u, \sigma_g(v))$.

$$\breve{\omega}_p(\xi_1, \dots, \xi_k) \coloneqq \psi_u\big(\omega_u(\zeta_1, \dots, \zeta_k)\big).$$

This definition involved several choices, so we must prove that $\breve{\omega}$ is well defined.

THEOREM 39.3. *If* $\omega \in \Omega^k_G(P, V)$ *then* $\breve{\omega}$ *is a well-defined element of* $\Omega^k(M, E)$*. Moreover the map*

$$\Omega^k_G(P, V) \mapsto \Omega^k(M, E), \qquad \omega \mapsto \breve{\omega}$$

is a linear isomorphism.

Proof. We prove the result in four steps.

1. To show that $\breve{\omega}$ is well defined we must check the value of $\psi_u(\omega_u(\zeta_1, \ldots, \zeta_k))$ does not depend on the choice of $u \in P_p$ and the choice of $\zeta_i \in T_uP$ such that $D\pi(u)\zeta_i = \xi_i$. In this step we deal with the ζ_i. If η_i was another tangent vector such that $D\pi(u)\eta_i = \xi_i$ then $\eta_i - \zeta_i$ is a vertical vector. Since ω is horizontal and k-linear we have

$$\begin{aligned} \omega_u(\zeta_1, \ldots, \zeta_k) &= \omega_u(\zeta_1 - \eta_1 + \eta_1, \ldots, \zeta_k - \eta_k + \eta_k) \\ &= \omega_u(\eta_1 + \text{vertical}, \ldots, \eta_k + \text{vertical}) \\ &= \omega_u(\eta_1, \ldots, \eta_k). \end{aligned}$$

2. Now let us deal with the choice of u. Suppose instead we choose $\tau_g(u)$. Since $\pi \circ \tau_g = \pi$, we have

Recall the orbits of τ are exactly the fibres of P.

$$D\pi(\tau_g(u)) \circ D\tau_g(u)\zeta_i = D\pi(u)\zeta_i = \xi_i, \tag{39.1}$$

so that $D\tau_g(u)\zeta_i \in T_{\tau_g(u)}P$ is a tangent vector that maps onto ξ_i. Thus it suffices to show that

$$\psi_u\big(\omega_u(\zeta_1, \ldots, \zeta_k)\big) = \psi_{\tau_g(u)}\big(\omega_{\tau_g(u)}\big(D\tau_g(u)\zeta_1, \ldots, D\tau_g(u)\zeta_k\big)\big). \tag{39.2}$$

By equivariance, we have

$$\begin{aligned} \omega_{\tau_g(u)}\big(D\tau_g(u)\zeta_1, \ldots, D\tau_g(u)\zeta_k\big) &= (\tau_g^*\omega)_u(\zeta_1, \ldots, \zeta_k) \\ &= \sigma_{g^{-1}}(\omega_u(\zeta_1, \ldots, \zeta_k)), \end{aligned}$$

Thus to complete the proof of (39.2) we need only observe that:

$$\psi_{\tau_g(u)} = \psi_u \circ \sigma_g, \tag{39.3}$$

which follows directly from the definition:

$$\begin{aligned} \psi_{\tau_g(u)}(v) &= [\tau_g(u), v] \\ &= [u, \sigma_g(v)] \\ &= \psi_u(\sigma_g(v)). \end{aligned}$$

3. We now know that $\breve{\omega}$ is well defined. To complete the proof, we build an inverse. Suppose $\alpha \in \Omega^k(M, E)$ is a bundle-valued form on M. Define $\widehat{\alpha} \in \Omega^k(P, V)$ by

$$\widehat{\alpha}_u(\zeta_1, \ldots, \zeta_k) := \psi_u^{-1}\Big(\alpha_{\pi(u)}\big(D\pi(u)\zeta_1, \ldots, D\pi(u)\zeta_k\big)\Big).$$

It is obvious that $\widehat{\alpha}$ is horizontal, so let us check that $\widehat{\alpha}$ is equivariant. To see this we argue as follows:

$$\begin{aligned}\big(\tau_g^*\widehat{\alpha}\big)_u(\zeta_1,\dots,\zeta_k) &= \widehat{\alpha}_{\tau_g(u)}\big(D\tau_g(u)\zeta_1,\dots,D\tau_g(u)\zeta_k\big)\\ &= \psi_{\tau_g(u)}^{-1}\Big(\alpha_p\big(D\pi(\tau_g(u))\circ D\tau_g(u)\zeta_1,\dots,D\pi(\tau_g(u))\circ D\tau_g(u)\zeta_k\big)\Big)\\ &= \sigma_{g^{-1}}\circ\psi_u^{-1}\Big(\alpha_p\big(D\pi(u)\zeta_1,\dots,D\pi(u)\zeta_k\big)\Big)\\ &= \sigma_{g^{-1}}\big(\widehat{\alpha}_u(\zeta_1,\dots,\zeta_k)\big).\end{aligned}$$

where the penultimate equality used (39.1) and (39.3).

4. It is clear that from the definitions that

$$\check{\widehat{\alpha}} = \alpha, \qquad \widehat{\check{\omega}} = \omega,$$

and thus the proof is complete. ■

Let us briefly consider the case $k = 0$. Every zero-form is vacuously horizontal, and the equivariance condition for a function $f\colon P \to V$ becomes

$$f(\tau_g(u)) = \sigma_{g^{-1}}(f(u)), \qquad \forall\, u \in P,\ g \in G. \tag{39.4}$$

Meanwhile $\Omega^0(M,E) = \Gamma(E)$. This proves:

COROLLARY 39.4. *Let P be a principal G bundle over M and let $E = P \times_G V$ denote an associated bundle. Then there is a one-to-one correspondence between $\Gamma(E)$ and functions $f\colon P \to V$ satisfying (39.4). Explicitly, given f satisfying (39.4) we define $s\colon M \to E$ via $s(p) := \psi_u(f(u))$, where u is any point in P_p. Conversely, given a section s, we define $f\colon P \to V$ by $f(u) = \psi_u^{-1}(s(\pi(u)))$.*

REMARK 39.5. Let E be a vector bundle of rank n. Corollary 39.4 tells us that section s of E can be identified with an equivariant function $f\colon \mathrm{Fr}(E) \to \mathbb{R}^n$.

Now let us move onto connections on principal bundles. Since a principal bundle is in particular a fibre bundle, we have already defined preconnections on principal bundles. Just as with vector bundles, a connection on a principal bundle is a preconnection that satisfies an additional condition.

DEFINITION 39.6. Let $\pi\colon P \to M$ be a principal G-bundle. Let τ denote the principal bundle action of G on P. A **connection** on P is a preconnection Δ which satisfies

$$D\tau_g(u)(\Delta_u) = \Delta_{\tau_g(u)}, \qquad \forall\, u \in P,\ g \in G. \tag{39.5}$$

Equation (39.5) is the natural analogue of (28.3) for connections on vector bundles.

As before, given a connection Δ we denote by

$$\zeta = \zeta^{\mathsf{h}} + \zeta^{\mathsf{v}}$$

the horizontal-vertical splitting of a tangent vector $\zeta \in TP$. The condition (39.5) implies that

$$D\tau_g(u)\zeta^{\mathsf{h}} = \big(D\tau_g(u)\zeta\big)^{\mathsf{h}}, \qquad \forall\, \zeta \in T_uP,\ g \in G. \tag{39.6}$$

We will shortly investigate the relationship between connections on principal bundles and connections on vector bundles. Before doing so, however, we look at parallel transport systems in principal bundles. The following definition is easier to remember if you follow the general mantra that you simply take the vector bundle version and replace "linear" with "equivariant" at every opportunity.

DEFINITION 39.7. Let $\pi\colon P \to M$ be a principal G-bundle. A **parallel transport system** $\mathbb{P}$ on P assigns to every point $u \in P$ and every curve $\gamma\colon [a,b] \to M$ with $\gamma(a) = \pi(u)$, a unique section $\mathbb{p}_{\gamma;u} \in \Gamma_\gamma(E)$ with initial condition u, i.e. such that $\mathbb{p}_{\gamma;u}(a) = u$. One calls $\mathbb{p}_{\gamma;u}$ the **parallel lift** of γ starting at u. This association should satisfy the following five axioms:

(i) **(Equivariance)**: For every smooth curve $\gamma\colon [a,b] \to M$ the map

$$\mathbb{P}_\gamma\colon P_{\gamma(a)} \to P_{\gamma(b)}, \qquad \mathbb{P}_\gamma(u) \coloneqq \mathbb{p}_{\gamma;u}(b)$$

is a diffeomorphism which is (τ,τ)-equivariant with respect to the G-action:

$$\mathbb{P}_\gamma(\tau_g(u)) = \tau_g\big(\mathbb{P}_\gamma(u)\big)$$

Moreover

$$\mathbb{P}_\gamma^{-1} = \mathbb{P}_{\gamma^-}$$

where $\gamma^-\colon [a,b] \to M$ is the reverse curve $t \mapsto \gamma(a-t+b)$.

(ii) (**Concatenation**): If $\gamma\colon [a,b] \to M$ is a smooth path and $c \in (a,b)$, then if we abbreviate

$$\gamma_1 \coloneqq \gamma|_{[a,c]}, \qquad \gamma_2 \coloneqq \gamma|_{[c,b]}, \qquad w \coloneqq \mathbb{p}_{\gamma;u}(c),$$

then

$$\mathbb{p}_{\gamma;u}(t) = \begin{cases} \mathbb{p}_{\gamma_1;u}(t), & t \in [a,c], \\ \mathbb{p}_{\gamma_2;w}(t), & t \in [c,b]. \end{cases}$$

This implies that

$$\mathbb{P}_\gamma = \mathbb{P}_{\gamma_2} \circ \mathbb{P}_{\gamma_1}.$$

(iii) (**Independence of parametrisation**): If $\gamma\colon [a,b] \to M$ is a smooth curve and $h\colon [a_1,b_1] \to [a,b]$ is a diffeomorphism such that $h(a_1) = a$ and $h(b_1) = b$ then for every point $u \in P_{\gamma(a)}$ and every $t \in [a_1,b_1]$, we have

$$\mathbb{p}_{\gamma\circ h;u}(t) = \mathbb{p}_{\gamma;u}(h(t)).$$

(iv) **(Smooth dependence on initial conditions)**: The section $\mathbb{p}_{\gamma;u}$ depends smoothly on both γ and u.

(v) **(Initial uniqueness)**: Suppose $\gamma, \delta\colon [a,b] \to M$ are two curves such that $\gamma(a) = \delta(a)$ and $\dot\gamma(a) = \dot\delta(a)$. Then for each $u \in P_{\gamma(a)}$, the two curves $\mathbb{p}_{\gamma;u}$ and $\mathbb{p}_{\delta;u}$ have the same initial tangent vector:

$$\dot{\mathbb{p}}_{\gamma;u}(a) = \dot{\mathbb{p}}_{\delta;u}(a)$$

As you can guess, connections on principal bundles are equivalent to parallel transport systems.

THEOREM 39.8. *Let $\pi\colon P \to M$ be a principal G-bundle. Then a connection on P (in the sense of Definition 39.6) determines and is uniquely determined by a parallel transport system on P (in the sense of Definition 39.7).*

The proof of Theorem 39.8 proceeds analogously to Theorem 30.1 and Theorem 30.2, and to avoid being repetitive, we omit the details. Instead let us now explain how connections on principal bundles are related to connections on vector bundles.

THEOREM 39.9. *Let $\pi\colon P \to M$ be a principal G-bundle, and suppose σ is a representation of G on a vector space V. Set $E = P \times_G V$. A connection Δ on P (in the principal bundle sense) induces a connection Δ_E on E (in the vector bundle sense).*

In the proof we denote by $\wp\colon P \times V \to E$ the map $(u, v) \mapsto [u, v]$. This is also a principal G-bundle by part *(ii)* of Theorem 18.3.

Proof. Although not strictly necessary, we will give three proofs, one from the point of view of parallel transport, one from the point of view of distributions, and one from the point of view of covariant derivatives.

- *Proof using parallel transport:* Let $\gamma\colon [0, 1] \to M$ be a smooth curve in M, and suppose $\rho \in \Gamma_\gamma(P)$ is a section along γ. Then for any fixed $v \in V$, $t \mapsto \wp(\rho(t), v)$ is a section of E along γ (not every section of E along γ is of this form though). We define a parallel transport system $\mathbb{P}^E$ on E by declaring that a section $\tilde{\rho}$ of E along γ is parallel if and only if $\tilde{\rho} = \wp(\rho, v)$ for ρ a parallel section of P along γ. In other words:

$$\mathbb{P}^E_\gamma[u, v] := [\mathbb{P}_\gamma(u), v] \,.$$

  This is well defined because $\mathbb{P}_\gamma$ is (τ, τ)-equivariant. Indeed, if (u_1, v_1) is another representative of $[u, v]$, then there exists $g \in G$ such that $u_1 = \tau_g(u)$ and $v_1 = \sigma_{g^{-1}}(v)$. Then

$$\begin{aligned} [\mathbb{P}_\gamma(u_1), v_1] &= \big[\mathbb{P}_\gamma(\tau_g(u)), \sigma_{g^{-1}}(v)\big] \\ &= \big[\tau_g(\mathbb{P}_\gamma(u)), \sigma_{g^{-1}}(v)\big] \\ &= [\mathbb{P}_\gamma(u), v] \,. \end{aligned}$$

  All the axioms for $\mathbb{P}^E$ follow from those of $\mathbb{P}$. For instance, to see that $\mathbb{P}^E_\gamma$ is a linear map we observe that for $c \in \mathbb{R}$ and $v, w \in V$:

$$\begin{aligned} \mathbb{P}^E_\gamma\big([u, v] + c[u, w]\big) &= \mathbb{P}^E_\gamma[u, v + cw] \\ &= [\mathbb{P}_\gamma(u), v + cw] \\ &= [\mathbb{P}_\gamma(u), v] + c\,[\mathbb{P}_\gamma(u), w] \\ &= \mathbb{P}^E_\gamma[u, v] + c\mathbb{P}^E_\gamma[u, w]. \end{aligned}$$

- *Proof using distributions:* Alternatively, in terms of distributions, we define
$$\Delta_E|_{[u,v]} := D\wp(u,v)\big(\Delta_u \times \{0\}\big).$$
It is clear this defines a preconnection on E. Let $\mu_c \colon E \to E$ denote the scalar multiplication $\mu_c[u,v] := [u,cv]$. Then
$$\mu_c \circ \wp(u,v) = [u,cv] = \wp(u,cv),$$
and hence
$$\begin{aligned} D\mu_c[u,v]\big(\Delta_E|_{[u,v]}\big) &= D\mu_c[u,v] \circ D\wp(p,v)\big(\Delta_u \times \{0\}\big) \\ &= D(\mu_c \circ \wp)(u,v)\big(\Delta_u \times \{0\}\big) \\ &= D\wp(u,cv)\big(\Delta_u \times \{0\}\big) \\ &= \Delta_E|_{[u,cv]}. \end{aligned}$$

- *Proof using covariant derivatives:* This is arguably the most interesting proof, since it uses Theorem 39.3. Firstly, if $\omega \in \Omega^k(P,V)$ is any V-valued k-form on P, we define the **horizontal component** of ω (with respect to Δ) to be the form $\omega^{\mathsf{h}} \in \Omega^k(P,V)$ given by
$$\omega^{\mathsf{h}}_u(\zeta_1,\dots,\zeta_k) := \omega_u(\zeta_1^{\mathsf{h}},\dots,\zeta_k^{\mathsf{h}}),$$
where as usual ζ^{h} denotes the horizontal component of ζ. Then ω^{h} is a horizontal vector-valued form. Now we claim:
$$\omega \in \Omega^k_G(P,V) \qquad \Rightarrow \qquad (d\omega)^{\mathsf{h}} \in \Omega^{k+1}_G(P,V). \tag{39.7}$$
We split the proof of (39.7) into two parts:
  - (i) If $\omega \in \Omega^k(P,V)$ is G-equivariant then so is $d\omega$.
  - (ii) If $\omega \in \Omega^k(P,V)$ is G-equivariant then so is ω^{h}.

  To prove (i), fix $g \in G$. Then
$$\begin{aligned} \tau_g^*(d\omega) &= d\tau_g^*(\omega) \\ &= d\sigma_{g^{-1}}(\omega) \\ &= \sigma_{g^{-1}}(d\omega) \end{aligned}$$
where the first line used Lemma 36.8 and the last line used the fact that $\sigma_{g^{-1}} \colon V \to V$ is a linear map. Next, to prove (ii), we take $g \in G$, $u \in P$ and $\zeta_1,\dots,\zeta_k \in T_uP$ and compute
$$\begin{aligned} (\tau_g^*\omega^{\mathsf{h}})_u(\zeta_1,\dots,\zeta_k) &= \omega^{\mathsf{h}}_{\tau_g(u)}\big(D\tau_g(u)\zeta_1,\dots,D\tau_g(u)\zeta_k\big) \\ &= \omega_{\tau_g(u)}\left(D\tau_g(u)\zeta_1^{\mathsf{h}},\dots,D\tau_g(u)\zeta_k^{\mathsf{h}}\right) \\ &= (\tau_g^*\omega)_u(\zeta_1^{\mathsf{h}},\dots\zeta_k^{\mathsf{h}}) \\ &= \sigma_{g^{-1}}\left(\omega_u(\zeta_1^{\mathsf{h}},\dots\zeta_k^{\mathsf{h}})\right) \\ &= \sigma_{g^{-1}}\left(\omega^{\mathsf{h}}_u(\zeta_1,\dots,\zeta_k)\right), \end{aligned}$$
where the second line used (39.6). Thus (39.7) is proved. We now use (39.7) to define an exterior covariant differential $d^\nabla \colon \Omega^k(M,E) \to \Omega^{k+1}(M,E)$ by
$$d^\nabla\alpha := \overbrace{(d\widehat{\alpha})^{\mathsf{h}}} \tag{39.8}$$

In particular, for $k = 0$, if $s \in \Gamma(E)$ then

$$\nabla s := \widetilde{(df)}^{\mathsf{h}}$$

where $f \colon P \to V$ is the function from Corollary 39.4. All the axioms of a covariant derivative operator are easy to check.

This completes the proof (three times over). ■

Wholesome exercise: Check that all three proofs give rise to the same connection on E.

If $\pi \colon E \to M$ is a vector bundle of rank k, then its frame bundle $\hat{\pi} \colon \mathrm{Fr}(E) \to M$ is a principal $\mathrm{GL}(k)$-bundle. In this special case, the converse to Theorem 39.9 is true.

Proposition 39.10. *Let $\pi \colon E \to M$ be a vector bundle. There is a bijective correspondence between connections on E (in the vector bundle sense) and connections on $\mathrm{Fr}(E)$ (in the principal bundle sense).*

Proof. We need only show that a connection on E determines one of $\mathrm{Fr}(E)$, thus providing an inverse to the construction from Theorem 39.9. This time we will give two proofs:

Only two, since – so far – we only have two different ways to view connections on principal bundles. A third will be introduced next lecture. . .

- *Proof using parallel transport:* Suppose $\gamma \colon [0,1] \to M$ is a smooth curve in M. Let $\ell \in \mathrm{Fr}(E_{\gamma(0)})$. Write $v_i := \ell(e_i)$, where e_i is the standard basis of $\mathbb{R}^n$. Let $\tilde{\ell} \in \mathrm{Fr}(E_{\gamma(1)})$ denote the frame defined by

  Here $n := \operatorname{rank} E$.

  $$\tilde{\ell}(e_i) := \mathbb{P}_\gamma(v_i).$$

  Then we define

  $$\mathbb{P}_\gamma^{\mathrm{Fr}(E)}(\ell) := \tilde{\ell}.$$

  We claim that $\mathbb{P}^{\mathrm{Fr}(E)}$ is a parallel transport system on $\mathrm{Fr}(E)$. To prove that $\mathbb{P}^{\mathrm{Fr}(E)}$ satisfies the equivariance axiom, we must show that

  $$\mathbb{P}_\gamma^{\mathrm{Fr}(E)}(\ell \circ A) = \mathbb{P}_\gamma^{\mathrm{Fr}(E)}(\ell) \circ A$$

  for $A \in \mathrm{GL}(n)$. This however is immediate from the fact that $\mathbb{P}_\gamma$ is a linear isomorphism.

- *Proof using distributions:* As before we consider the map $\wp \colon \mathrm{Fr}(E) \times \mathbb{R}^n \to E$ given by

  $$\wp(\ell, v) := \ell(v) \in E_p, \qquad \ell \in \mathrm{Fr}(E_p), \ v \in \mathbb{R}^n.$$

  We then define for $\ell \in \mathrm{Fr}(E)$

  $$\Delta_\ell^{\mathrm{Fr}(E)} := \left\{ \zeta \in T_\ell \,\mathrm{Fr}(E) \mid D\wp(\ell, 0)(\zeta, 0) \in \Delta_{\wp(\ell,0)} \right\}.$$

  The verification that this defines a connection on $\mathrm{Fr}(E)$ is left for you as another instructive exercise.

This completes the proof (twice over). ■

Remark 39.11. We have shown that there is a bijective correspondence between connections on $\mathrm{Fr}(E)$ and connections on E. For a general principal G-bundle P however, the passage given by Theorem 39.9 from connections on P to connections on an associated bundle $P \times_G V$ may neither be injective or surjective.

LECTURE 40

The Connection Form

In Lecture 31 we defined the connection map $\kappa \colon TE \to E$ associated to a connection on a vector bundle E. In this lecture we investigate the principal bundle analogue, and then use this to define the curvature of a principal bundle connection.

We being with a few general preliminaries on Lie group actions. These results are valid for an arbitrary right action of a Lie group on a manifold (i.e. we do *not* require a principal bundle action).

Definition 40.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and suppose τ is a smooth right action of G on a manifold P. Given $\xi \in \mathfrak{g}$, we associate a vector field Z_ξ on P via

$$Z_\xi(u) := \frac{d}{dt}\Big|_{t=0} \tau_{\exp(t\xi)}(u) \in T_u P.$$

We call Z_ξ a **fundamental vector field** on P.

We use right actions since this will later be applied to principal bundles. Nevertheless, with the usual modifications everything is also valid for left actions.

Let us unpack this a bit. Fix $u \in P$. Then the curve $\gamma_u(t) := \tau_{\exp(t\xi)}(u)$ is a curve in P with initial condition $\gamma_u(0) = \tau_e(u) = u$. Thus $\dot{\gamma}_u(0)$ belongs to T_uP, and this is the value of the vector field Z_ξ:

$$Z_\xi(u) = \dot{\gamma}_u(0).$$

If $f \in C^\infty(P)$ then (thought of a derivation), one has

$$Z_\xi(f)(u) = \frac{d}{dt}\Big|_{t=0} f \circ \gamma_u(t) = \frac{d}{dt}\Big|_{t=0} f(\tau_{\exp(t\xi)}(u)).$$

Of course, calling something a "vector field" does not make it one. Certainly Z_ξ is a section of TP, but it isn't immediate why it is smooth.

Lemma 40.2. *The fundamental vector field Z_ξ is smooth (and hence a vector field on P).*

Proof. It suffices to show by Proposition 8.2 that $Z_\xi(f)$ is a smooth function for each $f \in C^\infty(P)$. But this is clear from the formula above. To make it more transparent, let us write μ for the action. Then

$$Z_\xi(f)(u) = \frac{d}{dt}\Big|_{t=0} (f \circ \tau)(u, \exp(t\xi))$$

is the composition of smooth functions in both u and t. ∎

Example 40.3. Let G act on itself via right multiplication. Then by Proposition 12.2 the fundamental vector field associated to $\xi \in \mathfrak{g}$ is exactly the left-invariant vector field X_ξ.

In light of Example 40.3, the next result is the a generalisation of Problem E.2.

Proposition 40.4. *The flow of Z_ξ is given by $\Phi_t(u) := \tau_{\exp(t\xi)}(u)$. Thus Z_ξ is always complete.*

Proof. With γ_u as above, we need only show that γ_u is the integral curve of Z_ξ through p. This follows from:

$$\begin{aligned}\dot\gamma_u(t) &= \frac{d}{ds}\Big|_{s=0}\gamma_u(t+s)\\ &= \frac{d}{ds}\Big|_{s=0}\tau_{\exp((t+s)\xi)}(u)\\ &= \frac{d}{ds}\Big|_{s=0}\tau_{\exp(s\xi)}(\gamma_u(t))\\ &= Z_\xi(\gamma_u(t)).\end{aligned}$$

■

An alternative way to define the fundamental vector field Z_ξ is via the orbit map cf. (12.3).

$$\tau^u\colon G\to P,\qquad \tau^u(g) := \tau_g(u).$$

Then with γ_u as above,

$$\begin{aligned}D\tau^u(e)\xi &= \frac{d}{dt}\Big|_{t=0}\tau^u(\exp(t\xi))\\ &= \frac{d}{dt}\Big|_{t=0}\tau_{\exp(t\xi)}(u)\\ &= Z_\xi(u).\end{aligned}\tag{40.1}$$

On Problem Sheet O you will show:

Proposition 40.5. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and suppose G acts on a manifold P on the right. Then the map $\xi\mapsto Z_\xi$ is a Lie algebra homomorphism $\mathfrak{g}\to\mathfrak{X}(P)$.*

Our next result makes contact with the adjoint representation from Lecture 10.

Proposition 40.6. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and suppose G acts on a manifold P on the right. Then for $\xi\in\mathfrak{g}$ one has*

$$D\tau_g(u)Z_\xi(u) = Z_{\mathrm{Ad}_{g^{-1}}(\xi)}(\tau_g(u)).$$

Proof. For any $g,h\in G$ and $u\in P$, one has

$$\tau_g\circ\tau^u(h) = \tau_{hg}(u) = \tau^{\tau_g(u)}(g^{-1}hg).$$

Differentiating this identity at $h=e$ and using the fact that Ad is the differential of the conjugation action $h\mapsto gbg^{-1}$ at $h=e$, the claim follows from the chain rule and (40.1). ■

We now restrict to the principal bundle case.

Proposition 40.7. *Let $\pi\colon P\to M$ be a principal G-bundle. Then for any $u\in P$, the differential $D\tau^u(e)$ of the map τ^u from (40.1) at e is an isomorphism*

$$D\tau^u(e)\colon\mathfrak{g}\to V_uP.$$

Proof. We first show that any fundamental vector field Z_ξ is vertical. The map $\pi \circ \tau^u$ is constant, and thus by the chain rule

$$\begin{aligned} D\pi(u) Z_\xi(u) &= D\pi(u) \circ D\tau^u(e)\xi \\ &= D(\pi \circ \tau^u)(e)\xi \\ &= 0. \end{aligned}$$

Now suppose $\xi \in \ker D\tau^u(e)$. Then (40.1) and uniqueness of integral curves imply that u is a fixed point of $\tau_{\exp(t\xi)}$. But G acts freely on P, whence $\xi = 0$. To complete the proof we note that both $\mathfrak{g}$ and $V_u P$ have dimension equal to the dimension of G. Thus $D\tau^u(e)$ is an isomorphism, as claimed. ∎

We now define the principal bundle version of the connection map, which this time is called a *connection form*.

Definition 40.8. Let $\pi \colon P \to M$ be a principal G-bundle, and let Δ be a connection on P. The **connection form** ϖ of Δ is the $\mathfrak{g}$-valued 1-form $\varpi \in \Omega^1(P, \mathfrak{g})$ defined by

$$\varpi_p(\zeta) := D\tau^u(e)^{-1}\zeta^{\mathsf{v}}.$$

This does indeed define an element of $\mathfrak{g}$: the vertical component ζ^{v} belongs to $V_u P$ and hence by Proposition 40.7 there is a unique element $\varpi_p(\zeta) \in \mathfrak{g}$ such that $D\tau^u(e)\varpi_p(\zeta) = \zeta^{\mathsf{v}}$.

Of course, it must be proved that ϖ really is smooth. The next result establishes this, and shows that ϖ uniquely determines Δ. Recall that G acts on $\mathfrak{g}$ via the adjoint representation $\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g})$. In the following, whenever we talk about G acting on $\mathfrak{g}$, we will always implicitly assume that the action is the adjoint one.

Theorem 40.9 (Properties of the connection form). *Let $\pi \colon P \to M$ be a principal bundle with connection Δ. Then the connection form ϖ is smooth and (τ, Ad)-equivariant, i.e.*

$$\tau_g^* \varpi = \mathrm{Ad}_{g^{-1}}(\varpi), \qquad \forall\, g \in G,$$

and moreover satisfies

$$\varpi(Z_\xi) \equiv \xi, \qquad \forall\, \xi \in \mathfrak{g}. \tag{40.2}$$

Moreover if $\omega \in \Omega^1(P, \mathfrak{g})$ is any equivariant form satisfying (40.2) then $\ker \omega_p$ defines a connection on P.

Remark 40.10. The connection form does *not* belong to $\Omega^1_G(P, \mathfrak{g})$! Indeed, (40.2) is the "opposite" of being a horizontal form. We will see how that the *curvature form*, which is a $\mathfrak{g}$-valued 2-form, does belong to $\Omega^2_G(P, \mathfrak{g})$.

Proof of Theorem 40.9. We prove the theorem in three steps.

1. In this step we show that ϖ is equivariant and that (40.2) holds. We begin with the latter statement. By Proposition 40.7 for any $u \in P$ one has $Z_\xi(u) \in V_u P$ and thus $Z_\xi(u)^{\mathsf{v}} = Z_\xi(u)$; thus

$$\varpi_p(Z_\xi(u)) = D\tau^u(e)^{-1} Z_\xi(u)] = \xi$$

by (40.1). To verify equivariance, fix $u \in P$, $g \in G$, and $\zeta \in T_uP$. We wish to show that

$$\varpi_{\tau_g(u)}(D\tau_g(u)\zeta) = \mathrm{Ad}_{g^{-1}}(\varpi_u(\zeta)). \tag{40.3}$$

Since both sides of (40.3) are $\mathbb{R}$-linear and $\zeta = \zeta^{\mathsf{h}} + \zeta^{\mathsf{v}}$ is the sum of a horizontal and vertical vector, it suffices to prove (40.2) when ζ is horizontal and when ζ is vertical.

If ζ is horizontal then by (39.6) so is $D\tau_g(u)\zeta$. Thus $\varpi_p(\zeta)$ and $\varpi_{\tau_g(u)}(D\tau_g(u)\zeta)$ are both zero, and so (40.3) follows. If instead ζ is vertical then by Proposition 40.7 we may assume $\zeta = Z_\xi(u)$ for some $\xi \in \mathfrak{g}$. Then by Proposition 40.6 and (40.2) we have:

$$\begin{aligned}\varpi_{\tau_g(u)}\big(D\tau_g(u)Z_\xi(u)\big) &= \varpi_{\tau_g(u)}\big(Z_{\mathrm{Ad}_{g^{-1}}(\xi)}(\tau_g(u))\big)\\ &= \mathrm{Ad}_{g^{-1}}(\xi)\\ &= \mathrm{Ad}_{g^{-1}}(\varpi_u(Z_\xi(u)))\end{aligned}$$

which proves (40.3) for the vertical case.

2. In this step we prove that ϖ is smooth. Choose a basis $\{\xi_i\}$ of $\mathfrak{g}$. Then by Proposition 40.7 the vector fields $\{Z_{\xi_i}\}$ span the vertical subbundle. Now fix a point $u \in P$. Since Δ is a distribution, there exist vector fields Y_j on a neighbourhood of u that span Δ. Since Δ is complementary to VP, the collection $\{Z_{\xi_i}, Y_j\}$ span the entire tangent bundle to P near u. Thus if Z is any vector field on P we can write

$$Z = f^i\, Z_{\xi^i} + h^j\, Y_j$$

near p for smooth functions f^i, h^j. Then by (40.2) one has near p that

$$\varpi(Z) = f^i\, \xi_i.$$

The right-hand side is smooth, and since X was arbitrary this proves that ϖ is smooth at u (this is a special case of Theorem 36.3). Since u was also arbitrary, it follows that ϖ is smooth.

3. Finally we prove that any equivariant form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying (40.2) determines a connection via $\Delta := \ker\omega$. Indeed, $\ker\omega$ is automatically a subbundle (as ω is smooth), and (40.2) tells us that

$$TP = \ker\omega \oplus \ker D\pi = \ker\omega \oplus VP.$$

Thus Δ is a preconnection. Moreover since ω is equivariant we have

$$D\tau_g(\ker\omega) \subseteq \ker\omega.$$

Applying $D\tau_{g^{-1}}$ to both sides and using equivariance again we have

$$\ker\omega = D\tau_{g^{-1}} \circ D\tau_g(\ker\omega) \subseteq D\tau_{g^{-1}}(\ker\omega) \subseteq \ker\omega$$

which shows we have equality. Thus $\ker\omega$ is a connection. This completes the proof. ∎

Remark 40.11. We now have three different ways to specify a connection on a principal bundle: as a distribution, as a parallel transport system, and via a connection form. Just as with connections on vector bundles, it is useful to have a single fixed notation to refer to a connection, which can then be used to mean whichever viewpoint is convenient at the time. Thus from now on we will typically refer to a connection on a principal bundle with the symbol ϖ.

LECTURE 41

The Curvature Form

In this lecture we define the curvature form of a connection on a principal G-bundle P. This is a $\mathfrak{g}$-valued 2-form on P which is horizontal and equivariant, i.e. an element of $\Omega^2_G(P, \mathfrak{g})$.

We begin with some definitions. Let Δ be a connection on a principal G-bundle $\pi \colon P \to M$. We say a vector field Z on P is **horizontal** if $Z(u) \in \Delta_u$ for all u. Thus in particular given any vector field X on M, its horizontal lift (Definition 28.9) is horizontal.

REMARK 41.1. For any given $u \in P$ and any given $\zeta \in T_uP$, there exists a horizontal vector field Z on P such that $Z(u) = \zeta^{\mathsf{h}}$. Indeed, we can even take Z to be a horizontal lift: let X denote any vector field on M such that $X(\pi(u)) = D\pi(u)\zeta$ (such X exists by Problem D.2). Then $\overline{X}(u) = \zeta^{\mathsf{h}}$. Similarly Proposition 40.7 shows that for any $u \in P$ and any $\zeta \in T_uP$ we can find $\xi \in \mathfrak{g}$ such that $Z_\xi(u) = \zeta^{\mathsf{v}}$.

We now define the *curvature* of a connection on a principal bundle. Firstly, we define flatness in the same way.

DEFINITION 41.2. A connection Δ is said to be **flat** if Δ is integrable.

The curvature then measures how far away a connection is from being flat.

DEFINITION 41.3. Let $\pi \colon P \to M$ be a principal G-bundle with connection ϖ. The **curvature form** $\Omega \in \Omega^2(P, \mathfrak{g})$ of ϖ is defined by

$$\Omega_u(\zeta_1, \zeta_2) \coloneqq -\varpi_u\big([Z_1, Z_2](u)\big), \qquad p \in P,\ \zeta_1, \zeta_2 \in T_uP$$

where Z_1, Z_2 are any two horizontal vector fields on P such that $Z_i(u) = \zeta_i^{\mathsf{h}}$.

Such lifts exist by Remark 41.1. Of course, it must be proved that Ω is well-defined (i.e. independent of the choice of Z_1 and Z_2) and smooth. The negative sign is consistent with our original Definition 33.11 of the curvature of a connection on a vector bundle.

LEMMA 41.4. *The curvature form Ω is a well-defined horizontal $\mathfrak{g}$-valued 2-form. Moreover the connection is flat if and only if Ω is identically zero.*

Proof. Fix $u \in P$ and $\zeta_1, \zeta_2 \in T_uP$. Suppose Z_1 and Z_2 are any two horizontal vector fields on P such that $Z_i(u) = \zeta_i^{\mathsf{h}}$. Let f be a smooth function on P such that $f(u) = 0$, and let W denote any horizontal vector field on P. Then any $Z \coloneqq Z_1 + fW$ is another horizontal vector field on P such that $Z(u) = \zeta_1^{\mathsf{h}}$; moreover any horizontal vector field which agrees with Z_1 at u is locally a finite sum of vector fields of this form. Then

$$[Z, Z_2](u) = [Z_1, Z_2](u) + f(u)[W, Z_2](u) - Z_2(f)W(u),$$

and thus taking vertical components, we see that

$$[Z, Z_2](u)^{\mathsf{v}} = [Z_1, Z_2](u)^{\mathsf{v}},$$

and thus also

$$\varpi_u([Z, Z_2](u)) = \varpi_u([Z_1, Z_2](u)).$$

A similar argument shows that $\varpi_u([Z_1, Z_2](u))$ is independent of the choice of Z_2 as well. This proves that Ω is well defined. It is then obvious that Ω is smooth and anti-symmetric, and hence defines an element of $\Omega^2(P, \mathfrak{g})$. Moreover Ω is horizontal, since if say, $\zeta_1 \in T_u P$ is a vertical vector, then the horizontal vector field $Z_1 \equiv 0$ satisfies $Z_1(u) = \zeta_1^{\mathsf{h}}$. Finally, since $\Delta = \ker \varpi$, it is clear from the definition that the distribution Δ is integrable if and only if Ω is identically zero. ■

Here is another quantitative way to understand the curvature form. The proof is relegated to Problem Sheet O.

LEMMA 41.5. *Let $\pi \colon P \to M$ denote a principal G-bundle, and let ϖ denote a connection on P with curvature form Ω. Fix $X, Y \in \mathfrak{X}(M)$, and let $\overline{X}$ and $\overline{Y}$ denote their horizontal lifts. Then for any $u \in P$ one has*

$$\overline{[X, Y]}(u) - \left[\overline{X}, \overline{Y}\right](u) = D\tau^u(e)\Big(\Omega_p(\overline{X}(u), \overline{Y}(u))\Big).$$

In fact, the connection form belongs to $\Omega^2_G(P, \mathfrak{g}) \subset \Omega^2(P, \mathfrak{g})$.

THEOREM 41.6 (Properties of the curvature form). *Let $\pi \colon P \to M$ denote a principal bundle, and let ϖ denote a connection on P. Then:*

(i) *The curvature Ω belongs to $\Omega^2_G(P, \mathfrak{g})$.*

(ii) *The two forms ϖ and Ω satisfy* **Cartan's Structure Equation***:*

$$\Omega = d\varpi + \frac{1}{2}[\varpi, \varpi] \tag{41.1}$$

(iii) *The* **Bianchi Identity** *holds:*

$$d\Omega = [\Omega, \varpi]. \tag{41.2}$$

REMARK 41.7. The "Lie bracket" in (41.1) and (41.2) is the one from Example 36.6. This is not a true Lie bracket. Indeed, (36.1) tells us that for 1-forms it is *symmetric* instead of anti-symmetric. This can be seen directly as follows: if Z, W are two vector fields on P then

$$[\varpi, \varpi](Z, W) \overset{\text{def}}{=} [\varpi(Z), \varpi(W)] - [\varpi(W), \varpi(Z)].$$

The Lie bracket on the right-hand side is the (genuine) Lie bracket on $\mathfrak{g}$, and thus is anti-symmetric. Hence

$$[\varpi, \varpi](Z, W) = 2[\varpi(Z), \varpi(W)].$$

This is the reason for the factor of a $\frac{1}{2}$ on the right-hand side of (41.1).

We warn the reader that some textbooks are inconsistent with how $[\cdot, \cdot]$ is defined, and thus sometimes the factor of $\frac{1}{2}$ is incorrectly omitted.

Remark 41.8. The Bianchi Identity (41.2) for connections on principal bundles implies the Bianchi Identity for connections on vector bundles (Theorem 36.21), as you will prove on Problem Sheet O. Meanwhile Cartan's Structure Equation (41.1) is the principal bundle version of Theorem 35.10 – see Proposition 41.14 below.

The proof of Theorem 41.6 requires a preliminary lemma. We say a vector field Z is **τ-invariant** if $(\tau_g)_* Z = Z$ for every $g \in G$, that is:

$$D\tau_g(u)Z(u) = Z(\tau_g(u)), \qquad \forall\, u \in P,\ g \in G.$$

Lemma 41.9. *Let $\pi\colon P \to M$ be a principal bundle with connection ϖ. Then:*

(i) *If X is a vector field on M then the horizontal lift $\overline{X}$ of X is right-invariant.*

(ii) *If Z is a horizontal vector field on P then $[Z_\xi, Z]$ is also horizontal for any $\xi \in \mathfrak{g}$.*

(iii) *If Z is a τ-invariant vector field on P then $[Z_\xi, Z] = 0$ for any $\xi \in \mathfrak{g}$,*

Proof. To prove (i) we take $u \in P$ and $g \in G$. Since $\pi \circ \tau_g = \pi$, we have

$$\begin{aligned} D\pi(\tau_g(u))\big(D\tau_g(u)\overline{X}(u)\big) &= D\pi(u)\overline{X}(u) \\ &= X(\pi(u)) \\ &= X(\pi(\tau_g(u))) \\ &= D\pi(\tau_g(u))\overline{X}(\tau_g(u)) \end{aligned}$$

Since $D\pi(\tau_g(u))|_{\Delta_{\tau_g(u)}}$ is a linear isomorphism, we must have

$$D\tau_g(u)\overline{X}(u) = \overline{X}(\tau_g(u)).$$

Thus $\overline{X}$ is right-invariant, as claimed.

To prove (ii), we recall from Proposition 40.4 that the flow of Z_ξ is given by $\Phi_t(u) := \tau_{\exp(t\xi)}(u)$. Thus using Theorem 10.4 we have

$$\begin{aligned} [Z_\xi, Z](u) &= (\mathcal{L}_{Z_\xi} Z)(u) \\ &= \lim_{t\to 0} \frac{D\tau_{\exp(-t\xi)}(\tau_{\exp(t\xi)}(u))Z(\tau_{\exp(t\xi)}(u)) - Z(u)}{t}. \end{aligned}$$

Since τ preserves Δ and Z is horizontal, the numerator of the last equation belongs to Δ_u for all t. Thus also $[Z_\xi, Z](u) \in \Delta_u$.

Finally to prove (iii), if Z is right-invariant then the numerator above is identically zero, and thus $[Z_\xi, Z]$ is too. ∎

Proof of Theorem 41.6. We will prove the result in three steps.

1. In this step we prove Cartan's Structure Equation (41.1). This means that for any two vector fields Z, W on P we must show that

$$\Omega(Z, W) = d\varpi(Z, W) + [\varpi(Z), \varpi(W)] \tag{41.3}$$

as functions $P \to \mathfrak{g}$ (cf. Remark 41.7). Since both sides of (41.3) are point operators, it suffices to consider separately the three cases where one or both Z and W are horizontal or vertical respectively. By Remark 41.1, this in turn reduces to the case where Z and W are horizontal lifts, respectively fundamental vector fields. So let X, Y denote two vector fields on M and let $\xi, \zeta \in \mathfrak{g}$.

(i) *The case $Z = Z_\xi$ and $W = Z_\zeta$ (both sides vertical):*

In this case $\Omega(Z_\xi, Z_\zeta) = 0$ as Ω is horizontal. To compute the left-hand side we first start with:

$$\begin{aligned} d\varpi(Z_\xi, Z_\zeta) &= Z_\xi(\varpi(Z_\zeta)) - Z_\zeta(\varpi(Z_\xi)) - \varpi([Z_\xi, Z_\zeta]) \\ &= d(\varpi(Z_\zeta))Z_\xi - d(\varpi(Z_\xi))Z_\zeta - \varpi(Z_{[\xi,\zeta]}) \\ &= 0 - 0 - [\xi, \zeta], \end{aligned}$$

where the first line used Theorem 36.7, the second line used Problem O.2, and the third line used the fact that $\varpi(Z_\zeta)$ is the constant function $u \mapsto \zeta$ by (40.2), and thus $d(\varpi(Z_\zeta))$ is identically zero. Since

$$[\varpi(Z_\xi), \varpi(Z_\zeta)] = [\xi, \zeta]$$

by (40.2) again, this shows that the right-hand side of (41.3) is also identically zero.

(ii) *The case $Z = Z_\xi$ and $W = \overline{Y}$ (one side vertical, one side horizontal):* As before we have $\Omega(Z_\xi, \overline{Y}) = 0$ as $Z_\xi^{\mathfrak{h}} = 0$. Moreover by part (i) and part (ii) of Lemma 41.9 we have $[Z_\xi, \overline{Y}] = 0$, and thus

$$\begin{aligned} d\varpi(Z_\xi, \overline{Y}) &= Z_\xi(\underbrace{\varpi(\overline{Y})}_{=0}) - \overline{Y}(\varpi(Z_\xi)) - \varpi(\underbrace{[Z_\xi, \overline{Y}]}_{=0}) \\ &= 0 - d(\varpi(Z_\xi))\overline{Y} - 0 \\ &= 0 \end{aligned}$$

Similarly $[\varpi(Z_\xi), \varpi(\overline{Y})] = 0$ as $\varpi(\overline{Y}) = 0$. This proves (41.3) in this case too.

(iii) *The case $Z = \overline{X}$ and $W = \overline{Y}$ (both sides horizontal):*

In this case we have by

$$\begin{aligned} \Omega(\overline{X}, \overline{Y}) &= -\varpi([\overline{X}, \overline{Y}]) \\ &= d\varpi(\overline{X}, \overline{Y}) - \overline{X}(\varpi(\overline{Y})) + \overline{Y}(\varpi(\overline{X})) \\ &= d\varpi(\overline{X}, \overline{Y}) \\ &= d\varpi(\overline{X}, \overline{Y}) + [\varpi(\overline{X}), \varpi(\overline{Y})] \end{aligned}$$

where the second line used the Theorem 36.7 again and the last two lines used $\varpi(\overline{X}) = \varpi(\overline{Y}) = 0$. This proves (41.3) in this case, and hence in general.

2. In this step we prove the Bianchi Identity (41.2). For this we argue as follows:

$$\begin{aligned} d\Omega &\overset{(\dagger)}{=} d^2\varpi + \frac{1}{2}d[\varpi,\varpi] \\ &\overset{(\ddagger)}{=} \frac{1}{2}\left([d\varpi,\varpi] - [\varpi, d\varpi]\right) \\ &\overset{(\wr)}{=} [d\varpi,\varpi] \\ &\overset{(\dagger)}{=} [\Omega,\varpi] - \frac{1}{2}[[\varpi,\varpi],\varpi] \\ &\overset{(*)}{=} [\Omega,\varpi] \end{aligned}$$

where (†) used the Cartan Structure Equation (both times), (‡) used Problem O.1, (≀) used (36.1), and finally (∗) used Problem O.3. This proves the Bianchi Identity.

3. To complete the proof we show that Ω is equivariant, and hence defines an element of $\Omega^2_G(P,\mathfrak{g})$. For this we use Cartan's Structure Equation and the fact that ϖ is equivariant:

$$\begin{aligned} \tau_g^*\Omega &= \tau_g^*\left(d\varpi + \frac{1}{2}[\varpi,\varpi]\right) \\ &= \mathrm{Ad}_{g^{-1}}(d\varpi) + \frac{1}{2}\left[\mathrm{Ad}_{g^{-1}}(\varpi), \mathrm{Ad}_{g^{-1}}(\varpi)\right] \\ &= \mathrm{Ad}_{g^{-1}}(\Omega), \end{aligned}$$

where the second equality used the fact that $d\varpi$ is also equivariant (see claim (ii) from our third proof of Theorem 39.9). This finally completes the proof of the theorem. ■

Since the curvature belongs to $\Omega^2_G(P,\mathfrak{g})$, by Theorem 39.3 we can interpret it also as a bundle-valued form on M. But which bundle?

Definition 41.10. Let $\pi\colon P \to M$ denote a principal G-bundle. We denote by $\mathrm{Ad}(P) = P \times_G \mathfrak{g}$ the vector bundle over M corresponding to $\sigma = \mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ and call this the **adjoint bundle** of P.

Corollary 41.11. *Let $\pi\colon P \to M$ be a principal G-bundle and let ϖ denote a connection on P. Then the curvature Ω induces a bundle-valued 2-form $\breve{\Omega} \in \Omega^2(M, \mathrm{Ad}(P))$. Explicitly, for $p \in M$ and for $\xi_1, \xi_2 \in T_pM$, one has*

$$\breve{\Omega}_p(\xi_1,\xi_2) := [u, \Omega_u(\zeta_1,\zeta_2)]\,,$$

where u is any element in P_p and $\zeta_1, \zeta_2 \in T_uP$ are any two vectors such that $D\pi(u)\zeta_i = \xi_i$.

Proof. Apply Theorem 39.3 to Ω. ■

Let $\pi\colon P \to M$ denote a principal G-bundle, and let σ denote a representation of G on a vector space V. We have seen that a connection ϖ on a principal bundle P induces a connection ∇ on the associated bundle $E = P \times_G V$; moreover the third proof of Theorem 39.9 gave an explicit formula (39.8) for computing the covariant differential operator d^∇. We conclude this lecture by relating the curvature form Ω of ϖ to the curvature R^∇ of ∇. For this we first need to give another version of the formula (39.8) that explicitly uses the connection form ϖ.

The starting point for this discussion is the observation that the differential

$$\mu := D\sigma(e) \colon \mathfrak{g} \to \mathfrak{gl}(V), \qquad \xi \mapsto \mu_\xi$$

of σ is a Lie algebra representation of $\mathfrak{g}$ (this is a special case of Proposition 11.7). For example, if $\sigma = \mathrm{Ad}$ is the adjoint representation of G on $\mathfrak{g}$ then $\mu = \mathrm{ad}$.

The following lemma is on Problem Sheet O.

LEMMA 41.12. *Let $\pi \colon P \to M$ denote a principal G-bundle, and let $\sigma \colon G \to \mathrm{GL}(V)$ denote a smooth effective representation of G. Let $\mu := D\sigma(e)$, and suppose $f \colon P \to V$ is an equivariant smooth function. Then for any $\xi \in \mathfrak{g}$, one has*

$$Z_\xi(f) + \mu_\xi(f) = 0.$$

We can combine μ and ϖ to produce a 1-form out of a smooth (not necessarily equivariant) function $f \colon P \to V$. Namely, the map

$$\zeta \in T_u P \mapsto \mu_{\varpi_u(\zeta)}(f(u)) \in V$$

defines an element of $\Omega^1(P, V)$, which by a slight abuse of notation we denote by $\mu_\varpi(f)$.

PROPOSITION 41.13. *If $s \in \Gamma(E)$ corresponds to an equivariant function $f \colon P \to V$ (Corollary 39.4) then*

$$\nabla s = \check{\beta}, \qquad \text{where } \beta := df + \mu_\varpi(f)$$

Thus if $X \in \mathfrak{X}(M)$, the section $\nabla_X s$ of E corresponds to the equivariant function $\overline{X}(f)$, where $\overline{X}$ is the horizontal lift of X.

Proposition 41.13 is the special case $k = 0$ of a more general result that expresses the exterior covariant differential d^∇ in terms of ϖ. This result is somewhat technical (and messy) to state, and hence it is deferred to the bonus section below.

Next, observe that the representation μ induces a vector bundle homomorphism

$$\Phi = \Phi_\mu \colon \mathrm{Ad}(P) \to \mathrm{End}(E),$$

given explicitly by

$$\Phi_{[u,\xi]}\big([u, v]\big) := [u, \mu_\xi(v)], \qquad u \in P,\ \xi \in \mathfrak{g},\ v \in V.$$

This in turn induces a $C^\infty(M)$-linear map $\Phi_* \colon \Gamma(\mathrm{Ad}(P)) \to \Gamma(\mathrm{End}(E))$ by

$$\Phi_*(s)(p) := \Phi(s(p)), \qquad p \in M.$$

This is the easy direction of the Hom-Gamma Theorem 20.25.

Finally, we can also think of Φ_* as defining a map

$$\Phi_* \colon \Omega^k(M, \mathrm{Ad}(P)) \to \Omega^k(M, \mathrm{End}(E)),$$

by declaring that on a decomposable element $\omega \otimes s$, one has

$$\Phi_*(\omega \otimes s) := \omega \otimes \Phi_*(s). \tag{41.4}$$

With all of this notation out of the way, here is our desired result.

PROPOSITION 41.14. *Let $\pi \colon P \to M$ denote a principal G-bundle, and let $\sigma \colon G \to \mathrm{GL}(V)$ denote a smooth effective representation of G. Let ϖ denote a connection on P, and let ∇ denote the induced connection on E. Let Ω denote the curvature form of ϖ, and consider $\breve{\Omega} \in \Omega^2(M, \mathrm{Ad}(P))$ as in Corollary 41.11. Then with Φ_* defined as in (41.4), one has*

$$\Phi_*(\breve{\Omega}) = R^\nabla.$$

Proof. In this proof we will suppress the bijection of Corollary 39.4 between sections of E and equivariant functions $f \colon P \to V$, and treat it as an identification. Thus we write $s = f$ to indicate that a section s corresponds to f. Thus Proposition 41.13 can be stated more succinctly as

$$\nabla_X s = \overline{X}(f).$$

This will help keep the notation transparent. With this convention in mind, by Theorem 35.10 we have

$$R^\nabla(X,Y)(s) = \left[\overline{X}, \overline{Y}\right](f) - \overline{[X,Y]}(f).$$

By Lemma 41.5 and (40.1), one has

$$\left(\left[\overline{X}, \overline{Y}\right] - \overline{[X,Y]}\right)(u) = -Z_{\Omega_u\left(\overline{X}(u), \overline{Y}(u)\right)}(u)$$

Thus applying Lemma 41.12 we obtain

This is where it is crucial we defined Ω with a negative sign.

$$R^\nabla(X,Y)(s) = \mu_{\Omega\left(\overline{X},\overline{Y}\right)}(f),$$

which – after unravelling the notation – is exactly what we wanted to prove. ∎

Bonus Material for Lecture 41

First, a general definition. Let V and W by vector spaces and suppose $\mu \colon W \to \mathrm{GL}(V)$ is a representation.

DEFINITION 41.15. Suppose $\alpha \in \Omega^h(P, W)$ and $\beta \in \Omega^k(P, V)$ are vector-valued forms on W and V respectively. We define $\alpha \wedge_\mu \beta \in \Omega^{h+k}(P, V)$ by

$$\begin{aligned}(\alpha \wedge_\mu \beta)_u(\zeta_1, \dots, \zeta_{h+k}) \\ := \frac{1}{h!k!} \sum_{\varrho \in \mathfrak{S}_{h+k}} \mathrm{sgn}(\varrho) \mu_{\alpha_u(\zeta_{\varrho(1)}, \dots, \zeta_{\varrho(h)})}\left(\beta_u\left(\zeta_{\varrho(r+1)}, \dots \zeta_{\varrho(r+s)}\right)\right).\end{aligned} \tag{41.5}$$

This is similar (but not quite the same) as the construction of the wedge product $\wedge_\beta$ in Lecture 26.

Going back to the setting of Proposition 41.13, we are interested in the case $W = \mathfrak{g}$ and $\mu = D\sigma(e)$. Here is the main result of this bonus section.

THEOREM 41.16. *Let $\pi\colon P \to M$ denote a principal G-bundle, and let $\sigma\colon G \to \mathrm{GL}(V)$ denote a smooth effective representation of G. Abbreviate by $E = P \times_G V$ the associated vector bundle. Suppose ϖ is a connection on P, and let d^∇ denote the corresponding exterior covariant differential on E. Then for any $\alpha \in \Omega(M, E)$ we have*

$$d^\nabla \alpha = \check{\beta},$$

where $\beta := d\widehat{\alpha} + \varpi \wedge_\mu \widehat{\alpha}$.

As the proof will show, β is horizontal and equivariant, and hence $\check{\beta}$ is well-defined.

Proposition 41.13 is the special case $k = 0$ of Theorem 41.16:

Proof of Proposition 41.13. If $f \in \Omega^0(P, V)$ is a zero-form, (41.5) simplifies to

$$\varpi \wedge_\mu f = \mu_\varpi(f).$$

Moreover if $X \in \mathfrak{X}(M)$ then $\nabla_X s$ is the section of E corresponding to the equivariant function

$$df(Z) + \mu_{\varpi(Z)}(f),$$

where Z is any vector field on P such that $D\pi(Z) = X$ (this is independent of the choice of Z by equivariance). In particular, choosing $Z = \overline{X}$, the second term disappears, and thus $\nabla_X s$ corresponds to $\overline{X}(f)$, as required. ■

Proof of Theorem 41.16. We will prove the theorem in three steps.

1. In this step we set up notation and outline the strategy of the proof. Suppose $\alpha \in \Omega^k(M, E)$. By the definition (cf. (39.8)) of d^∇, we have

$$d^\nabla \alpha := \widecheck{(d\widehat{\alpha})^{\mathsf{h}}}.$$

Since the $\widehat{\cdot} \to \check{\cdot}$ correspondence is bijective, it suffices to show that if $\beta := \widehat{\alpha} \in \Omega^k_G(P, V)$ then for all $u \in P$ and all $\zeta_0, \dots, \zeta_k \in T_u P$, we have:

In the sum below, we think of elements $\varrho \in \mathfrak{S}_{k+1}$ as permutations of $\{0, 1, \dots, r\}$ (instead of the more usual $\{1, 2, \dots, r+1\}$.

$$d\beta_u\big(\zeta_0^{\mathsf{h}}, \dots, \zeta_k^{\mathsf{h}}\big) = d\beta_u\big(\zeta_0, \dots, \zeta_k\big) + \frac{1}{k!} \sum_{\varrho \in \mathfrak{S}_{k+1}} \operatorname{sgn}(\varrho) \mu_{\varpi_u(\zeta_{\varrho(0)})}\Big(\beta\big(\zeta_{\varrho(1)}, \dots \zeta_{\varrho(k)}\big)\Big). \tag{41.6}$$

Since both sides of (41.6) are linear in all variables, as in the proof of Cartan's Structure Equation (41.3) we may assume that each ζ_i is either vertical or horizontal. Moreover by Remark 41.1 we may assume $\zeta_i = Z_i(u)$ for some vector field Z_i on P, which is either of the form $Z_i = \overline{X}_i$ for some vector field X_i on M (if ζ_i is horizontal) or of the form $Z_i = Z_{\xi_i}$ for some $\xi_i \in \mathfrak{g}$ (if ζ_i is vertical). Define functions $\Psi_0, \Psi_1, \Psi_2\colon P \to V$ by

$$\Psi_0(u) := d\beta_u\big(Z_0(u)^{\mathsf{h}}, \dots, Z_k(u)^{\mathsf{h}}\big),$$

and

$$\Psi_1(u) \coloneqq d\beta_u\big(Z_0(u), \dots, Z_k(u)\big),$$

and

$$\Psi_2(u) \coloneqq \frac{1}{k!} \sum_{\varrho \in \mathfrak{S}_{k+1}} \operatorname{sgn}(\varrho) \mu_{\varpi_u(Z_{\varrho(0)}(u))} \Big(\beta\big(Z_{\varrho(1)}(u), \dots Z_{\varrho(k)}(u)\big)\Big).$$

It suffices to show that

$$\Psi_0 = \Psi_1 + \Psi_2, \qquad \text{as functions } P \to V.$$

2. In this step we deal with the two easy cases. If every single Z_i is horizontal then $\Psi_0 = \Psi_1$ by definition, and $\Psi_2 = 0$ since $\varpi(Z_i) = 0$ for every i. Next, suppose two or more of the Z_i are vertical. In this case without loss of generality we may assume $Z_0 = Z_{\xi_0}$ and $Z_1 = Z_{\xi_1}$ are vertical. In this case we again have $\Psi_0 = 0$, since $Z_0^{\mathsf{h}} = Z_1^{\mathsf{h}} = 0$. Also $\Psi_2 = 0$ as at least one of the arguments $Z_{\varrho(i)}$ for $i = 0, \dots, k$ is vertical and $\beta = \widehat{\alpha}$ is horizontal. Thus we need only show that $\Psi_1 = 0$. By Theorem 36.7 we have

$$\begin{aligned} \Psi_1 = &\sum_{i=0}^{k} (-1)^i Z_i\big(\beta(Z_0, \dots, \widehat{Z}_i, \dots, Z_k)\big) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \beta([Z_i, Z_j], Z_0, \dots \widehat{Z}_i, \dots, \widehat{Z}_j, \dots, Z_k). \end{aligned} \tag{41.7}$$

Every term in the first summand is zero, since at least one of the arguments is zero. The only term in the second summand that could possibly be non-zero is $i = 0$ and $j = 1$. But in this case by Problem O.2, $[Z_0, Z_1] = Z_{[\xi_0, \xi_1]}$ is also vertical, and hence this term is zero too.

3. In this step we deal with the hardest case, where exactly one of the Z_i is vertical. Without loss of generality assume that $Z_0 = Z_\xi$ and that $Z_i = \overline{X}_i$ for vector fields X_i on M for $i = 1, \dots, k$. As before, $\Psi_0 = 0$ (since $Z_0^{\mathsf{h}} = 0$). Now in (41.7) some of the terms survive, and we get

$$\Psi_1 = Z_\xi\big(\beta(Z_1, \dots, Z_k)\big) + \sum_{i=1}^{k} (-1)^i \beta([Z_\xi, Z_i], Z_1, \dots \widehat{Z}_i, \dots, Z_k).$$

But actually by part (iii) of Lemma 41.9, we have $[Z_\xi, Z_i] = [Z_\xi, \overline{X}_i] = 0$, and thus $\Psi_1 = Z_\xi\big(\beta(Z_1, \dots, Z_k)\big)$. Now if we look at Ψ_2, all the terms die apart from those permutations ϱ such that $\varrho(0) = 0$. Since $\varpi(Z_\xi) = \xi$, it follows that

$$\begin{aligned} \Psi_2 &= \frac{1}{k!} \sum_{\varrho \in \mathfrak{S}_{k+1} \text{ with } \varrho(0)=0} \operatorname{sgn}(\varrho) \mu_\xi \Big(\beta\big(Z_{\varrho(1)}, \dots Z_{\varrho(k)}\big)\Big) \\ &= \mu_\xi \left(\frac{1}{k!} \sum_{\varrho \in \mathfrak{S}_{r+1} \text{ with } \varrho(0)=0} \operatorname{sgn}(\varrho) \beta\big(Z_{\varrho(1)}, \dots Z_{\varrho(k)}\big) \right) \\ &= \mu_\xi\big(\beta(Z_1, \dots, Z_k)\big). \end{aligned}$$

Thus to complete the proof we need to show that

$$Z_\xi(\beta(Z_1, \dots, Z_k)) + \mu(v)[\beta(Z_1, \dots, Z_k)] = 0. \tag{41.8}$$

But since $Z_i = \overline{X}_i$ is right-invariant for each i and $\beta \in \Omega^k_G(P, V)$ is equivariant, it follows that $f \coloneqq \beta(Z_1, \ldots, Z_k)$ is itself equivariant. Thus (41.8) follows from Lemma 41.12. ■

LECTURE 42

The Ambrose–Singer Holonomy Theorem

In this lecture we define holonomy in principal bundles, and prove the principal bundle version of the Ambrose–Singer Holonomy Theorem. The vector bundle version (Theorem 35.6) is a simple corollary of the principal version, as you will prove on Problem Sheet O.

DEFINITION 42.1. Let $\pi \colon P \to M$ be a principal G-bundle with connection ϖ. The **holonomy group** $\mathrm{Hol}^{\varpi}(p)$ of ϖ at $p \in M$ is the group of equivariant diffeomorphisms of the fibre P_p of the form $\mathbb{P}_\gamma$, where γ is a piecewise smooth loop in M based at p. The **restricted holonomy group** $\mathrm{Hol}^{\varpi}_0(p) \subset \mathrm{Hol}^{\varpi}(p)$ is the subgroup consisting of parallel transport around null-homotopic loops γ.

The following result is the principal bundle analogue of Proposition 32.15. The key difference is that we can view the holonomy group $\mathrm{Hol}^{\varpi}(p)$ as being a subgroup of G itself.

PROPOSITION 42.2. *Let $\pi \colon P \to M$ be a principal G-bundle with connection ϖ.*

(i) *For each $u \in P$, there is a subgroup $H^{\varpi}(u) \subset G$ and a group isomorphism*
$$\phi_u \colon \mathrm{Hol}^{\varpi}(\pi(u)) \to H^{\varpi}(u).$$

(ii) *The subgroups $H^{\varpi}(u)$ and $H^{\varpi}(\tau_g(u))$ are conjugate in G.*

(iii) *The subgroups $H^{\varpi}(u)$ and $H^{\varpi}(\mathbb{P}_\gamma(u))$ coincide*

(iv) *There is a subgroup $H^{\varpi}_0(u) \subset H^{\varpi}(u)$ such that ϕ_u restricts to define an isomorphism $\mathrm{Hol}^{\varpi}_0(p) \to H^{\varpi}_0(u)$. This subgroup also satisfies the assertions of part (ii) and (iii).*

Proof. Let $p \in M$ and $u \in P_p$. If γ is a piecewise smooth loop based at p, we define $\phi_u\big(\mathbb{P}_\gamma\big)$ to be the unique element $g \in G$ such that
$$\tau_g(u) = \mathbb{P}_\gamma(u).$$
We set $H^{\varpi}(u)$ to be the image of ϕ_u. Suppose γ and δ are two piecewise smooth loops based at p, and set
$$g := \phi_u(\mathbb{P}_\gamma), \qquad \text{and} \qquad h := \phi_u(\mathbb{P}_\delta(u)).$$
Then by equivariance (Axiom (i) of Definition 39.7), we have
$$\begin{aligned} \mathbb{P}_\gamma \circ \mathbb{P}_\delta(u) &= \mathbb{P}_\gamma(\tau_h(u)) \\ &= \tau_h(\mathbb{P}_\gamma(u)) \\ &= \tau_h(\tau_g(u)) \\ &= \tau_{gh}(u). \end{aligned}$$

This shows that

$$\phi_u(\mathbb{P}_\gamma \circ \mathbb{P}_\delta) = gh = \phi_u(\mathbb{P}_\gamma) \circ \phi_u(\mathbb{P}_\delta).$$

Thus $H^\varpi(u)$ is a subgroup of G and that ϕ_u is a group homomorphism. This proves (i). Next, using the equivariance axiom again, if $\phi_u(\mathbb{P}_\gamma) = g$ then

$$\begin{aligned}\tau_{h^{-1}gh}(\tau_h(u)) &= \tau_{gh}(u) \\ &= \tau_h(\tau_g(u)) \\ &= \tau_h(\mathbb{P}_\gamma(u)) \\ &= \mathbb{P}_\gamma(\tau_h(u)),\end{aligned}$$

so that

$$\phi_{\tau_h(u)}(\mathbb{P}_\gamma) = h^{-1}gh.$$

Thus

$$H^\varpi(\tau_h(u)) = h^{-1}H^\varpi(u)h,$$

which proves (ii). To prove (iii), let $\gamma\colon [0,1] \to M$ be a path in M from $p := \gamma(0)$ to $q := \gamma(1)$. Let $u \in P_p$ and set $v := \mathbb{P}_\gamma(u)$. We claim that $H^\varpi(u) \subseteq H^\varpi(v)$. Indeed, suppose $g \in H^\varpi(u)$. Then there exists a piecewise smooth loop δ based at p such that

$$\mathbb{P}_\delta(u) = \tau_g(u).$$

Then $\gamma^- * \delta * \gamma$ is a loop based at q, and

$$\begin{aligned}\mathbb{P}_{\gamma^- * \delta * \gamma}(v) &= \mathbb{P}_\gamma \circ \mathbb{P}_\delta \circ \mathbb{P}_{\gamma^-}(v) \\ &= \mathbb{P}_\gamma \circ \mathbb{P}_\delta(u) \\ &= \mathbb{P}_\gamma(\tau_g(u)) \\ &= \tau_g(\mathbb{P}_\gamma(u)) \\ &= \tau_g(v).\end{aligned}$$

Thus $g \in H^\varpi(v)$. Applying the same argument with γ^- in place of γ shows that $H^\varpi(v) \subseteq H^\varpi(u)$, and thus $H^\varpi(u) = H^\varpi(v)$. This proves (iii). Finally, (iv) is proved in the same way, and we leave the details as an exercise. ■

The holonomy groups $H^\varpi(u) \subset G$ enjoy the same properties that the holonomy groups did for vector bundles. The next theorem summarises the key properties we will need. The proofs all proceed analogously to the corresponding statements about vector bundles.

THEOREM 42.3. *Let $\pi\colon P \to M$ be a principal G-bundle with connection ϖ. Then the holonomy group $H^\varpi(u)$ is a Lie subgroup of G. The connected component of $H^\varpi(u)$ containing the identity is exactly $H_0^\varpi(u)$. If M is simply connected then $H^\varpi(u) = H_0^\varpi(u)$. Finally, $H^\varpi(u)$ is the trivial subgroup $\{e\}$ for all $u \in P$ if and only if P is a trivial bundle and ϖ is the trivial connection.*

Meanwhile the proof of the next result is on Problem Sheet O.

Proposition 42.4. *Let $\pi\colon P \to M$ be a principal G-bundle. Let σ be a representation of G on a vector space V, and let $E = P \times_G V$ denote the associated vector bundle. Let ϖ denote a connection on P and let ∇ denote the associated connection on E. Fix $p \in M$. Then we can regard $\mathrm{Hol}^{\varpi}(p)$ and $\mathrm{Hol}^{\nabla}(p)$ as subgroups of G and $\mathrm{GL}(V)$ respectively, which are defined up to conjugation. Then (also up to conjugation)*

$$\sigma\big(\mathrm{Hol}^{\varpi}(p)\big) = \mathrm{Hol}^{\nabla}(p).$$

We now define the principal bundle version of Definition 34.2. This makes use of the notion of principal subbundles from Definition 17.21.

Definition 42.5. Let $\pi\colon P \to M$ be a principal G-bundle, let $H \subset G$ be a Lie subgroup and suppose $Q \subset P$ is a principal H-subbundle of P. A connection ϖ on P with associated distribution Δ is said to be **reducible** to Q if the distribution $\Delta \cap TQ$ defines a connection on Q. Equivalently, if $\iota\colon Q \hookrightarrow P$ denotes the inclusion this means that $\varpi|_Q = \iota^*\varpi$ is a connection form on Q with curvature $\Omega|_Q = \iota^*\Omega$.

Compare Problem M.6.

On Problem Sheet O you will investigate the relationship between this definition and the notion of a G-connection on a vector bundle (cf. Problem N.3). The next result is similar to Theorem 34.5.

Theorem 42.6 (The Reduction Theorem). *Let $\pi\colon P \to M$ be a principal G-bundle over a connected manifold M and let ϖ denote a connection on P. Fix a point $u_0 \in P$ and set $H := H^{\varpi}(u_0) \subset G$. Let Q denote the set of all points $u \in P$ which can be joined to u_0 via a piecewise smooth horizontal path. Then Q is a principal H-subbundle of P, and the connection ϖ is reducible to Q.*

i.e. $u = \mathbb{P}_\gamma(u_0)$ for some γ.

The proof of the Reduction Theorem is an application of Problem G.10. Since Problem G.10 was a bonus problem, the proof of the Reduction Theorem is non-examinable and hence relegated to the bonus section below.

We now state and prove the principal bundle version of the Ambrose–Singer Holonomy Theorem. The proof is another application of the Frobenius Theorem 15.4.

Theorem 42.7 (The Ambrose–Singer Holonomy Theorem Redux). *Let $\pi\colon P \to M$ be a principal G-bundle over a connected manifold M. Let ϖ denote a connection on P, and let $\Omega \in \Omega^2_G(P, \mathfrak{g})$ denote the curvature form. Fix $u_0 \in P$ and set $H = H^{\varpi}(u_0) \subset G$. Let Q denote the principal H-subbundle of P from the Reduction Theorem 42.6. Then the Lie algebra $\mathfrak{h}$ of H is the subalgebra of $\mathfrak{g}$ spanned by all elements of the form $\Omega_u(\zeta_1, \zeta_2)$ for $u \in Q$ and $\zeta_1, \zeta_2 \in T_uQ$.*

Proof. Let $\mathfrak{k}$ denote the Lie subalgebra of $\mathfrak{g}$ spanned by elements of the form $\Omega_u(\zeta_1, \zeta_2)$ for $u \in Q$ and $\zeta_1, \zeta_2 \in T_uQ$. Then certainly $\mathfrak{k} \subseteq \mathfrak{h}$. Let $k = \dim \mathfrak{k}$ and $m = \dim M$, so that $\dim \mathfrak{h} = \dim Q - m$. We will show that also $k = \dim Q - m$, which implies $\mathfrak{k} = \mathfrak{h}$.

Define a distribution $\widetilde{\Delta}$ on Q by

$$\widetilde{\Delta}_u := \Delta_u \oplus D\tau^u(e)(\mathfrak{k}),$$

where Δ is the connection distribution. To see that this is indeed a distribution on Q, we argue as in the proof of Step 2 of Theorem 40.9. Take a basis $\{\xi_i \mid i = 1, \dots, k\}$ of $\mathfrak{k}$, and let Z_{ξ_i} denote the fundamental vector fields associated to this basis. Fix $v \in Q$, and let $\{X_j \mid j = 1, \dots, m\}$ denote vector fields on M such that $\{X_j(q)\}$ is a basis of T_qM for all q near $\pi(v)$, and let $\overline{X}_j$ denote the horizontal lifts of X_j. Then $\{Z_{\xi_i}, \overline{X}_j\}$ spans $\widetilde{\Delta}$ near q, and thus $\widetilde{\Delta}$ is indeed a distribution of dimension $m+k$. Next. we claim $\widetilde{\Delta}$ is integrable. Using Lemma 14.10, we need only check:

(i) $[Z_{\xi_i}, Z_{\xi_j}]$ belongs to $\widetilde{\Delta}$.

(ii) $[Z_{\xi_i}, \overline{X}_j]$ belongs to $\widetilde{\Delta}$.

(iii) $[\overline{X}_i, \overline{X}_j]$ belongs to $\widetilde{\Delta}$.

Of these, (i) follows because $[Z_{\xi_i}, Z_{\xi_j}] = Z_{[\xi_i, \xi_j]}$ by Problem O.2 and because $\mathfrak{k}$ is (by definition) a subalgebra. Next, (ii) is immediate, since by part (iii) of Lemma 41.9 such a bracket is always zero. Finally, (iii) follows from Lemma 41.5. Thus by the Frobenius Theorem 15.4, $\widetilde{\Delta}$ induces a foliation of Q. Let L denote the leaf containing u_0. We claim that in fact $L = Q$ (and thus this is not a particularly thrilling foliation). Indeed, if $\rho(t)$ is a horizontal curve starting at u_0 then $\dot{\rho}(t) \in \Delta_{\rho(t)} \subset \widetilde{\Delta}_{\rho(t)}$ for each t, and thus $\operatorname{im}\rho$ is contained in an integral manifold of $\widetilde{\Delta}$. By maximality, $\operatorname{im}\rho$ is also contained in L. Since ρ was arbitrary, this shows that $Q \subseteq L$. Since $L \subseteq Q$ by definition, we have $L = Q$ as claimed. Since $\dim L = m + k$, this shows that $k = \dim Q - m = \dim \mathfrak{h}$. This completes the proof. ∎

We conclude this lecture by stating the following existence result. The proof is not too hard, but it is a long and somewhat uninspiring computation, and hence we will skip it.

See for instance Theorem 8.2 on p90 of Foundations of Differential Geometry Vol I. by Kobayashi and Nomizu.

Theorem 42.8. *Let $\pi\colon P \to M$ denote a principal G-bundle. Assume $\dim M \geq 2$ and that G is connected. Then there exists a connection ϖ on P with $H^\varpi(u) = G$ for all $u \in P$.*

As a corollary, we obtain the following converse to the Reduction Theorem 42.6.

Corollary 42.9. *Let $\pi\colon P \to M$ be a principal G-bundle, where $\dim M \geq 2$. Then for any connected Lie subgroup $H \subset G$, there exists a connection ϖ on P with $H^\varpi(u) = H$ (for some $u \in P$) if and only if P admits a principal H-subbundle.*

Remark 42.10. Corollary 42.9 means that the question as to when a given principal G-bundle admits a connection with holonomy equal to a prescribed subgroup H of G is not very geometrically interesting. Indeed, the existence (or non-existence) of a principal H-subbundle is a purely topological issue, and can be settled using tools from algebraic topology. We will see in Lecture 45 that the situation dramatically changes if we impose the additional condition that our connection is *torsion-free*.

Bonus Material for Lecture 42

In this bonus section we prove the Reduction Theorem 42.6.

Proof of the Reduction Theorem 42.6. The proof is an application of Problem G.10. Part (iii) of Proposition 42.2 tells us that Q is preserved by the action of H, and that the action of H on $P_q \cap Q$ for any point $q \in M$ is transitive. Moreover since M is connected the restriction of π to Q is surjective (compare this to the proof of Step 1 of Theorem 33.9). Thus to show that Q is a principal H-subbundle, by Problem G.10 we need only construct local sections of P that take values in Q.

This is a variation of the argument from the proof of Step 1 of Theorem 30.1. Set $p_0 = \pi(u_0)$ and fix an arbitrary point $p \in M$. Let $\psi\colon TM|_U \to M$ denote an adapted ray parametrisation at p, and write $\gamma_{q,\xi}(t) = \psi(q, t\xi)$ for $q \in U$ and $\xi \in T_qM$. Now for $u \in P_q$ we define a section $s_u \in \Gamma(U, P)$ by declaring that

$$s_u(\gamma_{q,\xi}(t)) = \mathbb{P}_{\gamma_{q,\xi};u}(t).$$

We claim that if $u \in Q_q \subset P_q$ then s_u takes values in Q. Indeed, if $u = \mathbb{P}_\gamma(u_0)$ for some path $\gamma\colon [0,1] \to M$ such that $\gamma(0) = p_0$ and $\gamma(1) := q$ then

$$s_u(\gamma_{q,\xi}(t)) = \mathbb{P}_{\gamma * \gamma^t_{q,\xi}}(u) \in Q,$$

where $\gamma^t_{q,\xi}(r) := \gamma_{q,\xi}(rt)$ for $0 \le r \le 1$.

Finally, to see that the connection is reducible to Q, we observe that by definition any horizontal curve starting in Q must remain in Q, and hence, if Δ denotes the connection distribution of ϖ, one has $\Delta_q \subset T_qQ$. Since clearly $V_qQ = V_qP \cap T_qQ$, it follows that $T_qQ = \Delta_q \oplus V_qQ$. Thus Δ is a preconnection on Q, and the equivariance condition is clear from above. This completes the proof. ■

Will J. Merry

LECTURE 43

Geodesics and Sprays

In this lecture we study *geodesics* and *sprays*. These are concepts normally associated with Riemannian geometry. However, as we will see, they make perfect sense for an arbitrary connection on a manifold. The word "geodesic" needs to be understood carefully however – in this more general setting there is no relation between geodesics and shortest paths, as we explain below.

For the remainder of the course we will almost exclusively work on the tangent bundle TM of a manifold M, rather than an arbitrary vector bundle. Thus we adopt the convention that a **connection on** M is, by definition, a connection on the vector bundle $\pi\colon TM \to M$.

We remind the reader that we (sometimes) write points in TM as pairs (p, ξ): this is simply notation to indicate that $\xi \in T_pM$.

DEFINITION 43.1. We define the **Christoffel symbols** of the chart x and the connection ∇ as

$$\Gamma^k_{ij}(p) := dx^k_p(\nabla_{\partial_i}\partial_j)(p)$$

Thus $\Gamma^k_{ij}\colon U \to \mathbb{R}$ is a smooth function on U.

Here and elsewhere, when there is no danger of confusion, we abbreviate the vector field $\frac{\partial}{\partial x^i}$ by ∂_i.

Pay attention to the indices—the Einstein Summation Convention is very useful here.

LEMMA 43.2. *The connection ∇ is uniquely determined on U by the Christoffel symbols.*

Proof. If X and Y are any two vector fields on U then we can write $X = a^i\,\partial_i$ and $Y = b^j\,\partial_j$ for smooth functions a^i, b^j. Abbreviate

$$\partial_i b^j := \frac{\partial b^j}{\partial x^i} = db^j(\partial_i).$$

Then by the axioms for a covariant derivative operator (Definition 31.6) we have

$$\begin{aligned}\nabla_X Y &= \nabla_{a^i\,\partial_i}(b^j\,\partial_j)\\ &= a^i\nabla_{\partial_i}(b^j\,\partial_j)\\ &= a^i b^j\nabla_{\partial_i}\partial_j + a^i\partial_i b^j\partial_j\\ &= \left(a^i\,b^j\,\Gamma^k_{ij} + a^i\partial_i b^k\right)\partial_k,\end{aligned}$$

where on the last line we replaced the dummy variable j by k. ∎

Lemma 43.2 gives yet another viewpoint on connections: they are determined locally by m^3 (where $m = \dim M$) smooth functions Γ^k_{ij}.

On Problem Sheet P you will investigate how the Christoffel symbols of two charts with overlapping domains are related.

Definition 43.3. Let ∇ be a connection on M. A curve γ in M is called a **geodesic** of ∇ if $\dot{\gamma} \in \Gamma_\gamma(TM)$ is a parallel curve:

$$\nabla^\gamma_T \dot{\gamma} = 0, \tag{43.1}$$

where $T = \frac{\partial}{\partial t}$ and ∇^γ denotes the pullback connection.

Example 43.4. Consider the connection ∇ on S^m introduced in Problem L.3. By Problem M.5, if p, q are two points in S^m such that $p \perp q$ then the great circle $\gamma\colon [0, 2\pi] \to S^m$ defined by $\gamma(t) = (\cos t)p + (\sin t)q$ is a geodesic.

The word "geodesic" was originally used to mean the shortest path between two points on the Earth's surface. As we will see in Lecture **??**, if M is endowed with a Riemannian metric g, and the connection ∇ is the *Levi-Civita* connection (see Theorem 46.1) of (M, g), then M admits a metric d_g (in the sense of point-set topology) for which every geodesic is locally a length-minimising curve. In this lecture however, we are working with arbitrary connections on manifolds, and thus geodesics do *not* need to locally minimise lengths – and indeed, without reference to a specific metric on M the idea of "length-minimising" simply does not make sense!

Geodesics always exist with prescribed initial conditions. The next result is a variation of Proposition 29.9.

Proposition 43.5. *Let ∇ be a connection on M, and let $(p, \xi) \in TM$. There exists a geodesic γ of ∇ in M with initial condition $\gamma(0) = p$ and $\dot{\gamma}(0) = \xi$. Moreover γ is unique up to the domain of definition.*

Proof. Let (U, x) be a chart on M and γ a smooth curve with image in U. Abbreviate $\gamma^i = x^i \circ \gamma$ and $\rho_i := \partial_i \circ \gamma \in \Gamma_\gamma(TM)$. Then $\dot{\gamma} = (\gamma^i)'\partial_i$, and by the chain rule (31.9) for covariant derivative operators, we have

$$(\nabla^\gamma_T \rho_i)(t) = \nabla_{\dot{\gamma}(t)} \partial_i.$$

Thus from Lemma 43.2 we obtain

$$(\nabla^\gamma_T \dot{\gamma})(t) = \Big((\gamma^k)''(t) + (\gamma^i)'(t)(\gamma^j)'(t)\, \Gamma^k_{ij}(\gamma(t)) \Big) \rho_k(t).$$

This means that (43.1) is locally equivalent to the following second-order system of ordinary differential equations:

$$(\gamma^k)'' + (\gamma^i)'(\gamma^j)'\, \Gamma^k_{ij}(\gamma) = 0, \qquad \forall\, 1 \le i, j, k \le m. \tag{43.2}$$

We refer to (43.2) as the **geodesic equation**. The conclusion now follows from standard existence and uniqueness theorems for ordinary differential equations. Note that in general we only get short-term existence (unless $\Gamma^k_{ij} = 0$). ∎

Lemma 43.6. *Let* $\gamma\colon (a,b) \to M$ *be a non-constant geodesic, and let* $h\colon (a_1,b_1) \to (a,b)$ *be a smooth map. Then* $\delta \coloneqq \gamma \circ h\colon (a_1,b_1) \to M$ *is a geodesic if and only if* h *is an affine map, i.e.* $h'' = 0$.

Proof. Using the chain rule for covariant derivative operators (31.9) once more we have

$$\begin{aligned}\nabla^{\delta}_{T}\dot\delta &= \nabla^{\delta}_{T}\big(h'(\dot\gamma \circ h)\big)\\ &= h''\,(\dot\gamma \circ h) + h'\nabla^{\delta}_{T}(\dot\gamma \circ h)\\ &= h''\,(\dot\gamma \circ h) + (h')^2\nabla^{\gamma}_{T}\dot\gamma\\ &= h''\,(\dot\gamma \circ h) + 0.\end{aligned}$$

Since $\dot\gamma$ is non-constant we see that $\nabla^{\delta}_{T}\dot\delta = 0$ if and only if $h'' = 0$. ■

In general it may not be possible to extend a geodesic to be defined on all of $\mathbb{R}$. The following definition is analogous to Definition 9.11.

Definition 43.7. A connection ∇ on a manifold is called **complete** if all geodesics have maximal domain of definition equal to $\mathbb{R}$.

Example 43.8. The connection ∇ on S^n from Problem M.5 is complete. Indeed, by Example 43.4 and Proposition 43.5, we see that any geodesic on S^n is a great circle of the form $\gamma(t) = (\cos t)p + (\sin t)q$ for two perpendicular points on S^m, and moreover any such geodesic may be extended to all of $\mathbb{R}$ by periodicity.

In fact, Definition 43.7 is more than analogous to Definition 9.11 – it is merely a special case, as we shall now see.

Convention. We (sometimes) write elements of TTM as triples (p,ξ,ζ): this is simply notation to indicate that $\zeta \in T_{(p,\xi)}TM$.

In Lemma 31.4 we showed how for a vector bundle $\pi\colon E \to M$, the differential of π gave rise to a new vector bundle $D\pi\colon TE \to TM$. When applied to $E = TM$, this means that we can see the total space of TTM (the tangent bundle of the tangent bundle) as a bundle over TM in two different ways:

(i) $\pi_{TM}\colon TTM \to TM$,

(ii) $D\pi\colon TTM \to TM$.

These are *not* the same structure – they do not even have the same fibre. We temporarily use the notation $T_{\text{(ii)}}TM$ to denote the total space of the bundle (ii). Then the fibre of $T_{\text{(ii)}}TM$ over $(p,\xi) \in TM$ is

$$T_{\text{(ii)}}TM_{(p,\xi)} = \{(p,\zeta,\eta) \mid D\pi(\zeta)\eta = \xi\}\,.$$

Let $\mu_c\colon TM \to TM$ and $\tilde\mu_c\colon TTM \to TTM$ denote scalar multiplication in the fibres in TM and TTM respectively, i.e.

$$\mu_c(p,\xi) \coloneqq (x, c\xi), \qquad \tilde\mu_c(p,\xi,\zeta) \coloneqq (p,\xi,c\zeta). \tag{43.3}$$

This should not be confused with scalar multiplication in the bundle $T_{\text{(ii)}}TM$, which is given by $c \bullet (x,\xi,\zeta) \coloneqq (x, c\,\xi, D\mu_c(\xi)\zeta)$. cf. (31.4)

Definition 43.9. Let M be a manifold. A vector field $\mathbb{S}$ on the tangent bundle TM is called a **spray** on M if the following two conditions hold:

$$D\pi \circ \mathbb{S} = \mathrm{id}_{TM} \tag{43.4}$$

$$\mathbb{S} \circ \mu_c = \tilde{\mu}_c \circ D\mu_c \circ \mathbb{S}. \tag{43.5}$$

Every vector field on TM satisfies (by definition) the section property for the bundle $\pi_{TM} \colon TTM \to TM$. Condition (43.4) is exactly the section property for the bundle $T_{(\mathrm{ii})}TM$. Thus a spray $\mathbb{S}$ is simultaneously a section of both bundles:

$$\mathbb{S} \in \Gamma(TTM) \cap \Gamma(T_{(\mathrm{ii})}TM)$$

We now prove that geodesics can be seen as integral curves of a spray. This theorem can be thought of as the motivation for the second condition (43.4) for a spray.

Theorem 43.10 (From Connections to Sprays). *Let ∇ be a connection on M. There is a unique spray $\mathbb{S}$ on M which is horizontal with respect to ∇. Moreover a curve γ in M is a geodesic if and only if $\dot{\gamma}$ is an integral curve of $\mathbb{S}$.*

The spray $\mathbb{S}$ constructed in Theorem 43.10 is called the **geodesic spray** of the connection ∇. The converse to Theorem 43.10 is also true: if we are given any spray $\mathbb{S}$ then there exists a connection ∇ for which $\mathbb{S}$ is the geodesic spray of ∇. We prove this next lecture as Theorem 44.5.

Caution: This connection is not unique. See the discussion after Theorem 44.5.

Proof. We prove the result in two steps.

1. Let $K \colon TTM \to TM$ denote connection map of ∇. The requirement that $\mathbb{S}$ is horizontal is equivalent to asking that $K \circ \mathbb{S} = 0$. Recall from Lemma 31.3 that $(D\pi, K) \colon T_{(\mathrm{ii})}TM \to TM \oplus TM$ is a vector bundle isomorphism. We can therefore define $\mathbb{S}$ by

$$\mathbb{S}(p, \xi) := (D\pi(p, \xi), K)^{-1} \big((p, \xi), (p, 0_p)\big).$$

Then $\mathbb{S}$ is smooth, since it is the composition

$$\mathbb{S} = (D\pi, K)^{-1}(\mathrm{id}_{TM}, \mathfrak{o} \circ \pi),$$

where $\mathfrak{o} \colon M \to TM$ is the zero section. Moreover $D\pi \circ \mathbb{S} = \mathrm{id}_{TM}$ and since $D\pi|_{\Delta}$ is an isomorphism, we see that $\mathbb{S}(p, \xi)$ is the only horizontal vector which is mapped to (p, ξ) under $D\pi(p, \xi)$. This shows there is at most one horizontal vector field on TM which satisfies the first condition of a spray. Thus if we can prove that $\mathbb{S}$ satisfies (43.5) we will have both existence and uniqueness for $\mathbb{S}$. For this, using Lemma 31.3 again, it suffices to show that

$$D\pi \circ \mathbb{S} \circ \mu_c = D\pi \circ \tilde{\mu}_c \circ D\mu_c \circ \mathbb{S} \tag{43.6}$$

and

$$K \circ \mathbb{S} \circ \mu_c = K \circ \tilde{\mu}_c \circ D\mu_c \circ \mathbb{S}. \tag{43.7}$$

To see (43.6) we observe that since $D\pi$ is a linear map,

$$D\pi \circ \tilde{\mu}_c = \mu_c \circ D\pi.$$

Next, since $\pi \circ \mu_c = \pi$ we have $D\pi \circ D\mu_c = D\pi$ and thus

$$D\pi \circ \tilde{\mu}_c \circ D\mu_c = \mu_c \circ D\pi.$$

Thus if we start with $(p, \xi) \in TM$, the right-hand side of (43.6) is

$$\begin{aligned} D\pi \circ \tilde{\mu}_c \circ D\mu_c \circ \mathbb{S}(p, \xi) &= \mu_c \circ D\pi \circ \mathbb{S}(p, \xi) \\ &= \mu_c(p, \xi) \\ &= (p, c\xi). \end{aligned}$$

Similarly if we feed (p, ξ) to the left-hand side we get $D\pi \circ \mathbb{S}(p, c\,\xi) = (p, c\xi)$, and thus (43.6) is proved. To prove (43.7), we again start from the fact that K is a linear map, and hence

$$K \circ \tilde{\mu}_c = \mu_c \circ K.$$

Moreover K is also a vector bundle morphism along π_{TM} (Theorem 31.5), which means that

cf. the second commutative diagram in (31.5)

$$K \circ D\mu_c = \mu_c \circ K.$$

Thus the right-hand side of (43.7) is equal to

$$\begin{aligned} K \circ \tilde{\mu}_c \circ D\mu_c \circ \mathbb{S} &= \mu_c \circ K \circ D\mu_c \circ \mathbb{S} \\ &= \mu_c \circ \mu_c \circ K \circ \mathbb{S} \\ &= 0 \end{aligned}$$

since $K \circ \mathbb{S} = 0$. Similarly the left-hand side of (43.7) is also zero. This proves that $\mathbb{S}$ is a spray.

2. It remains to show that the geodesics of ∇ are exactly the projections to M of integral curves of $\mathbb{S}$. Let δ be an integral curve of $\mathbb{S}$, and let $\gamma := \pi \circ \delta$. Since $\dot{\delta}$ is a curve in Δ, δ is parallel along γ. But

$$\begin{aligned} \dot{\gamma}(t) &= \frac{d}{ds}\Big|_{s=t} \pi(\delta(s)) \\ &= D\pi(\delta(t))\dot{\delta}'(t) \\ &= D\pi(\delta(t))\mathbb{S}(\delta(t)) \\ &= \delta(t), \end{aligned}$$

and thus $\dot{\gamma}$ is parallel along γ, so that γ is a geodesic. Conversely if γ is a geodesic then if δ denotes the unique integral curve of $\mathbb{S}$ with $\delta(0) = \dot{\gamma}(0)$ then the argument above shows that $\pi \circ \delta$ is another geodesic with the same initial condition as γ, and hence the uniqueness part of Proposition 43.5 shows that $\delta = \dot{\gamma}$. This completes the proof. ∎

We conclude this lecture by defining the *geodesic flow* of a connection.

Definition 43.11. Let ∇ denote a connection on M. The **geodesic flow** of ∇ is the maximal flow Φ_t of the geodesic spray $\mathbb{S}$ of ∇.

In general by Theorem 9.10, the geodesic flow is a smooth map $\Phi\colon \mathcal{D} \to TM$, where $\mathcal{D} \subset \mathbb{R} \times TM$ is an open set containing $\{0\} \times TM$. We have $\mathcal{D} = \mathbb{R} \times TM$ if and only if ∇ is complete. Explicitly, one has

$$\Phi_t(p, \xi) = \big(\gamma_{p,\xi}(t), \dot{\gamma}_{p,\xi}(t)\big),$$

where $\gamma_{p,\xi}$ is the unique geodesic from Proposition 43.5 with initial condition $\gamma_{p,\xi}(0) = p$ and $\dot{\gamma}_{p,\xi}(0) = \xi$.

LECTURE 44

The Exponential Map of a Spray

In this lecture we define the exponential map associated to a spray. We warn the reader this is *not* the same "exponential map" as the one discussed previously for Lie groups. They are however related in some cases, as we briefly explain in the bonus section below.

DEFINITION 44.1. Let $\mathbb{S}$ denote a spray on M with maximal flow $\Phi\colon \mathcal{D} \to TM$. Let $\mathcal{E}_p \subset TM$ denote the set of tangent vectors ξ such that $(1,(p,\xi)) \in \mathcal{D}$, and set $\mathcal{E} = \bigcup_{p\in M} \mathcal{E}_p$. Thus $\mathcal{E}$ consists of those points $(p,\xi) \in TM$ for which the maximal integral curve of $\mathbb{S}$ with initial condition (p,ξ) is defined for at least $t = 1$.

Since $\mathcal{D}$ is open by Theorem 9.10, so is $\mathcal{E}$. Moreover it follows from the second condition (43.5) from the definition of a spray that

$$\mathbb{S}(p, 0_p) = 0_p,$$

and thus 0_p is a fixed point of the flow Φ_t. Thus $\Phi_t(p,\xi)$ is defined for all $t \in \mathbb{R}$. This shows in particular that $\mathcal{E}_p$ is never empty: $(p, 0_p) \in \mathcal{E}_p$.

DEFINITION 44.2. We define the **exponential map** of $\mathbb{S}$ by

$$\exp =\colon \mathcal{E} \to M. \qquad \exp(p,\xi) = \pi(\Phi_1(p,\xi)).$$

Since Φ is smooth (Theorem 9.10), the map exp is smooth. We write

$$\exp_p := \exp|_{\mathcal{E}_p}\colon \mathcal{E}_p \to M.$$

Compare this to Theorem 12.3: there we had to work a bit here to prove smoothness, since we did not start with a vector field $\mathbb{S}$.

THEOREM 44.3 (Properties of the exponential map). *Let $\mathbb{S}$ be a spray on a smooth manifold M with exponential map* $\exp\colon \mathcal{E} \to M$. *Then:*

(i) *For each $p \in M$, $\mathcal{E}_p$ is a star-shaped neighbourhood of 0_p. Moreover if $\xi \in \mathcal{E}_p$ then*

$$\exp(p, t\xi) = \pi \circ \Phi_t(p,\xi), \qquad \forall\, t \in [0,1].$$

(ii) *For each $p \in M$, $\exp_p$ satisfies*

$$D\exp_p(0_p) \circ \mathcal{J}_{0_p} = \mathrm{id}_{T_pM},$$

and so $\exp_p$ has maximal rank m at 0_p. Thus $\exp_p$ maps a neighbourhood of 0_p in T_pM diffeomorphically onto a neighbourhood of $p \in M$.

This shows that the exponential map defines an adapted moving parametrisation about every point of M, cf. Remark 29.17.

(iii) *For every $p \in M$, the map $(\pi, \exp)\colon \mathcal{E} \to M \times M$ has rank $2m$ at 0_p, and thus maps a neighbourhood of 0_p in T_pM diffeomorphically onto a neighbourhood of (p,p) in $M \times M$. Moreover if $\mathfrak{o}\colon M \to TM$ denotes the zero section then there exists a neighbourhood U of $\mathfrak{o}(M)$ such that $(\pi, \exp)$ maps U diffeomorphically onto a neighbourhood of the diagonal in $M \times M$.*

The proof of this statement is non-examinable.

Proof. We prove the theorem in four steps.

1. In this step we prove part (i). Fix $(p,\xi) \in TM$ and let $\delta\colon (t^-,t^+) \to TM$ denote the maximal integral curve of $\mathbb{S}$ with initial condition (p,ξ). Let $\mu_c\colon TM \to TM$ and $\tilde{\mu}_c\colon TTM \to TTM$ denote scalar multiplication in the fibres in TM and TTM respectively (cf. (43.3)). For $c > 0$ we consider the curve

$$\delta_c\colon \left(\frac{t^-}{c}, \frac{t^+}{c}\right) \to TM, \qquad \delta_c(t) := \mu_c \circ \delta(ct).$$

Then

$$\begin{aligned}
\dot{\delta}_c(t) &= D\mu_c(\delta(ct))\big(c\,\dot{\delta}(ct)\big) \\
&= cD\mu_c(\delta(ct))\mathbb{S}(\delta(ct)) \\
&= \tilde{\mu}_c \circ D\mu_c(\delta(ct))\mathbb{S}(\delta(ct)) \\
&= \mathbb{S}(\mu_c(\delta(ct))) \\
&= \mathbb{S}(\delta_c(t)),
\end{aligned}$$

where the penultimate equation used (43.5). Thus δ_c is an integral curve of $\mathbb{S}$. Since $\delta_c(0) = c\,\xi$, it follows from uniqueness of integral curves that

$$\Phi_t(p, c\,\xi) = \mu_c \circ \Phi_{ct}(p,\xi), \qquad \text{for } ct \in (t^-,t^+).$$

In particular, if $\xi \in \mathcal{E}_p$ (so that $t^+ > 1$) then $c\,\xi \in \mathcal{E}_p$ for all $0 \le c \le 1$ and moreover

$$\begin{aligned}
\exp(p, c\,\xi) &= \pi \circ \Phi_1(p, c\,\xi) \\
&= \pi \circ \mu_c \circ \Phi_c(p,\xi) \\
&= \pi \circ \Phi_c(p,\xi).
\end{aligned}$$

This proves part (i).

2. In this step we prove (ii). Using (i), we have:

$$\begin{aligned}
D\exp_p(0_p)\big(\mathcal{J}_{0_p}(\xi)\big) &= \frac{d}{dt}\Big|_{t=0} \exp_p(0_p + t\xi) \\
&= \frac{d}{dt}\Big|_{t=0} \pi \circ \Phi_t(p,\xi) \\
&= D\pi(p,\xi)\mathbb{S}(p,\xi) \\
&= \xi,
\end{aligned}$$

Thus $D\exp_p(0_p) \circ \mathcal{J}_{0_p} = \mathrm{id}_{T_pM}$ as claimed. The Inverse Function Theorem 5.10 then completes the proof of (ii).

3. In this step we investigate the map $(\pi, \exp)$ in local coordinates. Let (U,x) denote a chart on M, and let $(TM|_U, y)$ denote the induced chart on TM, so that

Remember, this part of the proof is non-examinable!

In Theorem 5.6 this chart is denoted by $\tilde{x}$.

$$y^i = \begin{cases} x^i \circ \pi, & 1 \le i \le m, \\ dx^{i-m}, & m+1 \le i \le 2m. \end{cases}$$

Similarly we let $z := (x \circ \mathrm{pr}_1, x \circ \mathrm{pr}_2)$, so that $(U \times U, z)$ is a chart on $M \times M$, with local coordinates

$$z^i = \begin{cases} x^i \circ \mathrm{pr}_1, & 1 \le i \le n, \\ x^{i-m} \circ \mathrm{pr}_2, & m+1 \le i \le 2m. \end{cases}$$

By Lemma 4.4, we have

$$D(\pi,\exp)(0_p)\left(\left.\frac{\partial}{\partial y^j}\right|_{0_p}\right)=\sum_{i=1}^{2m}\left.\frac{\partial}{\partial y^j}\right|_{0_p}(z^i\circ(\pi,\exp))\left.\frac{\partial}{\partial z^i}\right|_{(p,p)}$$

We write the summation signs in this proof to make it clear exactly what index range we are summing over.

For $i\le m$ we have $z^i\circ(\pi,\exp)=x^i\circ\pi=y^i$ and for $i\ge m+1$ we have $z^i\circ(\pi,\exp)=x^{i-m}\circ\exp$. Thus if $1\le j\le m$ we have

$$D(\pi,\exp)(0_p)\left(\left.\frac{\partial}{\partial y^j}\right|_{0_p}\right)=\left.\frac{\partial}{\partial z^j}\right|_{(p,p)}+\sum_{i=1}^{m}\left.\frac{\partial}{\partial y^j}\right|_{0_p}(x^i\circ\exp)\left.\frac{\partial}{\partial z^{i+m}}\right|_{(p,x)} \tag{44.1}$$

meanwhile if $n+1\le j\le 2n$ we have

$$D(\pi,\exp)(0_p)\left(\left.\frac{\partial}{\partial y^j}\right|_{0_p}\right)=\sum_{i=m+1}^{2m}\left.\frac{\partial}{\partial y^j}\right|_{0_p}(x^{i-m}\circ\exp)\left.\frac{\partial}{\partial z^i}\right|_{(p,p)}. \tag{44.2}$$

We now claim that for $m+1\le j\le 2m$ one has

$$D\exp(0_p)\left(\left.\frac{\partial}{\partial y^j}\right|_{0_p}\right)=\left.\frac{\partial}{\partial x^{j-m}}\right|_x. \tag{44.3}$$

Indeed, if $\iota_p\colon T_pM\hookrightarrow TM$ denotes the inclusion then

$$\left.\frac{\partial}{\partial y^j}\right|_{0_p}=D\iota_p(0_p)\circ\mathcal{J}_{0_p}\left(\left.\frac{\partial}{\partial x^{j-m}}\right|_p\right) \tag{44.4}$$

One way to see this is to consider the curve ε_j in T_pM given by

$$\varepsilon_j(t):=\iota_p\left(t\left.\frac{\partial}{\partial x^{j-m}}\right|_x\right).$$

The left-hand side of (44.4) is $\dot\varepsilon_j(0)$ computed using (4.2), and the right-hand side of (44.4) is $\dot\varepsilon_j(0)$ computed using (4.1).

Since $\exp_p=\exp\circ\iota_p$, (44.3) follows from (44.4) and part (ii). Now inserting (44.3) into (44.2) and simplifying tells us that for $m+1\le j\le 2m$ we have

$$D(\pi,\exp)(0_p)\left(\left.\frac{\partial}{\partial y^j}\right|_{0_p}\right)=\left.\frac{\partial}{\partial z^j}\right|_{(p,p)} \tag{44.5}$$

4. We now complete the proof of (iii). By (44.1) and (44.5), the matrix of $D(\pi,\exp)(0_p)$ with respect to the bases $\{\frac{\partial}{\partial y^j}|_{0_p}\}$ and $\{\frac{\partial}{\partial z^i}|_{(p,p)}\}$ is of the form

$$D(\pi,\exp)(0_p)=\begin{pmatrix}\mathrm{id}&0\\ *&\mathrm{id}\end{pmatrix},$$

which has rank $2m$. Thus $(\pi,\exp)$ is a diffeomorphism on a neighbourhood of $(p,x$ by the Inverse Function Theorem 5.10. The final claim that $(\pi,\exp)$ is a diffeomorphism on a neighbourhood the zero-section is a formal point-set topological consequence of what we have already proved. ■

Namely: Suppose X and Y are locally compact, Hausdorff, and paracompact topological spaces and $f\colon X\to Y$ is a local homeomorphism. If $A\subset X$ is any closed set such that $f|_A$ is a homeomorphism then there exists an open set U containing A such that $f|_U$ is also a homeomorphism.

Remark 44.4. We will use part (ii) of Theorem 44.3 many times throughout the rest of the course. The stronger statement given by part (iii) will only be needed once: during our proof that the *injectivity radius* of a compact manifold is positive (see Proposition **??**).

We now prove a converse to Theorem 43.10.

Theorem 44.5 (From Sprays to Connections). *Let M be a smooth manifold and let $\mathbb{S}$ be a spray on M. There exists a connection ∇ on M such that $\mathbb{S}$ is the geodesic spray of ∇.*

Warning: This theorem is *not* asserting that there exists a unique connection ∇ for which $\mathbb{S}$ is the geodesic spray of ∇. In general, there can be many connections with the same geodesics (and hence the same geodesic spray). This annoyance will be rectified in Corollary 45.10 by restricting to torsion-free connections..

Proof. We prove the result in four steps.

1. In this step we define for each $(p,\xi) \in TM$ a subspace $\Delta_{(p,\xi)} \subset T_{(p,\xi)}TM$, which will form our desired connection distribution $\Delta \subset TM$. Fix $p \in M$. For any $\zeta \in T_pM$, the curve

$$\gamma_\zeta(t) := \exp_p(t\zeta).$$

is well-defined on some interval about 0 and satisfies $\gamma_\zeta(0) = p$. Now let $\xi \in T_pM$ denote another tangent vector at p (possibly equal to ζ). We define a section $\rho_{\xi,\zeta} \in \Gamma_{\gamma_\zeta}(TM)$ by

Pay attention to the order of ξ and ζ on the right-hand side!

$$\rho_{\xi,\zeta}(t) := D\exp_p(t\zeta)\big(\mathcal{J}_{t\zeta}(\xi)\big). \tag{44.6}$$

Then by part (ii) of Theorem 44.3, we have

$$\rho_{\xi,\zeta}(0) = \xi,$$

and thus in particular

$$\dot\gamma_\zeta(0) = \rho_{\zeta,\zeta}(0) = \zeta. \tag{44.7}$$

We define our connection Δ by declaring that these sections are all parallel:

$$\Delta_{(p,\xi)} := \{\dot\rho_{\xi,\zeta}(0) \mid \zeta \in T_pM\} \subset T_{(p,\xi)}TM, \tag{44.8}$$

and then set as usual

$$\Delta := \bigsqcup_{(p,\xi)\in TM} \Delta_{(p,\xi)}.$$

2. We now prove that Δ is a preconnection on TM. This proof is similar to the proof of Step 1 of Theorem 30.1, but simpler. Fix $(p,\xi) \in TM$ and consider the smooth map

$$C_\xi \colon \mathcal{E}_p \times [0,1] \to TM, \qquad C_\xi(\zeta,t) := \rho_{\xi,\zeta}(t).$$

Since $C_\xi(\zeta,t) = C_\xi(t\zeta,1)$ we have

$$\begin{aligned}\dot\rho_{\xi,\zeta}(0) &= \frac{d}{dt}\Big|_{t=0} C_\xi(t\zeta,1)\\ &= DC_\xi(0_p,1)\big(\mathcal{J}_{0_p}(\zeta),0\big),\end{aligned}$$

which shows that $\Delta_{(p,\xi)}$ is the image of a linear map $T_pM \to T_{(p,\xi)}TM$, and hence is a vector space of dimension at most m. But since

$$\begin{aligned}D\pi(p,\xi)\dot\rho_{\xi,\zeta}(0) &= \frac{d}{dt}\Big|_{t=0} \pi(\rho_{\xi,\zeta}(t))\\ &= \dot\gamma_\zeta(0)\\ &= \zeta\end{aligned}$$

by (44.7), we see that $D\pi(p,\xi)$ maps $\Delta_{(p,\xi)}$ surjectively onto T_pM. Thus $\Delta_{(p,\xi)}$ is a vector space of dimension m which is mapped isomorphically onto T_pM by $D\pi(p,\xi)$. The construction of vector bundle charts for Δ is similar (but again, easier) to Step 3 of Theorem 30.1, and we omit the details. We have thus proven that Δ is a preconnection.

3. In this step we show that Δ is a genuine connection. Let $\mu_c\colon (p,\xi) \mapsto (p, c\xi)$ denote the usual scalar multiplication on TM. We compute:

$$\begin{aligned} D\mu_c(p,\xi)\dot\rho_{\xi,\zeta}(0) &= \frac{d}{dt}\Big|_{t=0}\mu_c(\rho_{\xi,\zeta}(t)) \\ &= \frac{d}{dt}\Big|_{t=0} D\exp_p(t\zeta)\big(\mathcal{J}_{t\zeta}(c\,\xi)\big) \\ &= \dot\rho_{c\xi,w}(0). \end{aligned}$$

This shows that $D\mu_c(p,\xi)\big(\Delta_{(p,\xi)}\big) \subseteq \Delta_{(p,c\xi)}$. Since $\pi\circ\mu_c = \pi$, we have $D\pi(p,c\,\xi)\circ D\mu_c(p,\xi) = D\pi(p,\xi)$, and thus it follows that $D\pi(p,c\,\xi)$ maps both $D\mu_c(p,\xi)\big(\Delta_{(p,\xi)}\big)$ and $\Delta_{(p,c\xi)}$ isomorphically onto T_pM, and thus we must have equality:

$$D\mu_c(p,\xi)\big(\Delta_{(p,\xi)}\big) = \Delta_{(p,c\xi)}.$$

4. It remains to show that $\mathbb{S}$ is the geodesic spray of Δ. Since $\mathbb{S}$ is a spray and there is at most one horizontal spray with respect to a given connection by Theorem 43.10), it suffices to show that $\mathbb{S}(p,\xi) \in \Delta_{(p,\xi)}$ for each $(p,\xi)\in TM$. For this, let δ denote the integral curve of $\mathbb{S}$ with initial condition (p,ξ), and let $\gamma := \pi\circ\delta$. Then the argument from the last bit of the proof of Theorem 43.10 shows that $\delta = \dot\gamma$, and hence

$$\begin{aligned} \delta(s) &= \dot\gamma(s) \\ &= \frac{d}{dt}\Big|_{t=0}\gamma(s+t) \\ &= \frac{d}{dt}\Big|_{t=0}\exp_p(s\xi + t\xi) \\ &= D\exp_p(s\xi)\big(\mathcal{J}_{s\xi}(\xi)\big) \\ &= \rho_{\xi,\xi}(s). \end{aligned}$$

Therefore

$$\mathbb{S}(p,\xi) = \dot\delta(0) = \dot\rho_{\xi,\xi}(0) \in \Delta_{(p,\xi)}.$$

This completes the proof. ■

Bonus Material for Lecture 44

In general there is no relation between the two exponential maps (apart from sharing similar properties), and thus the terminology is a bit unfortunate.

Definition 44.6. A metric ρ on a Lie group G is said to be **left-invariant** if

$$l_g^*\rho = \rho, \qquad \forall\, g \in G,$$

and **right-invariant** if

$$r_g^*\rho = \rho, \qquad \forall\, g \in G.$$

A Riemannian metric is **bi-invariant** if it is both left and right-invariant.

For Lie groups, we can't use the letter "g" to denote a Riemannian metric, since g is reserved for an element of the group. Thus we use ρ instead.

For general Lie groups, such a metric need not exist. In the compact case, however, we have:

Theorem 44.7. *Let G be a compact Lie group. Then there exists a bi-invariant metric ρ on M.*

More generally, any connected semi-simple or reductive Lie group admits a bi-invariant **pseudo-Riemannian metric**. This is defined in the same way as a Riemannian metric, only instead of requiring ρ to be positive definite, we require ρ to have some fixed mixed signature.

Theorem 44.8. *Let G be a Lie group and let ρ be a bi-invariant (pseudo)-Riemannian metric on G. Let $\mathbb{S}$ denote the geodesic spray of the Levi-Civita connection of ρ. Then when restricted to $e \in G$, the exponential map of $\mathbb{S}$ agrees with the exponential map of the Lie group itself.*

cf. Theorem 46.1.

LECTURE 45

Torsion-free Connections

As we remarked last lecture, the correspondence between connections on M and sprays on M is not bijective, since different connections can have the same geodesics (and hence also the same geodesic spray). The aim of this lecture is to introduce a special type of connection, called a *torsion-free* connection, which *is* uniquely determined by its geodesics.

Recall from Problem N.1 that if ∇^1 and ∇^2 are two connections on M then their difference

$$A(X,Y) \coloneqq \nabla^1_X Y - \nabla^2_X Y$$

is an element of $\mathcal{T}^{1,2}(M)$, i.e. a tensor of type $(1,2)$.

LEMMA 45.1. *Two connections ∇^1 and ∇^2 have the same geodesic spray if and only if their difference A is skew-symmetric.*

Proof. From the proof of Theorem 43.10, if $\mathbb{S}_i$ is the geodesic spray of ∇^i then

$$\mathbb{S}_i(p,\xi) = D\pi(p,\xi)|^{-1}_{\Delta_i|_{(p,\xi)}}(\xi),$$

where $\Delta_i \subset TTM$ is the connection distribution of ∇^i. By part (ii) of Problem N.1, we have

$$\Delta_2|_{(p,\xi)} = \left\{\eta + \mathcal{J}_\xi\big(A(D\pi(p,\xi)\eta,\xi)\big) \mid \eta \in \Delta_1|_{(p,\xi)}\right\},$$

and hence

$$\begin{aligned}\mathbb{S}_2(p,\xi) &= D\pi(p,\xi)|^{-1}_{\Delta_2|_{(p,\xi)}}(\xi)\\ &= D\pi(p,\xi)|^{-1}_{\Delta_1|_{(p,\xi)}}(\xi) + \mathcal{J}_\xi(A(\xi,\xi))\\ &= \mathbb{S}_1(p,\xi) + \mathcal{J}_\xi(A(\xi,\xi)).\end{aligned}$$

Thus $\mathbb{S}_1 = \mathbb{S}_2$ if and only if $A(\xi,\xi) = 0$ for all ξ, i.e. that A is skew-symmetric. ∎

This motivates the following definition.

DEFINITION 45.2. Let ∇ be a connection on M. The **torsion tensor** T^∇ of ∇ is the tensor of type $(1,2)$ defined by

$$T^\nabla(X,Y) \coloneqq \nabla_X Y - \nabla_Y X - [X,Y], \qquad X,Y \in \mathfrak{X}(M).$$

As with the curvature tensor, merely calling T^∇ a tensor does not make it one. In contrast to Theorem 33.14 however, the verification that T^∇ really is a tensor is much easier.

LEMMA 45.3. *The torsion tensor T^∇ is an alternating tensor.*

Proof. By Theorem 21.5 we need only check that T^∇ is $C^\infty(M)$-linear in both variables. Take $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. Then

$$\begin{aligned} T^\nabla(fX, Y) &= \nabla_{fX} Y - \nabla_Y(fX) - [fX, Y] \\ &= f\nabla_X Y - Y(f)X - f\nabla_Y X - f[X,Y] + Y(f)X \\ &= fT^\nabla(X,Y) + 0, \end{aligned}$$

where we used Problem D.5 and the now familiar properties of a covariant derivative operator (parts (ii) and (iv) of Definition 31.6). It is clear that T^∇ is alternating, and thus T^∇ is also $C^\infty(M)$-linear in the second variable. ∎

DEFINITION 45.4. A connection ∇ is said to be **torsion-free** if $T^\nabla = 0$.

REMARK 45.5. Many textbooks call a torsion-free connection a "symmetric" connection. The motivation for this is the following: if ∇ is a torsion-free connection then the Christoffel symbols Γ^k_{ij} associated to any chart x on M (cf. Definition 43.1) are symmetric in i and j. Indeed, given any connection ∇ and any chart (U, x) on M then the local expression for T^∇ on U with respect to x is (cf. Definition 21.1):

$$T^\nabla = T^k_{ij}\, \partial_k \otimes dx^i \otimes dx^j$$

where the $T^k_{ij} \colon U \to \mathbb{R}$ are the smooth functions given by

$$T^k_{ij} = dx^k\big(T(\partial_i, \partial_j)\big).$$

But since $[\partial_i, \partial_j] = 0$ by Problem D.4, it follows that

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}.$$

In particular, $T^\nabla = 0$ if and only if for every local coordinate system (x^i) one has $\Gamma^k_{ij} = \Gamma^k_{ji}$.

REMARK 45.6. It is easy to turn any connection into a torsion-free one. Indeed, if ∇ is a connection then $\nabla_1 \coloneqq \nabla - \frac{1}{2}T^\nabla$ is another connection by Problem N.1, and it follows immediately from the definition that $T^{\nabla_1} = 0$.

The next theorem gives us yet another way to view connections: namely, specifying a connection on M is the same thing as specifying the geodesics and the torsion tensor.

THEOREM 45.7. *Let ∇^1 and ∇^2 denote two connections on M. Then $\nabla^1 = \nabla^2$ if and only if ∇^1 and ∇^2 have the same geodesics and the same torsion tensors.*

Proof. Let $A \coloneqq \nabla^1 - \nabla^2$, and decompose A into its symmetric and alternating parts: $A = A^s + A^a$, i.e.

$$\begin{aligned} A^s(X,Y) &\coloneqq \frac{1}{2}\left(A(X,Y) + A(Y,X)\right), \\ A^a(X,Y) &\coloneqq \frac{1}{2}\left(A(X,Y) - A(Y,X)\right) \end{aligned}$$

In Lemma 45.1 we already showed that $A^s = 0$ if and only if ∇^1 and ∇^2 have the same geodesics. Thus if suffices to show that $A^a = 0$ if and only if $T^{\nabla^1} = T^{\nabla^2}$. But this is immediate from:

$$\begin{aligned} 2A^a(X,Y) &= A(X,Y) - A(Y,X) \\ &= \nabla^1_X Y - \nabla^2_X Y - \nabla^1_Y X + \nabla^2_Y X \\ &= T^{\nabla^1}(X,Y) - T^{\nabla^2}(X,Y). \end{aligned}$$

This completes the proof. ∎

If $\varphi \colon M \to N$ is a smooth map and ∇ is a connection on N, then the pullback connection (also denoted by ∇) on M is a map

$$\nabla^\varphi \colon \mathfrak{X}(M) \times \Gamma_\varphi(TN) \to \Gamma_\varphi(TN).$$

If X is a vector field in M then $p \mapsto D\varphi(p)X(p)$ is a well-defined element of $\Gamma_\varphi(TN)$, which we write simply as $D\varphi(X)$. Thus the expression

If φ is a diffeomorphism then $\varphi_\star X = D\varphi(X) \circ \varphi^{-1}$ in this notation.

$$T^\nabla_\varphi \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma_\varphi(TN)$$

given by

$$T^\nabla_\varphi(X,Y) \coloneqq \nabla^\varphi_X(D\varphi(Y)) - \nabla^\varphi_Y(D\varphi(X)) - D\varphi[X,Y]$$

is well defined.

Since T^∇ is a point operator in both variables, the expression $T^\nabla(D\varphi(X), D\varphi(Y))$ is also a well defined section along φ. The next result is the analogue of Proposition 35.9 for the torsion tensor.

Proposition 45.8. *Let ∇ denote a connection on a smooth manifold N, and let $\varphi \colon M \to N$ denote a smooth map. Then for any $X, Y \in \mathfrak{X}(M)$, one has*

$$T^\nabla(D\varphi(X), D\varphi(Y)) = T^\nabla_\varphi(X,Y)$$

as elements of $\Gamma_\varphi(TN)$.

The proof of Proposition 45.8 is almost identical to that of Proposition 35.9, and we leave the details to you. Theorem 44.5 constructed a connection from a spray. In fact, this connection is torsion-free, as we now prove.

Proposition 45.9. *Let $\mathbb{S}$ denote a spray on M, and let ∇ denote the connection constructed in the proof of the Theorem 44.5. Then ∇ is torsion-free.*

This proof is non-examinable.

Proof. Fix $p \in M$ and $\xi, \zeta \in T_pM$. We will prove that $T^\nabla(\xi, \zeta) = 0$ in two steps.

1. In this step we derive an expression for $T^\nabla(\xi,\zeta)$. There is a well-defined vector field $\mathcal{J}(\xi) \in \mathfrak{X}(T_pM)$ defined by

This makes sense: if $\zeta \in T_pM$ then $\mathcal{J}_\zeta(\xi) \in T_\zeta T_pM$.

$$\mathcal{J}(\xi)(\zeta) \coloneqq \mathcal{J}_\zeta(\xi) = \frac{d}{dt}\Big|_{t=0} \zeta + t\xi.$$

Let exp denote the exponential map of $\mathbb{S}$ with domain $\mathcal{E} \subset TM$, and as usual let $\exp_p \colon \mathcal{E}_p \to M$ denote the restriction of exp to the fibre over p. Then for any $\xi \in T_pM$, we may regard $\mathcal{J}(\xi)$ as a vector field on $\mathcal{E}_p$, and hence (using the notation above), $D\exp_p(\mathcal{J}(\xi))$ is a vector field along $\exp_p$, which we abbreviate by $\mathbb{X}_\xi$. Moreover by part (ii) of Theorem 44.3, this vector field satisfies

$$\mathbb{X}_\xi(0_p) = D\exp_p(0_p)(\mathcal{J}_{0_p}(\xi)) = \xi.$$

By Proposition 45.8, we have

$$T^\nabla(\mathbb{X}_\xi, \mathbb{X}_\zeta) = T^\nabla_{\exp_p}(\mathcal{J}(\xi), \mathcal{J}(\zeta)),$$

and evaluating both sides at 0_p tells us that

$$T^\nabla(\xi, \zeta) = \nabla^{\exp_p}_\xi \mathbb{X}_\zeta - \nabla^{\exp_p}_\zeta \mathbb{X}_\xi - [\mathbb{X}_\xi, \mathbb{X}_\zeta](0_p). \tag{45.1}$$

2. In this step we compute the right-hand side of (45.1). Since $\mathcal{J}(\xi)$ and $\mathcal{J}(\zeta)$ are constant vector fields, the Lie bracket $[\mathcal{J}(\xi), \mathcal{J}(\zeta)]$ is zero by Problem D.4. Thus by Problem D.6 we also have $[\mathbb{X}_\xi, \mathbb{X}_\zeta] = 0$. Now let Δ denote the connection distribution from (44.8) and let $K \colon TTM \to TM$ denote the connection map of ∇. Then by definition (cf. Theorem 31.8) one has

This is a Lie bracket of vector fields on the vector space T_pM.

$$\begin{aligned}\nabla^{\exp_p}_\xi \mathbb{X}_\zeta &= K\big(D\mathbb{X}_\zeta(0_p)(\mathcal{J}(\xi)(0_p))\big) \\ &= K\left(\frac{d}{dt}\Big|_{t=0} D\exp_p(t\xi)(\mathcal{J}_{t\xi}(\zeta))\right).\end{aligned}$$

Using the notation from the proof of Theorem 44.5, this last term is exactly $K(\dot{\rho}_{\zeta,\xi}(0))$. Since $\dot{\rho}_{\zeta,\xi}(0) \in \Delta_{(p,\zeta)}$ by definition and $\Delta = \ker K$ we conclude that

cf. (44.6) – pay attention to the order of ξ and ζ!

$$\nabla^{\exp_p}_\xi \mathbb{X}_\zeta = 0.$$

Similarly $\nabla^{\exp_p}_\zeta \mathbb{X}_\xi = 0$. Thus by (45.1) we have $T^\nabla(\xi, \zeta) = 0$. This completes the proof. ■

This gives us the following strengthening of Theorem 44.5.

Corollary 45.10. *Let $\mathbb{S}$ be a spray on M and let T be an alternating tensor of type $(1, 2)$. There exists a unique connection on M with geodesic spray $\mathbb{S}$ and torsion tensor T.*

Proof. Let ∇ denote the connection on M given by Theorem 44.5. Then ∇ has geodesic spray $\mathbb{S}$ and ∇ is torsion-free by Proposition 45.9. The desired connection is then given by $\nabla_1 := \nabla + \frac{1}{2}T$ (apply Remark 45.6 backwards). This connection is unique by Theorem 45.7. ■

One of the most useful consequences of Corollary 45.10 is that if we start with a torsion-free connection we now have an explicit formula for the horizontal distribution in terms of the exponential map of the geodesic spray of ∇ (i.e. (44.8)). Here is an application of this, which will aid our forthcoming computations in Riemannian geometry.

Proposition 45.11. *Let ∇ be a torsion-free connection on M. Fix $p \in M$ and let $\{\xi_1, \dots, \xi_m\}$ be a basis of T_pM. There exists a chart (U, x) on M about p such that:*

(i) $x(p) = 0$,

(ii) $\partial_{i|p} = \xi_i$,

(iii) $\nabla_\zeta \partial_i = 0$ *for all* $\zeta \in T_pM$,

(iv) $\nabla^*_\zeta dx^i = 0$ *for all* $\zeta \in T_pM$ *(where* ∇^* *is the induced connection on* T^*M*).*

Note that by (iii) we have that in these coordinates the Christoffel symbols vanish at p: $\Gamma^k_{ij}(p) = 0$ for all i, j, k.

Proof. Let $\ell \colon T_pM \to \mathbb{R}^m$ denote the linear isomorphism determined by $\ell\xi_i = e_i$. Let exp denote the exponential map of the geodesic spray of ∇. Let $V \subset T_pM$ to be a neighbourhood of 0_p on which $\exp_p$ is a diffeomorphism. Let $U := \exp_p(V)$ and define

Such V exists by part (ii) of Theorem 44.3.

$$x := \ell \circ \exp_p|_V^{-1}.$$

Then (i) is clear. By construction we have

$$\partial_{i|\exp_p(\xi)} = D\exp_p(\xi) \circ \mathcal{J}_\xi(\xi_i),$$

and so taking $\xi = 0$ and applying part (ii) of Theorem 44.3 gives (ii). To prove (iii), we consider the curve $\rho(t) = t\zeta$ in T_pM. Then thinking of ∂_i as a smooth map $U \to TU$, we have

$$D\partial_i(p)\zeta = \frac{d}{dt}\Big|_{t=0} D\exp_p(t\zeta) \circ \mathcal{J}_{t\zeta}(\xi_i)$$

which belongs to the connection distribution Δ of ∇ at (p, ξ_i) by (44.8). Thus if K denotes the connection map of ∇ then

$$\nabla_\zeta \partial_i = K(D\partial_i(p)\zeta) = 0$$

as $\ker K = \Delta$. This proves property (iii). Finally property (iv) is immediate from (iii) and the definition (Problem M.3) of the induced connection on T^*M, since $dx^i(\partial_j) = \delta^i_j$. ∎

A torsion-free connection enjoys some additional symmetry properties of its curvature tensor.

Proposition 45.12. *Let ∇ be a torsion-free connection on M with curvature tensor R^∇. Then for all $X, Y, Z \in \mathfrak{X}(M)$, one has:*

(i) $R^\nabla(X,Y)(Z) + R^\nabla(Y,Z)(X) + R^\nabla(Z,X)(Y) = 0.$

(ii) $\big(\nabla_X R^\nabla\big)(Y,Z) + \big(\nabla_Y R^\nabla\big)(Z,X) + \big(\nabla_Z R^\nabla\big)(X,Y) = 0.$

Proof. Since R^∇ is a point operator in all three variables, it is sufficient to prove the result in the special case where $[X,Y] = [Y,Z] =$

In fact, it would suffice to prove the result in the special case $X = \partial_i$, $Y = \partial_j$ and $Z = \partial_k$, although the proof would be no simpler.

$[Z,X] = 0$. Then $\nabla_X(Y) = \nabla_Y(X)$, $\nabla_Y(Z) = \nabla_Z(Y)$, and $\nabla_Z(X) = \nabla_X(Z)$, and hence

$$\begin{aligned} &R^{\nabla}(X,Y)(Z) + R^{\nabla}(Y,Z)(X) + R^{\nabla}(Z,X)(Y) \\ &\quad = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_Y \nabla_Z X \\ &\qquad - \nabla_Z \nabla_Y X + \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y \\ &\quad = \nabla_X \big(\nabla_Y Z - \nabla_Z Y\big) + \nabla_Y \big(\nabla_Z X - \nabla_X Z\big) + \nabla_Z \big(\nabla_X Y - \nabla_Y X\big) \\ &\quad = 0 + 0 + 0. \end{aligned}$$

This proves (i). The proof of (ii) is on Problem Sheet P. ■

REMARK 45.13. In the literature the two identities (i) and (ii) are often somewhat confusingly referred to as the "First Bianchi Identity" and the "Second Bianchi Identity" respectively. We will avoid this nomenclature since we already have two "Bianchi Identities" (Theorem 36.21 and (41.2) from Theorem 41.6)!

Bonus Material for Lecture 45

In this bonus section we briefly survey how the torsion-free condition affects the possible holonomy groups that can arise. As explained at the end of Lecture 42, the general question as to which Lie groups can arise as the holonomy group of a connection on a given principal bundle is not very interesting (see Remark 42.10). If however we work with torsion-free connections, this dramatically changes.

Consider the following question:

- *Let M be a connected smooth manifold. What Lie subgroups $G \subset \mathrm{GL}(m)$ can occur as possible holonomy groups for torsion-free connections on M?*

This is an extremely difficult problem in general, and is an open problem for many manifolds M. We can simplify things by turning the question on its head and starting with the Lie group.

- *Let $G \subset \mathrm{GL}(m)$ be a Lie subgroup. Does there exist* **any** *smooth manifold M and a torsion-free connection ∇ on M such that G is the holonomy group of ∇?*

This is still very hard, but a complete classification is (mostly) understood. We conclude this lecture by outlining why. The key starting point is the two additional symmetries from Proposition 45.12.

DEFINITION 45.14. Let V be a vector space and suppose G is a Lie subgroup of $\mathrm{GL}(V)$ with Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$. We define two subspaces as follows:

$$\mathfrak{b} := \left\{ r \in \bigwedge^2 V^* \otimes \mathfrak{g} \mid r(u,v)(w) + r(v,w)(u) + r(w,u)(v) = 0,\ \forall\, u,v,w \in V \right\},$$

and

$$\mathfrak{c} := \{ \rho \in V^* \otimes \mathfrak{b} \mid \rho(u)(v,w) + \rho(v)(w,u) + \rho(w)(u,v) = 0,\ \forall\, u,v,w \in V \}$$

Finally define

$$\mathfrak{B}(\mathfrak{g}) := \{ r(u,v) \mid r \in \mathfrak{b},\ u,v \in V \}.$$

DEFINITION 45.15. We say that $G \subset \mathrm{GL}(V)$ is a **Berger subgroup** if its Lie algebra $\mathfrak{g}$ satisfies:

(i) $\mathfrak{c} \neq \{0\}$.

(ii) $\mathfrak{B}(\mathfrak{g}) = \mathfrak{g}$.

The next result gives a necessary condition for a Lie subgroup to occur as the holonomy group of a torsion-free connection.

THEOREM 45.16. *Let M be connected manifold and suppose $G \subset \mathrm{GL}(m)$ is a Lie subgroup. Assume that G is irreducible and M is not locally symmetric. If G is the holonomy group of a torsion-free connection then G is necessarily a Berger group.*

For the purposes of our discussion here, just ignore these two conditions. Defining them precisely would take us too far afield, and it is not necessary to understand the general "idea".

Proof (sketch). The idea is very simple: if G is the holonomy group of a torsion-free connection ∇, then the curvature tensor defines an element of $\mathfrak{c}$ by Proposition 45.12. Thus $\mathfrak{c}$ is not zero. On the other hand, the Ambrose-Singer Holonomy Theorem 35.6 tells us that $\mathfrak{B}(\mathfrak{g})$ is all of $\mathfrak{g}$. ■

Theorem 45.16 allows us to rule out many Lie groups (i.e. all the non-Berger groups). This however is merely the "easy" half of answering the second question posed above – to show that a Lie group really does appear as a holonomy group, one needs to explicitly construct a connection. Unlike Theorem 42.8, there is no easy way to construct a connection "by hand". In 1999, a complete classification of those groups that could appear was obtained by Merkulov and Schwachhöfer. The list is rather long, and we will not attempt to enumerate it here.

We remark however that the list gets *much* shorter if we require that ∇ is not only torsion-free, but in addition is Riemannian with respect to some Riemannian metric on M (in other words, that ∇ is a Levi-Civita connection with respect to some Riemannian metric on M). The holonomy groups that can arise for such ∇ are the so-called **Riemannian holonomy groups**. We will come back to this in the bonus section of the next lecture.

Will J. Merry

LECTURE 46

The Levi-Civita Connection

In this lecture we begin our study of Riemannian geometry proper, starting with the construction of the famous *Levi-Civita connection* of a Riemannian manifold. Let M be a smooth manifold, and suppose $g = \langle \cdot, \cdot \rangle$ is a Riemannian metric on M (i.e. a Riemannian metric on the vector bundle TM). Recall a connection ∇ on M is said to be Riemannian with respect to g if g is parallel with respect to the induced connection on $T^*M \otimes T^*M$: $\nabla g = 0$. By Proposition 37.12 this is equivalent to asking that the *Ricci Identity* holds:

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \qquad \forall\, X, Y, Z \in \mathfrak{X}(M). \tag{46.1}$$

In Proposition 37.14 we proved that metric connections always exist. Meanwhile Theorem 44.5 (together with Proposition 45.9) proved that torsion-free connections exist. But can we satisfy both conditions simultaneously? The following somewhat grandiosely named theorem asserts that the answer is yes in the best possible way: there is a *unique* connection on M with both these properties.

THEOREM 46.1 (The Fundamental Theorem of Riemannian Geometry). *Let $g = \langle \cdot, \cdot \rangle$ be a Riemannian metric on M. There exists a unique connection ∇ on M which is both torsion-free and metric with respect to g. We call ∇ the* **Levi-Civita connection** *of g.*

Proof. We first deal with uniqueness. Suppose that ∇ is a metric torsion-free connection on M. Let $X, Y, Z \in \mathfrak{X}(M)$. We combine the Ricci Identity (46.1) with the torsion-free condition:

$$\langle \nabla_X Y, Z\rangle - \langle \nabla_Y X, Z\rangle = \langle [X, Y], Z\rangle,$$

to obtain

$$\begin{aligned} X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle + \langle \nabla_Y Z, X\rangle \\ &\quad + \langle Z, \nabla_Y X\rangle - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle \\ &= 2\langle \nabla_X Y, Z\rangle - \langle [X, Y], Z\rangle \\ &\quad + \langle [X, Z], Y\rangle + \langle [Y, Z], X\rangle, \end{aligned}$$

and hence

$$\begin{aligned} \langle \nabla_X Y, Z\rangle = \frac{1}{2}\Big(& X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle + \langle [X, Y], Z\rangle\Big). \end{aligned} \tag{46.2}$$

With this in mind, let us define a function

$$\omega_{X,Y} \colon \mathfrak{X}(M) \to C^\infty(M)$$

by declaring that $\omega_{X,Y}(Z)$ is the right-hand side of (46.2). We claim that $\omega_{X,Y}$ is actually a one-form on M. By Theorem 21.5 we must show that $\omega_{X,Y}$ is $C^\infty(M)$-linear. For this we compute:

$$\begin{aligned}\omega_{X,Y}(fZ) =& \frac{1}{2}\Big(X\langle Y, fZ\rangle + Y\langle fZ, X\rangle - fZ\langle X, Y\rangle \\ & - \langle [Y, fZ], X\rangle + \langle [fZ, X], Y\rangle + \langle [X, Y], fZ\rangle\Big) \\ =& \ f\omega_{X,Y}Z + \frac{1}{2}\Big(X(f)\langle Y, Z\rangle + Y(f)\langle Z, X\rangle \\ & - X(f)\langle Y, Z\rangle - Y(f)\langle X, Z\rangle\big) \\ =& \ f\omega_{X,Y}Z + 0.\end{aligned}$$

Since $\omega_{X,Y}$ is a one-form, by Problem N.5 there is a unique well-defined vector field $(\omega_{X,Y})^\sharp$ on M obtained via the musical isomorphism with respect to g. Then

$$\nabla_X Y = (\omega_{X,Y})^\sharp. \tag{46.3}$$

Since $(\omega_{X,Y})^\sharp$ is defined independently of ∇, this establishes uniqueness.

For existence, we simply turn this argument on its head and *define* ∇ by (46.3). For this to make sense we need to prove that does indeed define a torsion-free connection which is Riemannian with respect to g. This is a series of straightforward, but rather lengthy computations. We must verify:

(i) $\nabla_{fX} Y = f\nabla_X Y$,

(ii) $\nabla_X(fY) = X(f)Y + f\nabla_X Y$,

(iii) $\nabla_X Y - \nabla_Y X = [X, Y]$,

(iv) $\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = X\langle Y, Z\rangle$,

as the remaining conditions are all trivial.

For (i), observe that

$$\begin{aligned}2\langle \nabla_{fX} Y, Z\rangle =& \ fX\langle Y, Z\rangle + Y\langle Z, fX\rangle - Z\langle fX, Y\rangle \\ & - \langle [Y, Z], fX\rangle + \langle [Z, fX], Y\rangle + \langle [fX, Y], Z\rangle \\ =& \ f\Big(X\langle Y, Z\rangle - Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle + \langle [X, Y], Z\rangle\Big) \\ & + Y(f)\langle Z, X\rangle - Z(f)\langle X, Y\rangle + Z(f)\langle X, Y\rangle - Y(f)\langle X, Z\rangle \\ =& \ 2f\langle \nabla_X Y, Z\rangle + 0.\end{aligned}$$

To prove (ii), we see that

$$\begin{aligned}2\langle \nabla_X(fY), Z\rangle =& \ X\langle fY, Z\rangle + fY\langle Z, X\rangle - Z\langle X, fY\rangle \\ & - \langle [fY, Z], X\rangle + \langle [Z, X], fY\rangle + \langle [X, fY], Z\rangle \\ =& \ f\Big(X\langle Y, Z\rangle - Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle + \langle [X, Y], Z\rangle\Big) \\ & + X(f)\langle Y, Z\rangle - Z(f)\langle X, Y\rangle + Z(f)\langle Y, X\rangle + X(f)\langle Y, Z\rangle \\ =& \ 2f\langle \nabla_X Y, Z\rangle + 2X(f)\langle Y, Z\rangle.\end{aligned}$$

To prove (iii), we compute

$$\begin{aligned}
2\langle \nabla_X Y, Z\rangle - 2\langle \nabla_Y X, Z\rangle &= X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\
&\quad - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle + \langle [X, Y], Z\rangle \\
&\quad - Y\langle X, Z\rangle - X\langle Z, Y\rangle + Z\langle Y, X\rangle \\
&\quad + \langle [X, Z], Y\rangle - \langle [Z, Y], X\rangle - \langle [Y, X], Z\rangle \\
&= - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle + \langle [X, Y], Z\rangle \\
&\quad - \langle [Z, X], Y\rangle + \langle [Y, Z], X\rangle + \langle [X, Y], Z\rangle \\
&= 2\langle [X, Y], Z\rangle,
\end{aligned}$$

and hence $\nabla_X Y - \nabla_Y X = [X, Y]$.

Finally, to prove (iv) we compute

$$\begin{aligned}
2\langle \nabla_X Y, Z\rangle + 2\langle Y, \nabla_X Z\rangle &= X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\
&\quad - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle + \langle [X, Y], Z\rangle \\
&\quad + X\langle Z, Y\rangle + Z\langle Y, X\rangle - Y\langle X, Z\rangle \\
&\quad - \langle [Z, Y], X\rangle + \langle [Y, X], Z\rangle + \langle [X, Z], Y\rangle \\
&= 2X\langle Y, Z\rangle.
\end{aligned}$$

This completes the proof of existence. ■

We can use (46.2) to express the Levi-Civita connection in local coordinates. Suppose (U, x) is a chart on M. Then we can write

$$g = g_{ij}\, dx^i \otimes dx^j$$

on U, where

$$g_{ij} \colon U \to \mathbb{R}, \qquad g_{ij} := \langle \partial_i, \partial_j\rangle$$

Note that the matrix $(g_{ij}(p))_{1\le i,j\le n}$ is symmetric and positive definite for every $x \in U$.

Lemma 46.2. *Let (M, g) be a Riemannian manifold and let ∇ denote the Levi-Civita connection of g. Let (U, x). Then the Christoffel symbols of ∇ are given by*

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}\right),$$

where $(g^{ij})_{1\le i,j\le m}$ is the inverse matrix to $(g_{ij})_{1\le i,j\le m}$.

Proof. Firstly we have (this is true for any connection)

$$2\langle \nabla_{\partial_i} \partial_j, \partial_l\rangle = 2\langle \Gamma_{ij}^k \partial_k, \partial_l\rangle = 2\Gamma_{ij}^k g_{kl}.$$

Now by (46.2) we have

$$\begin{aligned}
2\langle \nabla_{\partial_i} \partial_j, \partial_l\rangle &= \partial_i\langle \partial_j, \partial_l\rangle + \partial_j\langle \partial_l, \partial_i\rangle - \partial_l\langle \partial_i, \partial_j\rangle \\
&= \partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij},
\end{aligned}$$

since the Lie bracket terms $[\partial_i, \partial_j]$ all vanish by Problem D.4. Thus

$$2\Gamma_{ij}^k g_{kl} = \partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij},$$

Multiply both sides by $\frac{1}{2}g^{pl}$ and sum over l to get

The summation over l is forced by the Einstein Summation Convention.

$$\Gamma_{ij}^k g_{kl} g^{pl} = \frac{1}{2} g^{pl} \left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}\right). \tag{46.4}$$

But

$$g_{kl} g^{lp} = \delta_k^p.$$

(this is the definition of the inverse matrix) and hence the left-hand side of (46.4) is

$$\Gamma_{ij}^k g_{kl} g^{pl} = \Gamma_{ij}^k \delta_k^p.$$

Thus in particular taking $p = k$ on the right-hand side of (46.4) gives

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}\right)$$

as desired. ∎

Corollary 46.3. *Let (M, g) be a Riemannian manifold and let ∇ denote the Levi-Civita connection of g. For any point $p \in M$ there exists a chart (U, x) about p such that $\{\partial_{i|p}\}$ is an orthonormal basis at p and such that the Christoffel symbols vanish at p: $\Gamma_{ij}^k(p) = 0$ for all i, j, k.*

Such coordinates are called **normal coordinates** at p.

Proof. Choose an orthonormal basis $\{\xi_i\}$ of T_pM and apply Corollary 45.11. ∎

Remark 46.4. One can alternatively characterise normal coordinates in terms of the first derivatives of the metric. Indeed, in any local coordinates (x^i) one has

$$\begin{aligned}
\partial_k g_{ij} &= \partial_k \langle \partial_i, \partial_j \rangle \\
&= \langle \nabla_{\partial_k} \partial_i, \partial_j \rangle + \langle \partial_i, \nabla_{\partial_k} \partial_j \rangle \\
&= \langle \Gamma_{ki}^l \partial_l, \partial_j \rangle + \langle \partial_i, \Gamma_{kj}^l \partial_l \rangle \\
&= \Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{il},
\end{aligned}$$

where the second line used the Ricci identity (46.1). Thus coordinates (x^i) are normal at p if and only if $\{\partial_{i|p}\}$ is an orthonormal basis of T_pM and

$$\partial_k g_{ij}(p) = 0, \qquad \forall\, i, j, k. \tag{46.5}$$

Remark 46.5. If (U, x) are normal coordinates at p and $\xi = a^i\, \partial_{i|p}$ is a tangent vector in T_pM then the unique geodesic $\gamma_{p,\xi}$ with $\gamma_{p,\xi}(0) = p$ and $\dot{\gamma}_{p,\xi}(0) = \xi$ is given by

$$\gamma_{p,\xi}(t) = x^{-1}(ta^1, \ldots, ta^m),$$

for all t sufficiently small. This follows from the proof of Proposition 45.11.

The next result shows how the Levi-Civita connection behaves nicely with respect to pullbacks.

Proposition 46.6. *Let (N, g) be a Riemannian manifold, and let ∇ denote the Levi-Civita connection. Suppose $\varphi\colon M \to N$ is a smooth map. Then for $X, Y, Z \in \mathfrak{X}(M)$ the covariant derivative operator ∇^{φ} satisfies*

$$\begin{aligned}\langle \nabla^{\varphi}_X (D\varphi(Y)), D\varphi(Z)\rangle = \frac{1}{2}\Big(& X\langle D\varphi(Y), D\varphi(Z)\rangle + Y\langle D\varphi(Z), D\varphi(X)\rangle \\ & - Z\langle D\varphi(X), D\varphi(Y)\rangle - \langle D\varphi([Y,Z]), D\varphi(X)\rangle \\ & + \langle D\varphi([Z,X]), D\varphi(Y)\rangle + \langle D\varphi([X,Y]), D\varphi(Z)\rangle\Big).\end{aligned}$$

Proof. The pullback connection satisfies the Ricci Identity by Corollary 37.13. Thus the claim follows from the uniqueness of the Levi-Civita connection on (N, g) and Proposition 45.8. ■

In Proposition 46.6 the domain M of φ is not endowed with a Riemannian metric (only the target N is). We now consider the case where both M and N are equipped with metrics. Recall from Definition 37.7 that a vector bundle morphism between two Riemannian vector bundles is said to be an *isometric* vector bundle morphism if it preserves the Riemannian metrics. The following definition specialises this to Riemannian metrics on manifolds.

Definition 46.7. Let (M, g) and (N, h) be Riemannian manifolds. A smooth map $\varphi\colon M \to N$ is said to be **isometric** if $D\varphi\colon TM \to TN$ is an isometric vector bundle morphism in the sense of Definition 37.7. Explicitly, if we write $\langle \cdot, \cdot \rangle$ for (both) metrics, then φ is isometric if and only if

$$\langle \xi, \zeta\rangle = \langle D\varphi(p)\xi, D\varphi(p)\zeta\rangle, \qquad \forall\, p \in M,\ \xi, \zeta \in T_pM.$$

Equivalently, this means that the metric g is equal to the pullback tensor $\varphi^* h$ from Definition 21.8. Note that any isometric map is necessarily an immersion.

Note that an isometric map is necessarily an immersion. An isometric diffeomorphism is called an **isometry**.

Definition 46.8. We denote by $\mathrm{Iso}(M, g) \subset \mathrm{Diff}(M)$ the subgroup of isometries.

Isometries are much "rarer" than diffeomorphisms. This is encapsulated by the following theorem, which sadly goes beyond the scope of the course.

There is another theorem in Riemannian Geometry due to Myers and Steenrod. We state this result in Lecture **??** and label it the "(Easy) Myers-Steenrod Theorem". Here "easy" should be understood in a relative sense.

Theorem 46.9 (The (Hard) Myers-Steenrod Theorem). *Let (M, g) be a Riemannian manifold with at most finitely many components. Then $\mathrm{Iso}(M, g)$ is a Lie group. If M is compact then so is $\mathrm{Iso}(M, g)$.*

The space $\mathrm{Diff}(M)$ is an infinite-dimensional (Fréchet) manifold (cf. the bonus section to Lecture 13). Theorem 46.9 tells us that $\mathrm{Iso}(M, g)$ is a finite-dimensional subgroup.

Theorem 46.9 is too hard for us. However we can at least prove something in this direction:

Proposition 46.10. *Let φ be an isometry of a connected Riemannian manifold (M, g). Suppose there exists a point $p \in M$ such that $\varphi(p) = p$ and $D\varphi(p) = \mathrm{Id}_{T_pM}$. Then $\varphi(q) = q$ for all $q \in M$.*

We will prove Proposition 46.10 next lecture.

As already remarked, any isometric map between Riemannian manifolds is necessarily an immersion. In fact, there is a partial converse to this, as we now explain.

Suppose (N, g) is a Riemannian manifold and $\varphi \colon M \to N$ is a smooth map. Consider the pullback tensor $\varphi^* g \in \mathcal{T}^{0,2}(M)$. In general this will not define a metric on M – it will always be symmetric, but it need not be positive definite (for example, if φ is constant it is identically zero). If however φ is an immersion then $\varphi^* g$ is positive definite, and hence a Riemannian metric on M. This proves the following useful statement.

Lemma 46.11. *Let (N, g) be a Riemannian manifold and suppose $\varphi \colon M \to N$ is an immersion. Then $\varphi^* g$ is a Riemannian metric on M, and $\varphi \colon (M, \varphi^* g) \to (N, g)$ is an isometric map. Moreover $\varphi^* g$ is the unique Riemannian metric on M with this property.*

Definition 46.12. Let (N, g) be a Riemannian manifold. An embedded submanifold M of N is said to be a **Riemannian submanifold** if M is endowed with the pullback Riemannian metric $\iota^* g$ (where $\iota \colon M \hookrightarrow N$ denotes the inclusion).

Examples 46.13.

The standard Riemannian metric g_{Euc} on $\mathbb{R}^m$ is given by

$$\langle \mathcal{J}_u(v), \mathcal{J}_u(w) \rangle_{\mathrm{Euc}} := \langle v, w \rangle_{\mathrm{Euc}}, \qquad u, v, w \in \mathbb{R}^m,$$

where $\langle \cdot, \cdot \rangle_{\mathrm{Euc}}$ on the right-hand side denotes the Euclidean dot product.

Let $\iota \colon S^m \to \mathbb{R}^{m+1}$ denote the inclusion, and let $g_{\mathrm{round}} := \iota^* g_{\mathrm{Euc}}$. Then g_{round} is a Riemannian metric on S^m which we call the **round metric**. This is the unique metric on S^m that makes S^m into a Riemannian submanifold of $\mathbb{R}^{m+1}$. Our favourite connection on S^m (introduced originally in Problem L.3) is in fact the Levi-Civita connection by Problem P.4.

This is because the sphere looks "round" in this metric. The precise meaning of this will become clear in Lecture **??**.

If $\varphi \colon M \to N$ is an immersion then necessarily $\dim M \leq \dim N$. If $\dim M = \dim N$ then there are essentially two cases of interest:

(i) If φ is an injective immersion and $\dim M = \dim N$ then it follows from Proposition 6.3 and the Inverse Function Theorem 5.10 that φ is automatically an embedding onto its image. Such a map is often called an **open embedding**, since $\varphi(M)$ is then open in N.

Compare this to Problem C.3, which gives another (entirely unrelated) condition for an injective immersion to automatically be an embedding.

(ii) The other main case of interest is when φ is a **smooth covering map**. This means that φ is surjective and moreover every point $q \in N$ has a neighbourhood U_q such that φ maps each component of $\varphi^{-1}(U_q)$ diffeomorphically onto U_q.

Caution: Not all surjective submersions between manifolds of the same dimension are covering maps.

Covering maps are important in Algebraic Topology. We will not really have any cause to use them, other than to note they provide us with further examples of isometric maps.

DEFINITION 46.14. A **Riemannian covering** $\varphi\colon (M,g) \to (N,h)$ is an isometric map between Riemannian manifolds which is in addition a smooth covering map. Note this necessarily implies $\dim M = \dim N$.

Recall a covering map $f\colon X \to Y$ between topological spaces is **normal** if $f_*(\pi_1(X,x))$ is a normal subgroup of $\pi_1(Y, f(x))$. In particular, a universal cover is always normal. It is a standard result in covering space theory that a covering is normal if and only if the deck transformation group acts transitively on the fibres. If $\varphi\colon M \to N$ is a smooth normal covering map then φ is necessarily a submersion, and the deck transformations are diffeomorphisms of M. The proof of the next result is deferred to Problem Sheet P.

Don't worry if you are unfamiliar with covering space theory – we will not actually use any of this, it is just for interest.

PROPOSITION 46.15. *Let $\varphi\colon M \to N$ be a smooth normal covering map and let m be a Riemannian metric on M which is invariant under all deck transformations. Then there is a unique Riemannian metric on N such that φ is a Riemannian covering.*

Here are some (non)-examples.

EXAMPLES 46.16.

(i) We can think of the torus T^m as the quotient $\mathbb{R}^m/\mathbb{Z}^m$, and in fact this is the universal cover. By Proposition 46.15 there is a unique Riemannian metric on T^m such that the quotient map $\mathbb{R}^m \to T^m$ is a Riemannian covering, where $\mathbb{R}^m$ is equipped with its standard Euclidean metric g_{Euc}. We call this metric the **flat metric** on the torus and write it as g_{flat}.

As with the earlier "round" metric on S^m, the precise justification for the name "flat" will come in Lecture **??**.

(ii) Take $m = 2$. Then one can embed T^2 into $\mathbb{R}^3$ – think of a hollow doughnut. If $\iota\colon T^2 \to \mathbb{R}^3$ denotes the inclusion then $\iota^* g_{\mathrm{Euc}}$ is another Riemannian metric on T^2. As we will see next lecture, g_{flat} is *not* the same Riemannian metric as $\iota^* g_{\mathrm{Euc}}$. In fact, it is not possible to embed (T^2, g_{flat}) into $(\mathbb{R}^3, g_{\mathrm{Euc}})$.

(iii) The projection map $S^m \to \mathbb{R}P^m$ is a smooth normal covering. Thus there is a unique Riemannian metric m on $\mathbb{R}P^m$ such that $(S^m, g_{\mathrm{round}}) \to (\mathbb{R}P^m, m)$ is a Riemannian covering.

REMARK 46.17. Immersions are dual to submersions, and thus it won't surprise you to learn that there is a dual notion of a **Riemannian submersion** which allows for the case $\dim M \geq \dim N$. We won't have cause to study these in general (and they are a little messier to define), although see Problem P.8 for an important special case.

Bonus Material for Lecture 46

We conclude this lecture by briefly discussing Riemannian holonomy groups.

Definition 46.18. Let (M, g) be a connected Riemannian manifold. We define the **holonomy group of** g, written as $\mathrm{Hol}(g)$, to be the holonomy group Hol^{∇}, where ∇ is the Levi-Civita connection of g. As in Corollary 32.16, we think of $\mathrm{Hol}(g)$ as a subgroup of $\mathrm{GL}(m)$, which is defined only up to conjugation. Similarly we define the **restricted holonomy group of** g, written $\mathrm{Hol}_0(g)$.

It follows from Problem N.4 that $\mathrm{Hol}(g)$ is actually a subgroup of $\mathrm{O}(m)$ (and thus $\mathrm{Hol}_0(g)$ is a subgroup of $\mathrm{SO}(m)$). On Problem Sheet P you will extend this to the following statement:

Proposition 46.19. *Let M be a connected manifold and suppose ∇ is a torsion-free connection on M. Then ∇ is the Levi-Civita connection of a Riemannian metric g on M if and only if Hol^{∇} is conjugate in $\mathrm{GL}(m)$ to a subgroup of $\mathrm{O}(m)$.*

The following statement is much more difficult, and its proof goes beyond the scope of this course. It uses the Lie-theoretic fact that every connected Lie subgroup of $\mathrm{SO}(m)$ that acts *irreducibly* on $\mathbb{R}^m$ is in fact closed in $\mathrm{SO}(m)$.

Theorem 46.20. *Let (M, g) be a connected Riemannian manifold. Then $\mathrm{Hol}_0(g)$ is a closed connected subgroup of $\mathrm{SO}(m)$.*

Theorem 46.20, together with Theorem 45.16 (and lots and lots and lots of work) gives the following amazing result.

Theorem 46.21 (The Berger Classification Theorem). *Let M be a simply connected manifold and suppose g is an irreducible non-symmetric Riemannian metric on M. Then exactly one of the following options holds for the holonomy group $\mathrm{Hol}(g)$:*

As with Theorem 45.16, we won't define precisely what this means, as doing so would take us too far afield.

(i) $\mathrm{Hol}(g) = \mathrm{SO}(m)$.

(ii) $m = 2k$ *for* $k \geq 2$ *and* $\mathrm{Hol}(g) = \mathrm{U}(k) \subset \mathrm{SO}(2k)$.

(iii) $m = 2k$ *for* $k \geq 2$ *and* $\mathrm{Hol}(g) = \mathrm{SU}(k) \subset \mathrm{SO}(2k)$.

(iv) $m = 4k$ *for* $k \geq 2$ *and* $\mathrm{Hol}(g) = \mathrm{Sp^c}(k) \subset \mathrm{SO}(4k)$.

(v) $m = 4k$ *for* $k \geq 2$ *and* $\mathrm{Hol}(g) = \mathrm{Sp}(2k) \cdot \mathrm{Sp^c}(1) \subset \mathrm{SO}(4k)$.

(vi) $m = 7$ *and* $\mathrm{Hol}(g) = G_2 \subset \mathrm{SO}(7)$.

(vii) $m = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$.

$\mathrm{Sp^c}(k)$ is the **compact symplectic group** $\mathrm{Sp}(2k; \mathbb{C}) \cap \mathrm{U}(2k)$. One can think of $\mathrm{Sp^c}(k)$ as the quaternionic unitary group.
See here for the definition of G_2.

The group $\mathrm{Spin}(m)$ is the double cover of $\mathrm{SO}(m)$ (recall $\pi_1(\mathrm{SO}(m)) = \mathbb{Z}_2$). For $m \geq 3$ the group $\mathrm{Spin}(m)$ is simply connected, and thus is also the universal cover of $\mathrm{SO}(m)$.

Moreover all of these groups can occur as the holonomy group of an irreducible non-symmetric Riemannian metric.

As the name suggests, the fact that these are the only options is due to Berger in 1955. The proof that all of these groups really do occur took thirty more years to complete, and is the work of various mathematicians. This culminated in the work of Joyce, who in 1996 constructed compact Riemannian manifolds with holonomy the two so-called *exceptional holonomy groups* G_2 and $\mathrm{Spin}(7)$.

LECTURE 47

Symmetries of the Curvature Tensor

We begin this lecture by investigating what an isometric map does to the Levi-Civita connection. We then study the various symmetries the curvature tensor of a Levi-Civita connection enjoys.

DEFINITION 47.1. Let (N, h) be a Riemannian manifold and suppose $\varphi\colon M \to N$ is an immersion. Let $(\cdot)^\top\colon T_{\varphi(p)}N \to D\varphi(p)(T_pM)$ denote orthogonal projection with respect to the inner product $h_{\varphi(p)}$ onto the subspace $D\varphi(p)(T_pM)$ of $T_{\varphi(p)}N$.

For $\xi \in T_{\varphi(p)}N$ we write $\xi^\perp := \xi - \xi^\top$, so that

$$\xi = \xi^\top + \xi^\perp.$$

We call $\xi^\top$ the **tangential component** of ξ and $\xi^\perp$ the **orthogonal component**. Note if $\dim M = \dim N$ then $\xi^\top = \xi$ and $\xi^\perp = 0$ for all ξ, since projecting onto a subspace of full dimension doesn't do anything.

We can also think of $(\cdot)^\top$ as an operator

$$(\cdot)^\top\colon \Gamma_\varphi(TN) \to \Gamma_\varphi(TN), \qquad W^\top(p) = (W(p))^\top. \tag{47.1}$$

The metric h on N also defines for us a musical isomorphism between vector fields along φ and one-forms along φ: cf. Problem N.5.

$$W \in \Gamma_\varphi(TN) \mapsto W^\flat \in \Gamma_\varphi(T^*N), \qquad W^\flat_p(\xi) := \langle W(p), \xi\rangle\,. \tag{47.2}$$

Now suppose M also carries a Riemannian metric g and φ is isometric. In this case we can combine the musical isomorphism (47.2) with a musical isomorphism on M to obtain another description of the operator $(\cdot)^\top$.

LEMMA 47.2. *Let $\varphi\colon (M, g) \to (N, h)$ be an isometric map between Riemannian manifolds. The composition*

$$\Gamma_\varphi(TN) \xrightarrow{\flat_h} \Gamma_\varphi(T^*N) \xrightarrow{\varphi^*} \Omega^1(M) \xrightarrow{\sharp_g} \mathfrak{X}(M) \xrightarrow{D\varphi} \Gamma_\varphi(TN)$$

coincides with the operator $(\cdot)^\top$ from (47.1).

The proof of Lemma 47.2 is on Problem Sheet P.

PROPOSITION 47.3. *Let $\varphi\colon (M, g) \to (N, h)$ be an isometric map between Riemannian manifolds. Let ∇^g denote the Levi-Civita connection of (M, g), and let ∇^h denote the Levi-Civita connection of (N, h). Then:*

(i) *Write $\nabla^{h,\varphi}$ for the covariant derivative operator along φ induced by the Levi-Civita connection ∇^h on N. Then for all $X, Y \in \mathfrak{X}(M)$ one has*

$$\Big(\nabla^{h,\varphi}_X\big(D\varphi(Y)\big)\Big)^\top = D\varphi\big(\nabla^g_X Y\big)$$

(ii) *If in addition* $\dim M = \dim N$ *then the same thing holds without the "$\top$":*

$$\nabla_X^{h,\varphi}\big(D\varphi(Y)\big) = D\varphi\big(\nabla_X^g Y\big).$$

Proof. Since φ is isometric, it follows from (46.2) and Proposition 46.6 that for $X, Y, Z \in \mathfrak{X}(M)$ that

$$\left\langle \nabla_X^{h,\varphi}(D\varphi(Y)), D\varphi(Z) \right\rangle = \left\langle \nabla_X^g Y, Z \right\rangle.$$

Moreover as φ is isometric we have

$$\left\langle \nabla_X^g Y, Z \right\rangle = \left\langle D\varphi\big(\nabla_X^g Y\big), D\varphi(Z) \right\rangle,$$

which implies that

$$\Big(\nabla_X^{h,\varphi}\big(D\varphi(Y)\big)\Big)^\top = D\varphi\big(\nabla_X^g Y\big)$$

(both sides are elements of $\Gamma_\varphi(TN)$). This proves (i). The second statement is immediate consequence. ∎

Proposition 47.3 can also be reformulated in terms of the connection maps.

PROPOSITION 47.4. *Let* $\varphi\colon (M,g) \to (N,h)$ *be an isometric map between Riemannian manifolds. Let* $K_g\colon TTM \to TM$ *and* $K_h\colon TTN \to TN$ *denote the connection maps associated to the Levi-Civita connections on* (M,g) *and* (N,h) *respectively. Then:*

(i) *If* $\zeta \in T_{(p,\xi)}TM$ *then*

$$D\varphi(p)K_g(\zeta) = \Big(K_h\big(D(D\varphi)(p,\xi)\zeta\big)\Big)^\top,$$

where $D(D\varphi)(p,\xi)$ *denotes the differential of the map* $D\varphi\colon TM \to TN$ *at* $(p,\xi) \in TM$.

(ii) *If in addition* $\dim M = \dim N$ *then the same thing holds without the "$\top$":*

$$D\varphi(p)K_g(\zeta) = K_h\big(D(D\varphi)(p,\xi)\zeta\big).$$

Thus when $\dim M = \dim N$ *the following commutes:*

$$\begin{array}{ccc} TTM & \xrightarrow{D(D\varphi)} & TTN \\ {\scriptstyle K_g}\downarrow & & \downarrow{\scriptstyle K_h} \\ TM & \xrightarrow[D\varphi]{} & TN \end{array}$$

The proof of Proposition 47.4 is immediate from Proposition 47.3 and the definition (31.8) of the covariant derivative operators.

COROLLARY 47.5. *Let* $\varphi\colon (M,g) \to (N,h)$ *be an isometric map between Riemannian manifolds of the same dimension. Let* ∇^g *denote the Levi-Civita connection of* (M,g), *and let* ∇^h *denote the Levi-Civita connection of* (N,h). *Let* R^g *and* R^h *denote their curvature tensors. Then for all* $p \in M$ *and all* $\xi, \zeta, \eta \in T_pM$, *one has*

$$D\varphi(p)\big(R^g(\xi,\zeta)(\eta)\big) = R^h\big(D\varphi(p)\xi, D\varphi(p)\zeta\big)\big(D\varphi(p)\eta\big).$$

Proof. This follows from part (ii) of Proposition 47.4 together with Proposition 35.9. ∎

Definition 47.6. Let (M, g) be a Riemannian manifold. The **exponential map of** g is by definition the exponential map of the geodesic spray of the Levi-Civita connection of g.

The next result shows isometric maps between Riemannian manifolds of the same dimension behave similarly to Lie group homomorphisms for the exponential map of a Riemannian metric (compare this with Proposition 12.5).

Proposition 47.7. *Let $\varphi\colon (M, g) \to (N, h)$ be an isometric map between Riemannian manifolds of the same dimension. Let ∇^g denote the Levi-Civita connection of (M, g), and let ∇^h denote the Levi-Civita connection of (N, h). Let $\exp^g$ and $\exp^h$ denote the associated exponential maps. Then*

$$\exp^h \circ D\varphi = \varphi \circ \exp^g$$

Proof. It follows from part (ii) of Proposition 47.3 that if ρ is a parallel vector field along a curve γ in M then $D\varphi(\rho)$ is a parallel vector field along $\varphi \circ \gamma$ in N. Taking $\rho = \dot{\gamma}$ shows that φ maps geodesics in M to geodesics in N. The claim now follows from the uniqueness part of Proposition 43.5. ∎

The next corollary shows how restrictive the condition of being an isometric map is when the manifolds have the same dimension.

Corollary 47.8. *Let $\varphi, \psi\colon (M, g) \to (N, h)$ be two isometric maps between Riemannian manifolds of the same dimension. Assume M is connected and that there exists $o \in M$ such that $\varphi(p) = \psi(p)$ and $D\varphi(p) = D\psi(p)$. Then $\varphi = \psi$.*

Proof. Let

$$A := \{q \in M \mid \varphi(q) = \psi(q) \text{ and } D\varphi(q) = D\psi(q)\}.$$

Then A is non-empty as $p \in A$. Moreover A is closed as manifolds are Hausdorff and $D\varphi$ and $D\psi$ are continuous (actually, smooth). If $q \in A$ then by part (ii) of Theorem 44.3 there exists a neighbourhood V_q of $0_q \in T_qM$ such that $\exp^g_q$ maps V_q diffeomorphically onto its image. If $\xi \in V_q$ then by Proposition 47.7 we have

$$\begin{aligned}\varphi(\exp^g_q(\xi)) &= \exp^h_{\varphi(q)}(D\varphi(q)\xi)\\ &= \exp^h_{\psi(q)}(D\psi(q)\xi)\\ &= \psi(\exp^g_q(\xi)),\end{aligned}$$

and hence on V_q one has (as smooth maps)

$$\varphi \circ \exp^g_q = \psi \circ \exp^g_q,$$

which in particular implies that $\exp^g_q(V_q) \subset A$. Since $\exp^g_q(V_q)$ is open and q was arbitrary, it follows that A is also open, and hence $A = M$ as M is connected. ∎

Proposition 46.10 from the previous lecture is special case $M = N$ and $\psi = \mathrm{id}$.

We conclude this lecture by showing how a Riemannian metric allows us to view the curvature as a tensor of type $(0, 4)$ instead of type $(1, 3)$, and studying the various symmetries this tensor possesses.

Definition 47.9. Let $(M, g = \langle \cdot, \cdot \rangle)$ denote a Riemannian manifold, and suppose ∇ is a connection on M (not necessarily torsion-free or metric with respect to g). Then $R^\nabla \in \mathcal{T}^{1,3}(M)$. We use g to define a new tensor $\mathcal{R}_g^\nabla \in \mathcal{T}^{0,4}(M)$ by

This is another incarnation of the musical isomorphism – in general the metric defines musical isomorphisms $\mathcal{T}^{h,k} \to \mathcal{T}^{h\pm1,k\mp1}$.

$$\mathcal{R}_g^\nabla(X, Y, Z, W) := \left\langle R^\nabla(X, Y)(Z), W \right\rangle, \qquad \forall\, X, Y, Z, W \in \mathfrak{X}(M).$$

Suppose (U, x) is a chart on M. Then we can write

$$R^\nabla = R^l_{ijk}\, \partial_l \otimes dx^i \otimes dx^j \otimes dx^k$$

and

$$\mathcal{R}_g^\nabla = R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$

where R^l_{ijk} and R_{ijkl} are smooth functions on U given by

$$R^l_{ijk} = dx^l \big(R^\nabla(\partial_i, \partial_j)(\partial_k) \big)$$

and

$$R_{ijkl} := \mathcal{R}_g^\nabla(\partial_i, \partial_j, \partial_k, \partial_l).$$

If we write $g = g_{ij}\, dx^i \otimes dx^j$ then

$$R_{ijkl} = \left\langle R^\nabla(\partial_i, \partial_j)(\partial_k), \partial_l \right\rangle = \left\langle R^h_{ijk} \partial_h, \partial_l \right\rangle = g_{hl} R^h_{ijk}$$

The next result clarifies the symmetries of $\mathcal{R}_g^\nabla$.

Proposition 47.10 (Symmetries of $\mathcal{R}_g^\nabla$). *Let (M, g) be a Riemannian manifold and let ∇ be a connection on M. Then for any $X, Y, Z, W \in \mathfrak{X}(M)$:*

(i) $\mathcal{R}_g^\nabla(X, Y, Z, W) = -\mathcal{R}_g^\nabla(Y, X, Z, W)$.

i.e. flipping the first two variables

(ii) *If ∇ is metric with respect to g then*

i.e. flipping the last two variables

$$\mathcal{R}_g^\nabla(X, Y, Z, W) = -\mathcal{R}_g^\nabla(X, Y, W, Z).$$

(iii) *If ∇ is torsion-free then*

i.e. fixing the last variable and cyclically permuting the other three

$$\mathcal{R}_g^\nabla(X, Y, Z, W) + \mathcal{R}_g^\nabla(Y, Z, X, W) + \mathcal{R}_g^\nabla(Z, X, Y, W) = 0.$$

(iv) *If ∇ is the Levi-Civita connection of g then*

i.e. reversing all the variables

$$\mathcal{R}_g^\nabla(X, Y, Z, W) = \mathcal{R}_g^\nabla(W, Z, Y, X)$$

Proof. Property (i) is clear as R^∇ is alternating. Property (ii) is a restatement of Proposition 37.15. Property (iii) is a restatement of part (i) of Proposition 45.12.

Finally, property (iv) is an algebraic consequence of the other properties. Indeed,

$$\begin{aligned}\mathcal{R}_g^\nabla(X,Y,Z,W) &= -\mathcal{R}_g^\nabla(Y,X,Z,W)\\ &= \mathcal{R}_g^\nabla(X,Z,Y,W) + \mathcal{R}_g^\nabla(Z,Y,X,W)\end{aligned}$$

and also

$$\begin{aligned}\mathcal{R}_g^\nabla(X,Y,Z,W) &= -\mathcal{R}_g^\nabla(X,Y,W,Z)\\ &= \mathcal{R}_g^\nabla(Y,W,X,Z) + \mathcal{R}_g^\nabla(W,X,Y,Z)\end{aligned}$$

and so

$$\begin{aligned}2\mathcal{R}_g^\nabla(X,Y,Z,W) = &\underbrace{\mathcal{R}_g^\nabla(X,Z,Y,W)}_{=\text{(a)}} + \underbrace{\mathcal{R}_g^\nabla(Z,Y,X,W)}_{=\text{(b)}}\\ &+ \underbrace{\mathcal{R}_g^\nabla(Y,W,X,Z)}_{=\text{(c)}} + \underbrace{\mathcal{R}_g^\nabla(W,X,Y,Z)}_{=\text{(d)}}.\end{aligned}$$

Similarly

$$\begin{aligned}2\mathcal{R}_g^\nabla(W,Z,Y,X) = &\underbrace{\mathcal{R}_g^\nabla(W,Y,Z,X)}_{=\text{(e)}} + \underbrace{\mathcal{R}_g^\nabla(Y,Z,W,X)}_{=\text{(f)}}\\ &+ \underbrace{\mathcal{R}_g^\nabla(Z,X,W,Y)}_{=\text{(g)}} + \underbrace{\mathcal{R}_g^\nabla(X,W,Z,Y)}_{=\text{(h)}}.\end{aligned}$$

Then using

$$\begin{aligned}\text{(a)} &= \mathcal{R}_g^\nabla(X,Z,Y,W) = (-1)^2\mathcal{R}_g^\nabla(Z,X,W,Y) = \text{(g)},\\ \text{(b)} &= \mathcal{R}_g^\nabla(Z,Y,X,W) = (-1)^2\mathcal{R}_g^\nabla(Y,Z,W,X) = \text{(f)},\\ \text{(c)} &= \mathcal{R}_g^\nabla(Y,W,X,Z) = (-1)^2\mathcal{R}_g^\nabla(W,Y,Z,X) = \text{(e)},\\ \text{(d)} &= \mathcal{R}_g^\nabla(W,X,Y,Z) = (-1)^2\mathcal{R}_g^\nabla(X,W,Z,Y) = \text{(h)},\end{aligned}$$

we see that

$$2\mathcal{R}_g^\nabla(X,Y,Z,W) = 2\mathcal{R}_g^\nabla(W,Z,Y,X),$$

and this completes the proof. ∎

Bonus Material for Lecture 47

In this bonus section we state a theorem of Epstein, which, roughly speaking, can be thought of as a converse to Proposition 47.3.

DEFINITION 47.11. A **natural Riemannian connection** is an assignment of a connection $\nabla^{M,g}$ to every Riemannian manifold (M,m) which is natural in the following sense: If $\varphi\colon (M,g) \to (N,h)$ is an injective isometric map between Riemannian manifolds of the same dimension (i.e. an isometric open embedding) then

$$\varphi^*\nabla^{N,h} = \nabla^{M,g}.$$

We denote such a natural connection by $\boldsymbol{\nabla}$. A natural Riemannian connection is said to be **homogeneous** if it is invariant under scaling:

$$\nabla^{M,g} = \nabla^{M,cg}$$

for any $c > 0$.

This is easiest to explain with an example.

EXAMPLE 47.12. The assignment $\boldsymbol{\nabla}^{\mathrm{LC}}$ that assigns to each Riemannian manifold (M, g) its Levi-Civita connection is a natural Riemannian connection by Proposition 47.3. It is clear from (46.2) that the Levi-Civita connection is homogeneous.

The definition of a natural Riemannian connection can be phrased more concisely using categorical language. Here are the details. Consider the category $\mathsf{OpenEmb}$ whose objects are smooth manifolds and whose morphisms are *open embeddings*, i.e. embeddings that are diffeomorphisms onto their images (this is a subcategory of the category Man – note there are no morphisms from M to N in this category if $\dim M \neq \dim N$). Consider the contravariant functor $\mathscr{R}$ on $\mathsf{OpenEmb}$ that assigns to a manifold M the space $\mathscr{R}(M)$ of all Riemannian metrics on M, and assigns to an open embedding $\varphi\colon M \to N$ the induced map

$$\varphi^* \colon \mathscr{R}(N) \to \mathscr{R}(M), \qquad g \mapsto \varphi^* g.$$

The space $\mathscr{R}(M)$ is actually an infinite-dimensional (locally) Fréchet manifold, but we won't need or use this fact.

In a similar vein there is a contravariant functor $\mathscr{C}$ on $\mathsf{OpenEmb}$ that assigns to M the space $\mathscr{C}(M)$ of all connections on M, and on morphisms operates by pullback. Then a natural Riemannian connection $\boldsymbol{\nabla}$ is exactly a natural transformation from $\mathscr{R}$ to $\mathscr{C}$.

DEFINITION 47.13. Suppose $\boldsymbol{\nabla}$ is a natural Riemannian connection. We say that ∇ is of **polynomial type** if for each $m \geq 0$ there exist polynomials P_{ij}^k for $1 \leq i, j, k \leq m$ such that: For any m-dimensional Riemannian manifold (M, g), and for any chart (U, x) the Christoffel symbols Γ_{ij}^k are given as polynomials in the components g_{ij} of g relative to x, together with their inverse g^{ij}, and all derivatives of g_{ij} up to some finite order d, i.e.

$$\Gamma_{ij}^k = P_{ij}^k \left(g_{pq}; g^{rs}; \frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{hl} \right),$$

where α is any multi-index of degree at most d.

Again, this is easiest to explain with an example.

EXAMPLE 47.14. The natural Riemannian connection $\boldsymbol{\nabla}^{\mathrm{LC}}$ is a polynomial connection by Lemma 46.2 (with $d = 1$).

Here now is our promised theorem. One should think of it as a far-reaching complement of Theorem 46.1 (which in fact deserves the name "The Fundamental Theorem of Riemannian Geometry much better!)

THEOREM 47.15 (Epstein, 1978). *Let $\boldsymbol{\nabla}$ be a homogeneous natural Riemannian connection. Assume $\boldsymbol{\nabla}$ is of polynomial type. Then $\boldsymbol{\nabla} = \boldsymbol{\nabla}^{\mathrm{LC}}$ is the Levi-Civita connection.*

LECTURE 48

Sectional, Ricci, and Scalar Curvature

In this lecture we investigate various other curvatures that can be associated to a Riemannian manifold. In doing so we will finally make contact with the geometric intuition of the word "curvature": as we will see, the sphere S^m thought of as a Riemannian submanifold of $\mathbb{R}^{m+1}$ is positively curved, whereas the hyperbolic plane with its natural metric (see Definition 48.18) is negatively curved.

Definition 48.1. Let (M, g) be a Riemannian manifold. Let ∇ denote the Levi-Civita connection of g, and fix $p \in M$. Given two linearly independent tangent vectors $\xi_1, \xi_2 \in T_pM$ we define the **sectional curvature** of the 2-plane $\Pi = \operatorname{span}\{\xi_1, \xi_2\} \subseteq T_pM$ to be

$$\operatorname{sect}_g(p; \Pi) := \frac{\mathcal{R}_g^{\nabla}(\xi_1, \xi_2, \xi_2, \xi_1)}{\langle \xi_1, \xi_1 \rangle \langle \xi_2, \xi_2 \rangle - \langle \xi_1, \xi_2 \rangle^2}. \tag{48.1}$$

Note that this depends only on the 2-plane Π and not the choice of basis $\{\xi_1, \xi_2\}$, since both $\mathcal{R}_g^{\nabla}$ and g are linear and thus both the numerator and denominator of (48.1) are homogeneous of degree two. In particular, if e_1, e_2 are orthonormal vectors such that $\Pi := \operatorname{span}\{e_1, e_2\}$ then then

$$\operatorname{sect}_g(p; \Pi) = \mathcal{R}_g^{\nabla}(e_1, e_2, e_2, e_1).$$

Remark 48.2. If $\dim M = 2$ then there is only one two-plane in each tangent space (namely, the entire tangent space), and thus in this case the sectional curvature is simply a function $\operatorname{sect}_g \colon M \to \mathbb{R}$. For historical reasons in this case the sectional curvature is often called the **Gaussian curvature**.

Definition 48.3. Let (M, g) be a Riemannian manifold and let $\kappa \in \mathbb{R}$. We say that (M, g) has **constant curvature** κ if

$$\operatorname{sect}_g(p; \Pi) = \kappa, \qquad \forall\, p \in M,\ \forall\, \text{2-planes } \Pi \subset T_pM.$$

Example 48.4. If we consider $\mathbb{R}^m$ with its standard Euclidean metric (part 46.13 of Examples 46.13) then $\mathbb{R}^m$ has constant curvature with $\kappa = 0$.

Example 48.5. If we consider the sphere S^m as a Riemannian submanifold of $\mathbb{R}^{m+1}$ part 46.13 of Examples 46.13 then it follows from Problem M.5 that S^m has constant curvature with $\kappa = 1$. More generally, if $S^m(r)$ denotes the sphere of radius $r > 0$ then the same argument shows that $S^m(r)$ (as a Riemannian submanifold of $\mathbb{R}^{m+1}$) has constant curvature with $\kappa = \frac{1}{r^2}$.

We will discuss the case of $\kappa < 0$ in Definition 48.18 below.

Remark 48.6. The argument from Problem N.9 easily adapts to show that if M is any manifold that admits a metric of constant curvature then $p_r(TM) = 0$ for all $r > 0$.

In fact, the sectional curvature determines the full Riemannian curvature tensor. In order to prove this, we need the following algebraic lemma.

These are the four symmetries from Proposition 47.10.

LEMMA 48.7. *Let V be a vector space and $R_1, R_2 \colon V \times V \times V \times V \to \mathbb{R}$ two quadrilinear maps such that for all $x, y, z, w \in V$ and $i = 1, 2$:*

(i) $R_i(x, y, z, w) = -R_i(y, x, z, w)$,

(ii) $R_i(x, y, z, w) = -R_i(x, y, w, z)$,

(iii) $R_i(x, y, z, w) + R_i(y, z, x, w) + R_i(z, x, y, w) = 0$.

(iv) $R_i(x, y, z, w) = R_i(w, z, y, x)$.

Then if for all $x, y \in V$ we also have $R_1(x, y, y, x) = R_2(x, y, y, x)$, then in fact $R_1 \equiv R_2$.

Proof. It suffices to show that if a quadrilinear map R satisfies the four conditions of the lemma and in addition satisfies $R(x, y, y, x) = 0$ for all $x, y \in V$ then $R \equiv 0$. So suppose this is the case. Then

$$\begin{aligned} 0 &= R(x+z, y, y, x+z) \\ &= R(x, y, y, x) + R(z, y, y, x) + R(x, y, y, z) + R(z, y, y, z) \\ &= R(x, y, y, z) + R(z, y, y, x) + 0 \\ &= 2R(x, y, y, z), \end{aligned}$$

and hence R is also alternating with respect to the second and third variables:

$$R(x, y, z, w) = -R(x, z, y, w)$$

Then

$$\begin{aligned} 0 &= R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) \\ &= R(x, y, z, w) - R(y, x, z, w) - R(x, z, y, w) \\ &= 3R(x, y, z, w). \end{aligned}$$

This completes the proof. ■

COROLLARY 48.8. *The sectional curvatures determine the full curvature tensor.*

The next corollary tells us that if the sectional curvatures at a given point are independent of the choice of two-plane then the full curvature tensor takes a particularly nice form. First, a definition:

DEFINITION 48.9. Let (M, g) denote a Riemannian manifold. Define a tensor $\mathcal{S}_g \in \mathcal{T}^{0,4}(M)$ by

$$\mathcal{S}_g(X, Y, Z, W) := \langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle.$$

COROLLARY 48.10. *Suppose that (M, g) is a Riemannian manifold and ∇ is the Levi-Civita connection on M. Suppose there exists a function $f \in C^\infty(M)$ such that*

$$\mathrm{sect}_g(p; \Pi) = f(p), \qquad \forall \text{ 2-planes } \Pi \subset T_pM.$$

Then $\mathcal{R}_g^\nabla = f\mathcal{S}_g$.

Proof. Apply Lemma 48.7 to $\mathcal{R}_g^\nabla$ and $f\mathcal{S}_g$. ∎

If M is 2-dimensional then the hypotheses of Corollary 48.10 are automatically satisfied (cf. Remark 48.2), and hence we obtain:

COROLLARY 48.11. *Let (M, g) be a two-dimensional Riemannian manifold, and let ∇ denote the Levi-Civita connection of g. Then*

$$\mathcal{R}_g^\nabla = \mathrm{sect}_g \, \mathcal{S}_g,$$

where $\mathrm{sect}_g \in C^\infty(M)$ denotes the sectional (or Gaussian) curvature.

In higher dimensions the situation dramatically changes: if M is connected then the hypotheses of Corollary 48.10 force g to have to constant curvature.

THEOREM 48.12 (Schur's Theorem, Version I). *Let (M, g) be a connected Riemannian manifold of dimension $n \geq 3$. Suppose there exists a function $f \in C^\infty(M)$ such that $\mathrm{sect}_g(p; \Pi) = f(p)$ for all 2-planes $\Pi \subset T_pM$. Then f is a constant function, and hence (M, g) is a space of constant curvature.*

The proof of Theorem 48.12 requires the following preliminary technical lemma.

LEMMA 48.13. *Let (M, g) be a Riemannian manifold, and let ∇ denote the Levi-Civita connection of g. Fix $p \in M$ and let (U, x) be normal coordinates at p. Then for all $1 \leq h, i, j, k, l \leq m$ one has*

$$\partial_i R_{jklh}(p) + \partial_j R_{kilh}(p) + \partial_k R_{ijlh}(p) = 0. \tag{48.2}$$

Lemma 48.13 is essentially just a restatement of part (ii) of Proposition 45.12.

Proof. The following computation is only valid at the point p, but to keep the notation simple in the following computation we omit the p from both sides:

$$\begin{aligned}
\big(\nabla_{\partial_i} R^\nabla\big)(\partial_j, \partial_k)(\partial_l) &= \nabla_{\partial_i}\big(R^\nabla(\partial_j, \partial_k)\big)(\partial_l) - R^\nabla\big(\nabla_{\partial_i}\partial_j, \partial_k\big)(\partial_l) \\
&\quad - R^\nabla\big(\partial_j, \nabla_{\partial_i}\partial_k\big)\partial_l \\
&= \nabla_{\partial_i}\big(R^\nabla(\partial_j, \partial_k)\big)(\partial_l) + 0 \\
&= \nabla_{\partial_i}\big(R^\nabla(\partial_j, \partial_k)\partial_l\big) - R^\nabla(\partial_j, \partial_k)\big(\nabla_{\partial_i}\partial_l\big) \\
&= \nabla_{\partial_i}\big(R_{jkl}^h \partial_h\big) + 0 \\
&= \partial_i\big(R_{jkl}^h\big)\partial_h + R_{jkl}^h \nabla_{\partial_i}\partial_h \\
&= \partial_i\big(R_{jkl}^h\big)\partial_h + 0,
\end{aligned}$$

where the first equality used the definition of the induced connection on the tensor bundle $T^{1,3}(TM) \to M$ and the third equality used the definition of the induced connection on $T^{1,1}(TM)$. Thus part (ii) of Proposition 45.12 tells us that in these coordinates we have

$$\partial_i\big(R_{jkl}^h\big)(p) + \partial_j\big(R_{kil}^h\big)(p) + \partial_k\big(R_{ijl}^h\big)(p) = 0. \tag{48.3}$$

We now translate this into a statement about R_{ijkl}. Firstly, by definition:

$$R_{ijkl} = \langle R^{\nabla}(\partial_i, \partial_j)(\partial_k), \partial_l\rangle = \langle R^h_{ijk}\partial_h, \partial_l\rangle = g_{hl}R^h_{ijk}.$$

By (46.5) the first derivatives of the g_{hl} vanish at p, and hence for any $1 \le \nu \le m$ we have

$$\begin{aligned}\partial_\nu R_{ijkl}(p) &= \partial_\nu\big(g_{hl}R^h_{ijk}\big)(p)\\ &= \partial_\nu g_{hl}(p)R^h_{ijk}(p) + g_{hl}(p)\partial_\nu\big(R^h_{ijk}\big)(p)\\ &= \partial_\nu\big(R^l_{ijk}\big)(p).\end{aligned}$$

Inserting this into (48.3) gives the desired equation. ■

We can now prove Schur's Theorem 48.12.

Proof of Theorem 48.12. Fix $p \in M$, and let (U, x) be normal coordinates about p. Applying Corollary 48.10 to the coordinate vector fields ∂_i we see that on U we have

$$R_{jklh} = f\big(g_{jh}g_{kl} - g_{jl}g_{kh}\big).$$

Differentiating both sides of this equation and evaluating at p gives

$$\partial_i R_{jklh}(p) = \partial_i f(p)\big(\delta_{jh}\delta_{kl} - \delta_{jl}\delta_{kh}\big),$$

where we again used the fact that the first derivatives of the g_{jk} vanish at p. Inserting this equation into (48.2) (together with the analogous statements for ∂_j and ∂_k) gives us

$$\begin{aligned}0 &= \partial_i R_{jklh}(p) + \partial_j R_{kilh}(p) + \partial_k R_{ijlh}(p)\\ &= \partial_h f(p)\big(\delta_{jh}\delta_{kl} - \delta_{jl}\delta_{kh}\big) + \partial_j f(p)\big(\delta_{kh}\delta_{il} - \delta_{kl}\delta_{ih}\big)\\ &\quad + \partial_k f(p)\big(\delta_{ih}\delta_{jl} - \delta_{il}\delta_{jh}\big).\end{aligned}$$

Now fix an arbitrary $1 \le i \le m$. Since $m \ge 3$, we can choose j, k such that i, j, k are all distinct. Then setting $h = k, l = j$ in the previous equation gives

$$0 = -\partial_i f(p).$$

Since i was arbitrary, it follows that $df_p = 0$. Since p was arbitrary, f must be locally constant. Since M is connected, f is constant. This completes the proof. ■

We next investigate how the sectional curvatures change when one changes the metric. This will lead us to the hyperbolic plane.

Definition 48.14. Let M be a smooth manifold. Two Riemannian metrics g_1 and g_2 on M are **conformally equivalent** if there exists a smooth positive function $\phi\colon M \to (0, \infty)$ such that $g_2 = \phi g_1$.

Let us compute (or more accurately, state) how the Levi-Civita connection and its curvature tensor change under conformal equivalence. In the following we let $g_1 = \langle\cdot,\cdot\rangle$ denote a Riemannian metric on M and we let $g_2 = \phi g_1$ denote a conformally equivalent metric. Set

$$\psi := \log\sqrt{\phi} \qquad \text{so that} \qquad g_2 = e^{2\psi}g_1$$

Lemma 48.15. *Let ∇^i be the Levi-Civita connection of g_i. Then for $X, Y \in \mathfrak{X}(M)$ one has*

$$\nabla^2_X Y - \nabla^1_X Y = X(\psi)Y + Y(\psi)X - \langle X, Y\rangle d\psi^\sharp.$$

Here $\sharp$ denotes the musical isomorphism with respect to $g_1 = \langle \cdot, \cdot \rangle$.

Note that if ϕ is a constant function then $\nabla^2 = \nabla^1$ – this once again shows that the Levi-Civita connection is homogeneous in the sense of Definition 47.11. Next, we have:

Lemma 48.16. *Let R^i be curvature tensor of the Levi-Civita connection ∇^i of g_i. Then for $X, Y, Z \in \mathfrak{X}(M)$ one has*

$$\begin{aligned} R^2(X,Y)(Z) - R^1(X,Y)(Z) = {} & \langle \nabla^1_X(d\psi^\sharp), Z\rangle Y - \langle \nabla^1_Y(d\psi^\sharp), Z\rangle X \\ & - \langle X, Z\rangle \nabla^1_Y(d\psi^\sharp) - \langle Y, Z\rangle \nabla^1_X(d\psi^\sharp) \\ & + Y(\psi)Z(\psi)X - \langle Y, Z\rangle \|d\psi^\sharp\|^2 X \\ & - X(\psi)Z(\psi)Y + \langle X, Z\rangle \|d\psi^\sharp\|^2 Y \\ & + X(\psi)\langle Y, Z\rangle d\psi^\sharp - Y(\psi)\langle X, Z\rangle d\psi^\sharp. \end{aligned}$$

The proof of Lemma 48.16 is an easy, albeit lengthy computation, which we leave to the conscientious reader as a wholesome exercise.

Corollary 48.17. *Let $p \in M$ and let $\Pi = \operatorname{span}\{e_1, e_2\} \subset T_pM$, where the e_i are orthonormal with respect to g_1. Then*

$$\begin{aligned} \phi(p)\operatorname{sect}_{g_2}(p;\Pi) - \operatorname{sect}_{g_1}(p;\Pi) = & - \langle \nabla^1_{e_1}(d\psi^\sharp), e_1\rangle - \langle \nabla^1_{e_2}(d\psi^\sharp), e_2\rangle \\ & - \|d\psi^\sharp(p)\|^2 + e_1(\psi)^2 + e_2(\psi)^2. \end{aligned}$$

Recall our notation for a half-space from Definition 24.18.

Definition 48.18. Let $\mathbb{H}^m := \mathbb{R}^m_{u^m>0}$. We equip $\mathbb{H}^m$ with the metric $g_{\mathrm{hyp}} := \phi g_{\mathrm{Euc}}$ where ϕ is the smooth positive function $\phi(u^1, \dots, u^m) = \frac{1}{(u^m)^2}$. Thus $\psi = -\log u^m$ and Corollary 48.17 becomes

$$\phi \operatorname{sect}_{g_{\mathrm{hyp}}}(p;\Pi) - 0 = -\phi.$$

Thus $(\mathbb{H}^m, g_{\mathrm{hyp}})$ is a space with constant curvature $\kappa = -1$. We call $(\mathbb{H}^m, g_{\mathrm{hyp}})$ the m-dimensional **hyperbolic plane**. More generally if we take $\phi = \frac{r^2}{(u^m)^2}$ then we get a space with constant curvature $\kappa = -\frac{1}{r^2}$. We denote this metric by $g_{\mathrm{hyp};r}$.

We conclude the our discussion on sectional curvature with the following theorem. In the following we say a Riemannian manifold (M, g) is **complete** if the Levi-Civita connection ∇ of g is complete in the sense of Definition 43.7.

In Lecture **??** we will see that this is equivalent to asking that M is complete as a metric space.

Theorem 48.19 (Killing-Hopf). *Let (M, g) be a connected, simply connected and complete Riemannian manifold with constant curvature κ. Then (M, g) is isometric to exactly one of the following three manifolds:*

(i) $(\mathbb{R}^m, g_{\mathrm{Euc}})$ *if* $\kappa = 0$,

(ii) $(S^m(r), g_{\mathrm{round}})$ *if* $\kappa > 0$, *where* $r := \frac{1}{\sqrt{\kappa}}$.

(iii) $(\mathbb{H}^m, g_{\mathrm{hyp};r})$ *if* $\kappa < 0$, *where* $r := \frac{1}{\sqrt{-\kappa}}$.

Sadly we won't have enough time to prove Theorem 48.19. We will however prove several related results in Lecture **??**, starting with the famous *Cartan-Hadamard Theorem* (Theorem **??**).

Instead for now we move onto our next variant of the curvature tensor.

Definition 48.20. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. The **Ricci** tensor of g is the $(0, 2)$-tensor defined as followed: for $p \in M$ and $\xi, \zeta \in T_pM$,

In more formal language, the Ricci tensor is the **trace** of the full curvature tensor. See Definition **??**.

$$\mathrm{Ric}_g(\xi, \zeta) := \sum_{i=1}^{m} \mathcal{R}_g^{\nabla}(e_i, \xi, \zeta, e_i), \tag{48.4}$$

where (e_i) is an orthornomal basis of T_pM.

Note that Ric_g is symmetric by part (iv) of Proposition 47.10. In a chart (U, x) if we write

$$\mathrm{Ric}_g = r_{ij}\, dx^i \otimes dx^j,$$

where $r_{ij} = \mathrm{Ric}_g(\partial_i, \partial_j)$, then

$$r_{ij}(p) = \sum_{h=1}^{m} R_{hijh}(p) \tag{48.5}$$

When performing computations with the Ricci tensor we typically need to include the summation signs (i.e. the Einstein Summation Convention "doesn't work"). This is because the definition (48.4) has a sum over the index i, which appears twice as a lower index.

Remark 48.21. Unlike the sectional curvatures, if $\dim M \geq 4$, the full curvature tensor $\mathcal{R}_g^{\nabla}$ is in general not completely determined by the Ricci tensors. This should not surprise you, as one typically cannot recover a matrix from its trace. When $\dim M = 2$ or $\dim M = 3$ however it is possible to recover $\mathcal{R}_g^{\nabla}$ from Ric_g, as you show on Problem Sheet P.

The Ricci tensor is a symmetric tensor of type $(0, 2)$. The metric is another symmetric tensor of type $(0, 2)$, and it therefore makes sense to ask whether the two are related. In general the answer is "no": for instance, there is no reason why Ric_g should be positive definite.

Definition 48.22. We say that a metric g is an **Einstein metric** on M if there exists a constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric}_g = \lambda g.$$

We will discuss the motivation for this condition (together with an explanation of the name) in the bonus section of this lecture. For now let us note that this notion is only interesting when $\dim M \geq 4$. Indeed, on Problem Sheet P you will prove that if $\dim M = 2$ or $\dim M = 3$ then a metric g is Einstein if and only if g has constant curvature.

Here is the Ricci Curvature version of Schur's Theorem 48.12.

Theorem 48.23 (Schur's Theorem, Version II). *Let (M, g) be a connected Riemannian manifold of dimension $m \geq 3$. Assume there exists a smooth function f on M such that $\mathrm{Ric}_g = fg$. Then f is a constant function, and hence g is an Einstein metric.*

The proof of Theorem 48.23 again uses Lemma 48.13, and is similar to that of Theorem 48.12.

Proof. Fix $p \in M$ and let (U, x) be normal coordinates about p. By assumption, we have

$$r_{ij}(p) = f(p)g_{ij}(p), \qquad \forall\, 1 \le i, j \le m.$$

As in the proof of Lemma 48.13, the following computations are only valid at the point p. Nevertheless we omit the p on both sides to avoid over-complicating the notation. We will also once again suspend our use of the summation convention, as it will prove confusing in this proof. Fix some $\nu \in \{1, \dots, m\}$. Using (48.5) together with the fact that the first derivatives of g_{ij} vanish at p we obtain

i.e. (46.5)

$$\delta_{ij}\partial_\nu f = \partial_\nu r_{ij} = \sum_{h=1}^{m} \partial_\nu R_{hijh}. \tag{48.6}$$

Set $i = j$ and sum over both i and h to obtain

$$m\partial_\nu f = \sum_{i=1}^{m} \delta_{ii}\partial_\nu f = \sum_{i=1}^{m}\sum_{h=1}^{m} \partial_\nu R_{hiih}. \tag{48.7}$$

Next, using (48.2) with $j = h, k = \nu$ and $l = i$ we have

$$\partial_i R_{hpih} + \partial_h R_{\nu iih} + \partial_\nu R_{ihih} = 0.$$

Using parts (i) and (ii) of Proposition 47.10 we rewrite this as

$$\partial_i R_{h\nu ih} + \partial_h R_{i\nu hi} = \partial_\nu R_{hiih}, \tag{48.8}$$

and hence summing (48.8) over i and h and inserting into (48.7), we have

$$\begin{aligned}
m\partial_\nu f &= \sum_{i=1}^{m}\sum_{h=1}^{m} \partial_i R_{h\nu ih} + \sum_{i=1}^{m}\sum_{h=1}^{m} \partial_h R_{i\nu hi} \\
&= \sum_{i=1}^{m}\left(\sum_{h=1}^{m} \partial_i R_{h\nu ih}\right) + \sum_{h=1}^{m}\left(\sum_{i=1}^{m} \partial_h R_{i\nu hi}\right) \\
&= \sum_{i=1}^{m} \partial_i r_{\nu i} + \sum_{h=1}^{m} \partial_h r_{\nu h} \\
&= \sum_{i=1}^{m} \delta_{\nu i}\partial_i f + \sum_{h=1}^{m} \delta_{\nu h}\partial_h f && \text{by (48.6)} \\
&= \partial_\nu f + \partial_\nu f \\
&= 2\partial_\nu f.
\end{aligned}$$

Since $m \ne 2$ we must have $\partial_\nu f(p) = 0$. Since ν was arbitrary we have $df_p = 0$, and then since p was arbitrary it follows that f is locally constant. Since M is connected, f is constant. ■

We can repeat the trick we used to obtain the Ricci curvature from the full curvature. This gives us a tensor of type $(0, 0)$, i.e. a smooth function.

i.e. taking the trace

DEFINITION 48.24. The **scalar curvature** $\mathrm{scal}_g \in C^\infty(M)$ is defined by

$$\mathrm{scal}_g(p) := \sum_{j=1}^{m} \mathrm{Ric}_g(e_j, e_j) = \sum_{i=1}^{m}\sum_{j=1}^{m} \mathcal{R}_g^{\nabla}(e_i, e_j, e_j, e_i),$$

where (e_i) is an orthonormal basis of T_pM.

Despite the fact that the scalar curvature is "only" a function, it still caries a lot of information about the Riemannian manifold (M, g). Some applications of this are covered on Problem Sheet Q.

Bonus Material for Lecture 48

In several reasonable senses, Einstein metrics are the "best" sort of Riemannian metric a manifold can carry. Here are three explanations as to why:

(i) A naive guess as to what a "best" metric might look like would be to ask that g has constant curvature. But Theorem 48.19 (together with the Cartan-Hadamard Theorem **??**) shows that this is too restrictive, in the sense that many manifolds M cannot admit such a metric. Indeed, if the universal cover $\widetilde{M}$ of M is not diffeomorphic to $\mathbb{R}^m$ or S^m, then no such metric exists. On the other hand, asking for a metric to have constant scalar curvature is not restrictive enough: one can show that if M is any compact manifold of dimension $m \geq 3$ then M admits an infinite dimensional family of metrics with constant scalar curvature. However the Einstein condition is "just right", in the sense that when Einstein metrics exist, they always occur in finite-dimensional families. It is known that some compact manifolds admit no Einstein metrics, but it is hoped that "most" high-dimensional manifolds do admit them. This is an active field of current research,

(ii) Consider the space $\mathscr{R}_1(M)$ of all Riemannian metrics g on M with volume 1. This space can be seen as an infinite-dimensional Fréchet manifold. Now consider the functional

$$\mathscr{S} : \mathscr{R}_1(M) \to \mathbb{R}, \qquad \mathscr{S}(g) := \int_{M,g} \mathrm{scal}_g \, .$$

See Lecture **??** for the definition the volume of a metric and for the integral $\int_{M,g} f$ of a function f.

This functional is differentiable, and with a little bit of work one can show that a metric g is a critical point of $\mathscr{S}$ if and only if g is an Einstein metric. Thus Einstein metrics are obtained by doing calculus of variations on the space of metrics.

i.e. $d\mathscr{S}_g = 0$

(iii) The name "Einstein metric" comes from physics (no surprises there!) In general relativity, one posits that physical spacetime is a four-dimensional manifold equipped with a **Lorentz metric** (this is like a Riemannian metric, apart from instead of being positive definite, it has signature $(3,1)$ – it is negative definite on the time direction). The **Einstein Field Equation** states that

$$\mathrm{Ric}_g - \frac{1}{2}\,\mathrm{scal}_g \cdot g = T, \tag{48.9}$$

where T is the so-called **stress-energy tensor**. If $T \equiv 0$ then we obtain the **Einstein field equation in a vacuum**. In fact in this case one necessarily has $\mathrm{scal}_g = 0$, and thus the Einstein field equation in a vacuum is equivalent to asking that $\mathrm{Ric}_g = 0$. However from a mathematical point of view, it is then a natural generalisation the vacuum version of (48.9) to look at what we have deemed Einstein metrics.

A wonderful book on this subject (and a gateway to advanced Riemannian geometry in general) is the monograph Einstein Manifolds by Besse. I highly recommend it.

Problem Sheet A

Problem A.1. Let $\varphi\colon M \to N$ be a continuous map between two smooth manifolds. Prove that the following two statements are equivalent:

(i) For every point $p \in M$, if (U, x) is **any** chart on M with $p \in U$ and (V, y) is **any** chart on N with $\varphi(U) \subseteq V$, the composition

$$y \circ \varphi \circ x^{-1} \colon x(U) \to y(V)$$

is of class C^k.

(ii) For every point $p \in M$, **there exists** a chart (U, x) on M with $p \in U$ and a chart (V, y) on N with $\varphi(U) \subseteq V$ such that the composition

$$y \circ \varphi \circ x^{-1} \colon x(U) \to y(V)$$

is of class C^k.

Problem A.2. Prove that the set $\mathrm{GL}(m)$ of invertible $m \times m$ matrices is a smooth manifold of dimension m^2.

Problem A.3. Let M and N be two smooth manifolds of dimension m and n respectively. Prove that $M \times N$ is a smooth manifold of dimension $m + n$. Deduce that the m-dimensional torus:

$$T^m := \underbrace{S^1 \times \cdots \times S^1}_{m \text{ times}} \subset \mathbb{R}^{2m}.$$

is a compact smooth manifold of dimension m.

Problem A.4. Let $\mathbb{R}P^m$ denote m-dimensional real projective space, i.e. the space of lines through the origin in $\mathbb{R}^{m+1}$. Prove that $\mathbb{R}P^m$ is a compact smooth manifold of dimension m.

Problem A.5. Let $G(k, m)$ denote the set of k-dimensional linear subspaces of $\mathbb{R}^m$. We call $G(k, m)$ a Grassmannian manifold. Prove that $G(k, m)$ is a compact smooth manifold and compute its dimension.

Problem A.6. Let X denote the union of the x-axis and the y-axis in $\mathbb{R}^2$. Prove that X is *not* a topological manifold.

Problem A.7. Let Y denote the "pinched 2-dimensional torus", as shown in Figure A.1. Prove that Y is *not* a topological manifold.

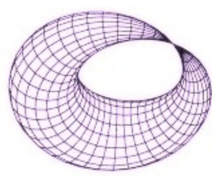


Figure A.1: The pinched torus.

Problem A.8. Show that the smooth atlas on $\mathbb{R}$ consisting of the single chart $x\colon \mathbb{R} \to \mathbb{R}$ given by $x(t) = t^3$ defines a smooth structure that is *different* to the "standard" smooth structure (the latter is the smooth structure containing the identity map as a chart). Prove however that both the smooth structures belong to the same diffeomorphism class.

Bonus Problem(s) for Sheet A

These problem(s) are hard, and are included for enthusiasts only.
Solutions will not be provided.

Problem A.9. Let M be a compact connected topological manifold of dimension one. Prove that M is homeomorphic to S^1.

Problem Sheet B

PROBLEM B.1. Let $\{V_a \mid a \in A\}$ be a family of vector spaces indexed by a set A, and let W be a fixed set. Suppose that for each $a \in A$ we are given a bijection $\ell_a \colon V_a \to W$ such that for any $a, b \in A$, the composition $\ell_b^{-1} \circ \ell_a \colon V_a \to V_b$ is a linear isomorphism. Prove that there is a unique vector space structure on W such that each ℓ_a is a linear isomorphism.

PROBLEM B.2. Let M be a smooth manifold of dimension m with maximal smooth atlas $\mathcal{X}$. Given a point $p \in M$, let $\mathcal{X}_p \subset \mathcal{X}$ denote the set of charts (U, x) such that $p \in U$. Define an equivalence relation on $\mathbb{R}^m \times \mathcal{X}_p$ by saying

This problem is non-examinable.

$$(\xi, x) \sim (\zeta, y) \qquad \Leftrightarrow \qquad D(y \circ x^{-1})(x(p))\xi = \zeta.$$

(i) Prove that this is indeed a well-defined equivalence relation.

(ii) Write $[\xi, x]$ denote the equivalence class of (ξ, x), and let $\mathcal{T}_p M$ denote the set of equivalence classes. Prove that for every $x \in \mathcal{X}_p$ the map $\ell_x \colon \mathbb{R}^m \to \mathcal{T}_p M$ given by

$$\ell_x \xi \coloneqq [\xi, x]$$

is a bijection. Deduce that $\mathcal{T}_p M$ admits a unique vector space structure such that each ℓ_x is a linear isomorphism.

(iii) Let $(U, x) \in \mathcal{X}_p$, and let $\tilde{\ell}_x \colon \mathbb{R}^m \to T_p M$ denote the linear isomorphism defined by

$$\tilde{\ell}_x e_i = \frac{\partial}{\partial x^i}\Big|_p.$$

Prove that there exists a linear isomorphism $\kappa_p \colon \mathcal{T}_p M \to T_p M$ which in addition satisfies

$$\kappa_p \circ \ell_x = \tilde{\ell}_x,$$

for every $(U, x) \in \mathcal{X}_p$. Deduce that $\mathcal{T}_p M$ is another equivalent way to define the tangent space of a manifold.

PROBLEM B.3. Let E and F be vector spaces and assume that $\ell \colon E \to F$ is a linear map. Prove that for any $p \in E$ the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\ \ell\ } & F \\ {\scriptstyle \mathcal{J}_p}\downarrow & & \downarrow{\scriptstyle \mathcal{J}_{\ell p}} \\ T_p E & \xrightarrow[D\ell(p)]{} & T_{\ell p} F \end{array}$$

PROBLEM B.4. Let M be a smooth manifold of dimension m. Prove that the cotangent bundle $T^* M$ is naturally a smooth manifold of dimension $2m$.

Bonus Problem(s) for Sheet B

These problem(s) are hard, and are included for enthusiasts only. Solutions will not be provided.

Recall from Proposition 1.32 that a locally Euclidean space M is a topological manifold if and only if it is Hausdorff, paracompact, and has at most countably many components. The following three problems show that no two of these conditions imply the third.

PROBLEM B.5. Consider the subspace $S \coloneqq \mathbb{R} \times \{-1\} \cup \mathbb{R} \times \{1\} \subseteq \mathbb{R}^2$ together with its subspace topology and define an equivalence relation on S by setting

$$(u_1, v_1) \sim (u_2, v_2) \quad \Leftrightarrow \quad u_1 = u_2 \quad \text{and} \quad u_1, u_2 \neq 0.$$

Equip $M = S/\sim$ with the quotient topology. Prove that M is locally Euclidean and paracompact, but not Hausdorff.

PROBLEM B.6. Consider $\mathbb{R}^2$ as a set and equip it with the topology $\mathcal{T}$ generated by the basis

$$\mathcal{B} = \{U \times \{a\} \mid U \subseteq \mathbb{R} \text{ open}, \, a \in \mathbb{R}\}.$$

Define

$$x_a \colon \mathbb{R} \times \{a\} \to \mathbb{R}, \qquad x_a(p, a) = p$$

and set $\mathcal{X} = \{x_a \mid a \in \mathbb{R}\}$. Prove that the topological space $(\mathbb{R}^2, \mathcal{T})$ is locally Euclidean, Hausdorff and paracompact, but that it has an uncountable number of connected components.

PROBLEM B.7. Let $H \subset \mathbb{R}^2$ be the right half plane

$$H \coloneqq \left\{(u, v) \in \mathbb{R}^2 \mid u > 0\right\},$$

endowed with the subspace topology from $\mathbb{R}^2$. Given $c \in \mathbb{R}$, let $M_c \subset \mathbb{R}^3$ be the set

$$M_c \coloneqq \{(u, v, c) \mid u \leq 0\},$$

endowed with the subspace topology from $\mathbb{R}^3$. Then set

$$M \coloneqq H \sqcup \bigsqcup_{c \in \mathbb{R}} M_c,$$

Next, given $a, b, c, \varepsilon \in \mathbb{R}$ with $a < b$ and $\varepsilon > 0$, let

$$U(a, b, c, \varepsilon) \coloneqq \{(u, v) \in H \mid 0 < u < \varepsilon, \; c + au < v < c + bu\} \subset H,$$

and

$$V(a, b, c, \varepsilon) \coloneqq \{(u, v, c) \mid -\varepsilon < u \leq 0, \; a < v < b\}.$$

Define a topology on M by declaring that a basis is given by all sets of the following three forms:

(i) open sets in H,

(ii) open sets in $\operatorname{int} M_a$,

(iii) each union $U(a, b, c, \varepsilon) \cup V(a, b, c, \varepsilon)$.

Prove that M is locally Euclidean, connected and Hausdorff but not paracompact.

Problem Sheet C

PROBLEM C.1. Let M and N be smooth manifolds. Prove that for all $(p, q) \in M \times N$ there is a canonical isomorphism

$$T_{(p,q)}(M \times N) = T_p M \times T_q N.$$

PROBLEM C.2. Let $\varphi \colon M \to N$ be a smooth map. Prove that $D\varphi \colon TM \to TN$ is also smooth. Prove that if $\varphi \colon M \to N$ is an embedding then so is $D\varphi \colon TM \to TN$.

PROBLEM C.3. Let $\varphi \colon M \to N$ be an injective immersion with M compact. Prove that φ is an embedding. Give an example to show that this need not be true if M is not compact.

PROBLEM C.4. Let $\mathcal{O}$ be an open subset in $\mathbb{R}^m$ and suppose $f \colon \mathcal{O} \to \mathbb{R}$ is smooth. Define $g \colon \mathcal{O} \to \mathbb{R}^{m+1}$ by

$$g(x) = (x, f(x)).$$

Prove that g is a smooth embedding, and hence that $g(\mathcal{O})$ is a smooth embedded m-dimensional submanifold of $\mathbb{R}^{m+1}$.

We call $g(\mathcal{O})$ the **graph** of f.

PROBLEM C.5. Let $i \colon S^m \hookrightarrow \mathbb{R}^{m+1}$ denote the inclusion. Prove that

$$Di(p)[T_p S^m] = \mathcal{J}_p(p^\perp),$$

where $\mathcal{J}_p \colon \mathbb{R}^{m+1} \to T_p\mathbb{R}^{m+1}$ is the dash-to-dot map and

$$p^\perp := \left\{ q \in \mathbb{R}^{m+1} \mid \langle p, q \rangle = 0 \right\},$$

for $\langle \cdot, \cdot \rangle$ the standard Euclidean dot product.

PROBLEM C.6. Let M be an embedded submanifold of $\mathbb{R}^n$. We define the **normal space to** M **at** p to be the $(n-m)$-dimensional subspace $\mathrm{Nor}_p M \subset T_p\mathbb{R}^n$ consisting of all vectors that are orthogonal to $T_p M$ with respect to the Euclidean dot product. We define the **normal bundle** of M as the set

This problem is non-examinable.

$$\mathrm{Nor}\, M := \{(p, \xi) \in T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \mid p \in M, \xi \in \mathrm{Nor}_p M\}.$$

Prove that $\mathrm{Nor}\, M$ is an embedded n-dimensional submanifold of $T\mathbb{R}^n = \mathbb{R}^{2n}$.

PROBLEM C.7. Let $\varphi \colon M \to N$ be a smooth map. Assume that the rank of φ is constant on all of M.

(i) Assume that φ is injective. Prove that φ is an immersion.

(ii) Assume that φ is surjective. Prove that φ is a submersion.

(iii) Assume that φ is bijective. Prove that φ is a diffeomorphism.

Bonus Problem(s) for Sheet C

These problem(s) are hard, and are included for enthusiasts only. Solutions will not be provided.

Problem C.8. Let $\varphi\colon M \to N$ be smooth, and let $L \subset N$ be an embedded submanifold. We say that φ is **transverse** and **regular** at L if

$$D\varphi(p)(T_pM) + T_{\varphi(p)}L = T_{\varphi(p)}N, \qquad \forall\, p \in \varphi^{-1}(L).$$

Assume that $\varphi^{-1}(L) \neq \emptyset$. Prove that if φ is transverse and regular at L then $\varphi^{-1}(L)$ is a smooth embedded submanifold of M of dimension $m - n + l$.

The Implicit Function Theorem 6.10 is the special case where L is a point.

Problem C.9. Let M be a smooth manifold and let N denote a covering space of M. Prove that N is a topological manifold, and moreover that there exists a unique smooth structure on N such that N is a smooth manifold and the covering projection $\pi\colon N \to M$ is smooth.

Problem Sheet D

Problem D.1. Let $\mathcal{O} \subset \mathbb{R}^m$ be an open set.

(i) Prove that the dash-to-dot maps induce a diffeomorphism between $T\mathcal{O}$ and $\mathcal{O} \times \mathbb{R}^m$.

(ii) Prove that there is a bijective correspondence between vector fields on $\mathcal{O}$ and smooth functions $\mathcal{O} \to \mathbb{R}^m$. Namely, given a vector field X associate the function f defined by

$$f(p) := \mathcal{J}_p^{-1}(X(p)), \qquad \forall\, p \in \mathcal{O}.$$

(iii) Let X and f be associated as above, and let γ be a smooth curve in $\mathcal{O}$. Prove that γ is an integral curve of f in the sense of (9.1), i.e. $\gamma' = f(\gamma)$, if and only if γ is an integral curve of X in the sense of (9.2), i.e. $\dot{\gamma} = X(\gamma)$.

Problem D.2. Let M be a smooth manifold, let $p \in M$, and let $\xi \in T_pM$. Let U be any open set containing p. Prove that there exists a vector field $X \in \mathfrak{X}(U)$ such that $X(p) = \xi$.

Problem D.3. Let M be a smooth manifold and let $W \subset M$ be a non-empty open set. Prove that the Lie bracket $[\cdot, \cdot]$ on $\mathfrak{X}(W)$ satisfies the Jacobi identity.

Problem D.4. Let M be a smooth manifold and let (U, x) be a chart on M with local coordinates (x^i). Fix $X, Y \in \mathfrak{X}(U)$, and write $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^i \frac{\partial}{\partial x^i}$. Prove that

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j},$$

where $\frac{\partial Y^j}{\partial x^i}$ and $\frac{\partial X^j}{\partial x^i}$ are the functions from Definition 8.4.

Problem D.5. Let M be a smooth manifold and let $W \subset M$ be a non-empty open set. Let $X, Y \in \mathfrak{X}(W)$, and let $f, g \in C^\infty(W)$. Prove that

$$[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X.$$

Problem D.6. Let $\varphi\colon M \to N$ be a smooth map. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. We say that X and Y are **φ-related** if

If φ is a diffeomorphism then any $X \in \mathfrak{X}(M)$ is φ-related to $\varphi_* X$.

$$D\varphi(p)X(p) = Y(\varphi(p)), \qquad \forall\, p \in M.$$

(i) Prove that $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are φ-related if and only if for every open set $V \subset N$ and every smooth function $f \in C^\infty(V)$, one has

$$X(f \circ \varphi) = Y(f) \circ \varphi.$$

(ii) Let $X_i \in \mathfrak{X}(M)$ and $Y_i \in \mathfrak{X}(N)$ for $i = 1, 2$ be vector fields. Assume X_i is φ-related to Y_i for each $i = 1, 2$. Prove that $[X_1, X_2]$ is φ-related to $[Y_1, Y_2]$.

This problem is non-examinable.

PROBLEM D.7. Let $M \subset N$ be an (immersed or embedded) submanifold and let $p \in M$. We say that a vector field $Y \in \mathfrak{X}(N)$ is **tangent to M at p** if $Y(p) \in T_pM \subset T_pN$. We say Y is **tangent to M** if it is tangent to M at every point $p \in M$.

(i) Assume $M \subset N$ is an embedded submanifold. Prove that $Y \in \mathfrak{X}(N)$ is tangent to M if and only if $Y(f)|_M \equiv 0$ for every function $f \in C^\infty(N)$ such that $f|_M \equiv 0$.

(ii) Now assume $M \subset N$ is merely an immersed submanifold. Let $\iota\colon M \hookrightarrow N$ denote the inclusion. Assume that $Y \in \mathfrak{X}(N)$ is tangent to M. Prove there exists a unique $X \in \mathfrak{X}(M)$ such that X is $\imath$-related to Y.

(iii) Continue to assume that $M \subset N$ is an immersed submanifold. Suppose $Y_1, Y_2 \in \mathfrak{X}(N)$ are tangent to M. Prove that $[Y_1, Y_2]$ is tangent to M.

Bonus Problem(s) for Sheet D

These problem(s) are hard, and are included for enthusiasts only. Solutions will not be provided.

This question asks you to remove the compactness hypothesis from Theorem 7.2. This result is commonly known as the Weak Whitney Embedding Theorem.

PROBLEM D.8. Let M be a smooth manifold of dimension m. Prove that there exists a proper embedding $\varphi\colon M \to \mathbb{R}^{2m+1}$.

The next problem shows that the Strong Whitney Embedding Theorem 7.1 is sharp.

PROBLEM D.9. Let $m = 2^k$. Prove that $\mathbb{R}P^m$ does not smoothly embed in $\mathbb{R}^{2m-1}$.

Problem Sheet E

Problem E.1. Let $J_0 \in \mathrm{Mat}(2n)$ denote the matrix

$$J_0 := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where I is the $n \times n$ identity matrix. The **symplectic linear group** $\mathrm{Sp}(2n)$ consists of the matrices A such that $A^T J_0 A = J_0$. Prove that $\mathrm{Sp}(2n)$ is a Lie group. Compute its dimension, and compute its Lie algebra $\mathfrak{sp}(2n)$.

Problem E.2. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose $\gamma\colon \mathbb{R} \to G$ be a smooth curve with $\gamma(0) = e$. Set $\xi := \dot{\gamma}(0)$. Prove that γ is a one-parameter subgroup if and only if

$$\Phi_t^\xi = r_{\gamma(t)}$$

(this is an equality of diffeomorphisms of G).

Here $\Phi_t^\xi\colon G \to G$ denotes the flow of the unique left-invariant vector field X_ξ satisfying $X_\xi(e) = \xi$.

Problem E.3. Prove that the Lie bracket on $\mathfrak{gl}(n)$ is given by matrix commutation, i.e.

$$[A, B] = AB - BA, \qquad \forall\, A, B \in \mathfrak{gl}(n) = \mathrm{Mat}(n).$$

Problem E.4. Let $\varphi\colon M \to N$ be a smooth map between two smooth manifolds. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, and assume X and Y are φ-related in the sense of Problem D.6. Let Φ_t and Ψ_t denote the respective flows, with domains $M_t \subset M$ and $N_t \subset N$ respectively. Prove that $\varphi(M_t) \subset N_t$ and that

$$\Psi_t \circ \varphi = \varphi \circ \Phi_t, \qquad \text{on } M_t.$$

Deduce that if φ is a diffeomorphism then for any vector field X with flow Φ_t, the flow of $\varphi_* X$ is given by $\varphi \circ \Phi_t \circ \varphi^{-1}$.

Problem E.5. Let X and Y be vector fields on a smooth manifold M with flows Φ_t and Ψ_t respectively. Prove that $[X, Y] \equiv 0$ if and only if the two flows commute, i.e. $\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t$ for all s, t small.

This problem is non-examinable.

Problem E.6. Prove that if Lie group is abelian then its Lie algebra is abelian.

On Problem Sheet F you will prove that if a *connected* Lie group has abelian Lie algebra, then it is an abelian Lie group.

Problem E.7. Let G be a Lie group and suppose H is a subgroup of G which is also an embedded submanifold. Prove that H is closed in G (as a subspace).

This problem is non-examinable.

Bonus Problem(s) for Sheet E

These problem(s) are hard, and are included for enthusiasts only. Solutions will not be provided. Both $\mathbb{R}^m$ and the torus T^m have the

abelian Lie algebra $\mathbb{R}^m$. This shows that the functor $G \mapsto \mathfrak{g}$ from the category of Lie groups to the category of Lie algebras is not injective. If however we restrict to the subcategory of simply connected Lie groups, this problem goes away:

Problem E.8. Let G and H be simply connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Assume $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic (as Lie algebras). Prove that G and H are isomorphic as Lie groups.

Problem Sheet F

PROBLEM F.1.

(i) Let M be a smooth manifold. Assume there exist vector fields $X_1, \dots, X_m \in \mathfrak{X}(M)$ such that $\{X_i(p)\}$ is a basis of T_pM for every $p \in M$. Prove that the tangent bundle TM is diffeomorphic to $M \times \mathbb{R}^m$.

(ii) Let G be a Lie group with Lie algebra $\mathfrak{g}$. Prove that TG is diffeomorphic to $G \times \mathfrak{g}$.

PROBLEM F.2. Let $A \in \mathfrak{gl}(m)$. Prove that the matrix exponential

This problem is non-examinable.

$$e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

converges and defines an element of $\mathrm{GL}(m)$. Prove that $A \mapsto e^A$ is the exponential map of $\mathrm{GL}(m)$.

PROBLEM F.3. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Prove that for $\xi, \zeta \in \mathfrak{g}$ one has $\mathrm{ad}_\xi(\zeta) = [\xi, \zeta]$.

PROBLEM F.4. Let σ be a smooth action of a Lie group G on a smooth manifold M.

(i) Prove that σ is proper if and only if the following condition holds: if (p_k) is a sequence in M and (g_k) is a sequence in G such that both (p_k) and $(\sigma_{g_k}(p_k))$ converge, then a subsequence of (g_k) converges.

(ii) Deduce that if G is compact then every smooth action is proper.

PROBLEM F.5. Let G be a Lie group and H be a closed subgroup (possibly equal to G). Let H act on G via left (or right) translations. Prove that this action is proper.

PROBLEM F.6. Let σ be a proper smooth action of a Lie group G on a smooth manifold M. Prove that the orbits $\mathsf{orb}_\sigma(p)$ are closed subsets of M.

PROBLEM F.7. Let Δ be an integrable distribution on a smooth manifold M, and let L be a connected integral manifold of Δ. Assume that L is closed in M. Prove that L is a leaf of foliation induced by Δ.

PROBLEM F.8. Let $\varphi \colon M \to N$ be a surjective submersion. Prove that the connected components of the preimages $\varphi^{-1}(p)$ as p ranges over N defines an $(m-n)$-dimensional foliation of M.

PROBLEM F.9. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Prove that the centre of G is the kernel of the adjoint representation $\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g})$. Deduce that G is abelian if and only if $\mathfrak{g}$ is abelian.

This is a partial converse to Problem E.6.

Bonus Problem(s) for Sheet F

These problem(s) are hard, and are included for enthusiasts only. Solutions will not be provided. A **topological group** G is a topological space that is also a group in the algebraic sense, with the property that the group multiplication

$$\mu\colon G \times G \to G, \qquad \mu(g,h) = gh,$$

and group inversion

$$i\colon G \to G, \qquad i(g) = g^{-1},$$

are both **continuous** maps. The goal of the next problem is to show that if G is a topological space that simultaneously carries the structure of a topological manifold and a topological group, then G admits at most one diffeomorphism class of smooth structures that turns G into a Lie group.

PROBLEM F.10.

(i) Let G be a Lie group. Suppose $\gamma\colon \mathbb{R} \to G$ is a continuous group homomorphism. Prove that γ is necessarily smooth, and hence is a one-parameter subgroup.

(ii) Let G and H be Lie groups, and suppose $\varphi\colon G \to H$ is a continuous group homomorphism. Prove that φ is automatically smooth, and hence is a Lie group homomorphism. *Hint:* Use the previous part.

(iii) Let G be a topological space which is simultaneously a topological group and a topological manifold. Prove that G admits at most one diffeomorphism class of smooth structures that turns G into a Lie group.

Remark: The converse to (iii) was Hilbert's Fifth Problem, famously posed by David Hilbert in 1900. It was eventually proved in 1952 by Montgomery and Zippen.

PROBLEM F.11. Let G be a connected space that satisfies all the conditions for a Lie group apart from not necessarily being second countable. Prove that G is automatically second countable (and hence a Lie group).

PROBLEM F.12. Let G be a connected Lie group. Let $\mathrm{Aut}(G)$ denote the set of Lie group isomorphisms $\varphi\colon G \to G$. Prove that $\mathrm{Aut}(G)$ admits the structure of a Lie group. *Hint:* First prove this in the case where G is simply connected.

Problem Sheet G

Problem G.1. Show that the real projective space $\mathbb{R}P^{m-1}$ can be seen as the homogeneous space $\mathrm{SO}(m)/\,\mathrm{O}(m-1)$.

Problem G.2. Let σ be a smooth free action of G on M. Assume that the quotient space M/G admits the structure of a smooth manifold such that the quotient map $\rho\colon M \to M/G$ is a smooth submersion. Prove that σ is a proper action.

This question is a partial converse to the Quotient Manifold Theorem 13.6.

Problem G.3. Let $\pi_1\colon P \to M$ and $\pi_2\colon Q \to N$ be two G-principal bundles. Suppose (φ, Φ) is a principal bundle morphism from P to Q such that φ is a diffeomorphism. Prove that Φ is also a diffeomorphism.

Problem G.4. Let σ be an effective action of a Lie group G on a smooth manifold L. Assume we are given two fibre bundles

$$L \to E \xrightarrow{\pi_1} M, \qquad \text{and} \qquad L \to F \xrightarrow{\pi_2} M$$

Let $\{U_a \mid a \in A\}$ be an open cover of M such that both E and F admit G-bundle atlases over the U_a. Let

This can always be achieved by taking intersections.

$$g^1_{ab}\colon U_a \cap U_b \to G, \qquad \text{and} \qquad g^2_{ab}\colon U_a \cap U_b \to G$$

denote the transition functions of E and F with respect to these bundle atlases. Prove that E and F are isomorphic as (G, σ)-fibre bundles if and only if there exists a family $f_a\colon U_a \to G$ of smooth functions such that

$$f_a(p) \circ g^1_{ab}(p) = g^2_{ab}(p) \circ f_b(p), \qquad \forall\, p \in U_a \cap U_b,\ \forall\, a, b \in A.$$

Problem G.5. Let $V \to E \xrightarrow{\pi} M$ be a vector bundle. Prove that it is possible to reduce the structure group from $\mathrm{GL}(V)$ to $\mathrm{O}(V)$. Find an example where it is not possible to reduce the structure group from $\mathrm{GL}(V)$ to $\mathrm{GL}^+(V)$.

Problem G.6. Prove that there are exactly two vector bundles of rank 1 over S^1 (up to vector bundle isomorphism).

This problem is non-examinable.

Problem G.7. Prove that the Klein bottle is a fibre bundle over S^1 with fibre S^1. Prove however that the Klein bottle is not a principal S^1-bundle over S^1.

This problem is non-examinable.

Problem G.8. Find a non-trivial principal S^1-bundle over $\mathbb{R}P^2$.

This problem is non-examinable.

Problem G.9. Suppose σ is a smooth transitive action of a Lie group G on M, so that M is the homogeneous space G/H for an appropriate subgroup H of G. Prove that the subgroup of G acting trivially on M is the largest normal subgroup $N(H)$ of G contained in H. Let $\overline{G}$ and $\overline{H}$ denote the quotient groups $G/N(H)$ and $H/N(H)$ respectively. Prove that $\overline{G}$ acts effectively and transitively on M, and M is the homogeneous space $\overline{G}/\overline{H}$.

PROBLEM G.10. Let $\pi\colon P \to M$ be a principal G-bundle with corresponding right action τ. Let $H \subset G$ be a Lie subgroup, and let $Q \subset P$ be a subset such that:

This problem is non-examinable.

(i) The restriction $\pi|_Q\colon Q \to M$ is surjective.

(ii) If $q \in Q$ and $h \in H$ then $\tau_h(q) \in Q$.

(iii) For all $p \in M$, the action of H on $Q \cap P_p$ is transitive.

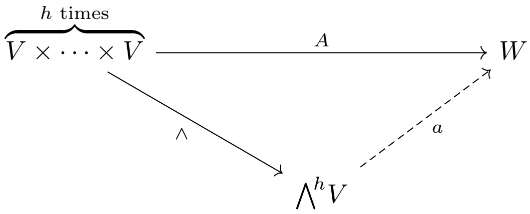
(iv) For all $p \in M$, there exists a neighbourhood U of p and a smooth local section $\psi\colon U \to P$ of π such that $\psi(q) \in Q$ for all $q \in U$.

Prove that $\pi|_Q\colon Q \to M$ is a principal H-bundle, and that moreover Q is a principal H-subbundle of P.

Problem Sheet H

PROBLEM H.1. Let V, W and U be vector spaces. Prove there are natural isomorphisms $V \otimes W \cong W \otimes V$ and $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$.

PROBLEM H.2. Let V and W be vector spaces. Prove that for any $A \in \mathrm{Alt}_h(V, W)$ there is a unique linear map $a \colon \bigwedge^h V \to W$ such that the following diagram commutes:



Prove moreover that $\bigwedge^h V$ is uniquely characterised by this property.

PROBLEM H.3. Let V be a vector space of dimension n with basis $\{e_1, \dots, e_n\}$. Prove that

$$\{e_{i_1} \wedge \cdots \wedge e_{i_h} \mid 1 \le i_1 < \cdots < i_h \le n\}$$

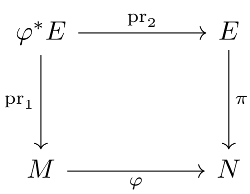
is a basis of $\bigwedge^h V$ and $\bigwedge^h V = 0$ for $h > n$. Deduce that $\dim \bigwedge^h V = \binom{n}{h}$ and that $\dim \bigwedge V = 2^n$.

PROBLEM H.4. Let $\varphi \colon M \to N$ be a smooth map, and suppose $L \to E \xrightarrow{\pi} N$ is a fibre bundle. Set

$$\varphi^* E := \big\{(p, u) \in M \times E \mid \varphi(p) = \pi(u)\big\},$$

with projection $\mathrm{pr}_1 \colon \varphi^* E \to M$.

(i) Prove that $\varphi^* E$ is a fibre bundle over M with fibre L such that (φ, pr_2) is a fibre bundle morphism:



(ii) Prove that

$$T_{(p,u)}(\varphi^* E) = \big\{(\xi, \zeta) \in T_p M \times T_u E \mid D\varphi(p)\xi = D\pi(p)\zeta\big\}.$$

(iii) Suppose $\psi \colon K \to M$ is another smooth map. Prove that $\psi^*(\varphi^* E)$ and $(\varphi \circ \psi)^* E$ are isomorphic fibre bundles over K.

(iv) Assume now that E is has structure group G. Prove that $\varphi^* E$ has structure group a Lie subgroup of G. Deduce that the pullback of a vector bundle is a vector bundle, and the pullback of a principal bundle is a principal bundle.

i.e. E is a (G, σ)-fibre bundle for some effective action σ of G on L.

(v) Prove that the isomorphism from part (iii) can be taken to be an isomorphism of G-fibre bundles (or a subgroup thereof).

Problem H.5. Let $L \to E \xrightarrow{\pi_1} M$ and $K \to F \xrightarrow{\pi_2} N$ be fibre bundles with structure groups G and H respectively. Prove that $(\pi_1, \pi_2)\colon E \times F \to M \times N$ is another fibre bundle with fibre $L \times K$ and structure group $G \times H$. We call this the **external product** of E and F.

Problem H.6. Let $\iota\colon M \to M \times M$ denote the diagonal map $p \mapsto (p, p)$. This problem is non-examinable.

(i) Let E and F be two vector bundles over M. Prove that

$$E \oplus F = \iota^*(E \times F).$$

(ii) Let P and Q be two principal bundles over M. Prove that

$$P \star Q = \iota^*(P \times Q).$$

Problem H.7. Let E be a vector bundle over M and F a vector bundle over N. Suppose $\Psi\colon E \to F$ is any smooth map that maps each fibre E_p for $p \in M$ linearly onto some fibre F_q for $q \in N$. Prove that $\Psi = \Phi_2 \circ \Phi_1$ where Φ_1 is a vector bundle homomorphism and Φ_2 is a vector bundle isomorphism along a map $M \to N$.

Problem H.8. Let $V \to E \xrightarrow{\pi_1} M$ and $W \to F \xrightarrow{\pi_2} M$ be two vector bundles over M, and let $\Phi\colon E \to F$ be a vector bundle homomorphism.

(i) Assume Φ is injective on each fibre. Consider the quotient vector space

$$\overline{E}_p \coloneqq F_p / \Phi_p(E_p).$$

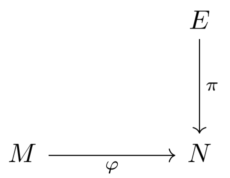
Prove that $\overline{E} \coloneqq \bigsqcup_{p \in M} \overline{E}_p$ is a vector bundle over M with fibre W/V. Deduce that $\operatorname{im}\Phi$ is a vector subbundle of F.

(ii) Assume that Φ is surjective on each fibre. Let

$$Z_p \coloneqq \ker \Phi_p \subset E_p.$$

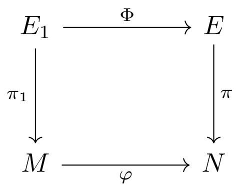
Prove that $Z \coloneqq \bigsqcup_{p \in M} Z_p$ is a vector bundle over M. What is the fibre?

Problem H.9. Let $\varphi\colon M \to N$ be a smooth map and suppose $\pi\colon E \to N$ is a vector bundle, which we illustrate pictorially as: This problem is non-examinable.

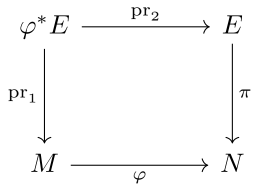


(δ)

A **solution** of the diagram (δ) is a vector bundle $\pi_1\colon F \to M$ together with a vector bundle morphism $\Phi\colon F \to E$ along φ:

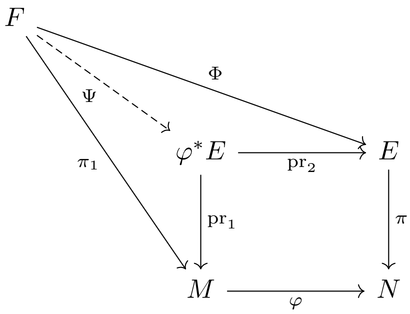


As we have seen, one possible solution is the pullback bundle $\varphi^* E$:

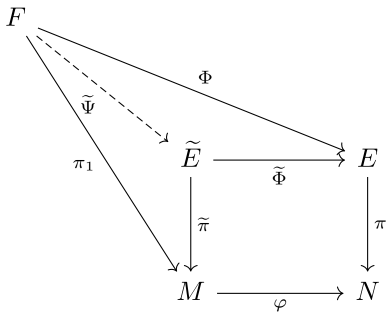


The aim of this problem is to prove that the pullback bundle can be characterised as a solution to a *universal mapping problem*: namely, that $\varphi^* E$ is the "most efficient" solution in the following sense:

Suppose $\pi_1 \colon F \to M$ and Φ is *any* solution to (δ). Prove there exists a *unique* vector bundle homomorphism $\Psi \colon F \to \varphi^* E$ such that the following diagram commutes:



Prove moreover that $\varphi^* E$ is uniquely determined by this property. Explicitly this means that if $\widetilde{\pi} \colon \widetilde{E} \to M$ and $\widetilde{\Phi}$ is another solution to the diagram (δ) with the property that for *any* solution $\pi_1 \colon F \to M$ and Φ of (δ) there exists a *unique* vector bundle homomorphism $\widetilde{\Psi} \colon F \to \widetilde{E}$ such that the following commutes:



then in fact $\widetilde{E}$ is isomorphic as a vector bundle over M to $\varphi^* E$.

Bonus Problem(s) for Sheet H

These problem(s) are hard, and are included for enthusiasts only. Solutions will not be provided.

Problem H.10. Let M be a smooth compact connected manifold of dimension 4. Let $G = \mathrm{SU}(2)$. How "many" isomorphism classes of principal G-bundles $P \to M$ are there?

This problem is relevant in gauge theory; a topic we hope to return to in Differential Geometry II.

Problem Sheet I

PROBLEM I.1. Let $\varphi\colon M \to N$ be a smooth map.

This problem is non-examinable.

(i) Let $A \in \mathcal{T}^{h,k}(M)$ denote a tensor of type (h,k). Let (U,x) and (V,y) denote two charts on M with $U \cap V \neq \emptyset$. Then one can write

$$A = f^{i_1 \cdots i_h}_{j_1 \cdots j_k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_h}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_k}$$

and

$$A = g^{i_1 \cdots i_h}_{j_1 \cdots j_k} \frac{\partial}{\partial y^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial y^{i_h}} \otimes dy^{j_1} \otimes \cdots \otimes dy^{j_k}$$

for smooth functions $f^{i_1 \cdots i_h}_{j_1 \cdots j_k} \in C^\infty(U)$ and $g^{i_1 \cdots i_h}_{j_1 \cdots j_k} \in C^\infty(V)$. Investigate the relationship between

$$f^{i_1 \cdots i_h}_{j_1 \cdots j_k}|_{U \cap V} \qquad \text{and} \qquad g^{i_1 \cdots i_h}_{j_1 \cdots j_k}|_{U \cap V}.$$

(ii) Let $\omega \in \Omega^k(M)$ denote a differential k-form. Let (U,x) and (V,y) denote two charts on M with $U \cap V \neq \emptyset$. Then one can write

$$\omega = f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

and

$$\omega = g_{i_1 \cdots i_k} dy^{i_1} \wedge \cdots \wedge dy^{i_k}$$

for smooth functions $f_{i_1 \cdots i_k} \in C^\infty(U)$ and $g_{i_1 \cdots i_k} \in C^\infty(V)$. Investigate the relationship between

$$f_{i_1 \cdots i_k}|_{U \cap V} \qquad \text{and} \qquad g_{i_1 \cdots i_k}|_{U \cap V}.$$

(iii) Assume φ is a diffeomorphism, and let $A \in \mathcal{T}^{h,k}(N)$. Let (U,x) be a chart on M and (V,y) a chart on N with $\varphi(U) \subset V$. Write

$$\varphi^* A = f^{i_1 \cdots i_h}_{j_1 \cdots j_k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_h}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_k}$$

and

$$A = g^{i_1 \cdots i_h}_{j_1 \cdots j_k} \frac{\partial}{\partial y^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial y^{i_h}} \otimes dy^{j_1} \otimes \cdots \otimes dy^{j_k}$$

for smooth functions $f^{i_1 \cdots i_h}_{j_1 \cdots j_k} \in C^\infty(U \cap \varphi^{-1}(V))$ and $g^{i_1 \cdots i_h}_{j_1 \cdots j_k} \in C^\infty(V)$. Investigate the relationship between

$$f^{i_1 \cdots i_h}_{j_1 \cdots j_k} \qquad \text{and} \qquad g^{i_1 \cdots i_h}_{j_1 \cdots j_k}.$$

(iv) Let $\omega \in \Omega^k(N)$. Let (U,x) be a chart on M and (V,y) a chart on N such that $\varphi(U) \subset V$. Then one can write

$$\varphi^*\omega = f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

and

$$\omega = g_{i_1 \cdots i_k} dy^{i_1} \wedge \cdots \wedge dy^{i_k}$$

for smooth functions $f_{i_1 \cdots i_k} \in C^\infty(U \cap \varphi^{-1}(V))$ and $g_{i_1 \cdots i_k} \in C^\infty(V)$. Investigate the relationship between

$$f_{i_1 \cdots i_k} \qquad \text{and} \qquad g_{i_1 \cdots i_k}.$$

(v) Conclude that local coordinates are horrible.

Problem I.2. Let $\pi\colon E \to M$ be a vector bundle. An operator $\zeta\colon \Gamma(E) \to \Gamma(E)$ is said to satisfy the **Leibniz rule** if there exists a vector field X on M such that for any $f \in C^\infty(M)$ and $s \in \Gamma(E)$ one has

$$\zeta(fs) = (Xf)s + f\zeta(s).$$

Prove that an operator satisfying the Leibniz rule is a local operator but – provided $X \neq 0$ – is not a point operator.

Problem I.3. Let M be a smooth manifold and let $E_1, \dots, E_k$ and E be vector bundles over M. Let $\zeta\colon \Gamma(E_1) \times \cdots \times \Gamma(E_k) \to \Gamma(E)$ be a $C^\infty(M)$-multilinear operator. Prove that for each $p \in M$ there is a unique $\mathbb{R}$-multilinear map

$$\Phi_p\colon E_{1|p} \times \cdots \times E_{k|p} \to E_p$$

such that for all $s_i \in \Gamma(E_i)$ one has

$$\Phi_p(s_1(p), \dots s_k(p)) = \zeta(s_1, \dots, s_k)(p).$$

Problem I.4. Let M be a smooth manifold and let $U \subset M$ be a non-empty open set. Prove that there is a canonical identification between $\mathscr{T}^{h,k}(U)$ and $C^\infty(U)$-multilinear functions

$$\underbrace{\Omega^1(U) \times \cdots \times \Omega^1(U)}_{h \text{ copies}} \times \overbrace{\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U)}^{k \text{ copies}} \to C^\infty(U).$$

Problem I.5. This problem introduces the **vertical bundle** of a fibre bundle.

This problem is non-examinable. Vertical bundles will play a major role in Differential Geometry II.

(i) Let $\pi\colon E \to M$ be a fibre bundle with fibre L. Let

$$VE := \bigsqcup_{u \in E} \left\{\ker D\pi(u)\colon T_uE \to T_{\pi(u)}M\right\}$$

with projection map $\pi_V\colon VE \to E$. Prove that VE is a vector bundle over E of rank $l = \dim L$.

(ii) Assume now that $\pi\colon E \to M$ is a vector bundle. Prove that the vertical bundle VE is isomorphic as a vector bundle to the pullback bundle $\pi^*E \to E$.

(iii) Continue to assume that $\pi\colon E \to M$ is a vector bundle. Prove that the composition $\pi \circ \pi_V\colon VE \to M$ is another vector bundle over M which is isomorphic to the direct sum bundle $E \oplus E$.

(iv) Continue to assume that $\pi\colon E \to M$ is a vector bundle. View M as an embedded submanifold of TM via the zero section. Prove that the composite bundle $\pi \circ \pi_V\colon VE \to M$ is a vector subbundle of $D\pi\colon TE \to TM$.

Problem I.6. Let M be a smooth manifold.

(i) Suppose $A \in \mathscr{T}^{1,1}(M) \cong \Gamma(\mathrm{End}(TM))$. Prove there exists a unique tensor derivation ζ_A on M with the property that $\zeta_A(Y)(p) = A_p(Y(p))$ for any vector field Y and satisfies $\zeta_A(f) = 0$ for any function f.

(ii) Let ξ be an arbitrary tensor derivation. Prove that there exists a vector field X on M and $A \in \mathcal{T}^{1,1}(M)$ such that $\xi = \mathcal{L}_X + \zeta_A$. Deduce that the space of tensor derivations on M can be identified with $\mathfrak{X}(M) \times \Gamma(\mathrm{End}(TM))$.

Definition. Let $X \in \mathfrak{X}(M)$ with flow Φ_t. Define an operator $\widetilde{\mathcal{L}}_X$ on $\mathcal{T}(M)$ by

$$\widetilde{\mathcal{L}}_X A := \frac{d}{dt}\Big|_{t=0} \Phi_t^* A = \lim_{t \to 0} \frac{\Phi_t^* A - A}{t}.$$

Proposition 22.14 states that $\widetilde{\mathcal{L}}_X$ coincides with the Lie derivative $\mathcal{L}_X$. The proof of Proposition 22.14 uses Problem I.7. Thus to avoid a circular argument, you **cannot** use the fact that $\widetilde{\mathcal{L}}_X = \mathcal{L}_X$ while solving Problem I.7!
This problem is non-examinable.

Problem I.7. Let (h_0, k_0), (h_1, k_1) and (h_2, k_2) be three pairs of non-negative integers. Suppose we are given a $C^\infty(M)$-bilinear operator

$$\mathcal{A} \colon \mathcal{T}^{h_0,k_0}(M) \times \mathcal{T}^{h_1,k_1}(M) \to \mathcal{T}^{h_2,k_2}(M).$$

Assume in addition that $\mathcal{A}$ has the property that if $\varphi \colon U \to V$ is a local diffeomorphism between open sets of M then the corresponding local operators

$$\varphi^*\big(\mathcal{A}^V(A,B)\big) = \mathcal{A}^U(\varphi^* A, \varphi^* B).$$

Prove that for every vector field X on M, one has

$$\widetilde{\mathcal{L}}_X\big(\mathcal{A}(A,B)\big) = \mathcal{A}\big(\widetilde{\mathcal{L}}_X A, B\big) + \mathcal{A}\big(A, \widetilde{\mathcal{L}}_X B\big).$$

Bonus Problem(s) for Sheet I

These problem(s) are hard, and are included for enthusiasts only. Solutions will not be provided.

Problem I.8. Let R be a commutative ring and let V be a finitely generated projective R-module. Prove that for all $h, k \geq 0$,

$$T^{h,k}V \cong \mathrm{Mult}_{k,h}(V).$$

Problem I.9. Let $\pi \colon E \to M$ be a vector bundle and let $U \subset M$ be an arbitrary open set (possibly equal to M). Prove that the space $\Gamma(U, E)$ is a finitely generated projective $C^\infty(U)$-module.

Problem Sheet J

PROBLEM J.1. Let E and F be two vector bundles over M and let $\{U_a \mid a \in A\}$ be an arbitrary open covering of M. Suppose we are given a collection

$$\{\zeta_a \colon \Gamma(U_a, E) \to \Gamma(U_a, F) \mid a \in A\}$$

of local operators such that

$$\zeta_a^{U_a \cap U_b} \equiv \zeta_b^{U_a \cap U_b}, \qquad \text{if } U_a \cap U_b \neq \emptyset.$$

Prove there exists a unique local operator $\zeta \colon \Gamma(E) \to \Gamma(F)$ such that

$$\zeta^{U_a} = \zeta_a, \qquad \forall\, a \in A.$$

PROBLEM J.2. Let $\varphi \colon M \to N$ denote a smooth map. Let $A \in \mathcal{T}^{0,k}(N)$. Using the Tensor Criterion Theorem 21.5, regard A as a $C^\infty(N)$-multilinear function

$$\underbrace{\mathfrak{X}(N) \times \cdots \times \mathfrak{X}(N)}_{k \text{ copies}} \to C^\infty(N).$$

and similarly regard $\varphi^*(A)$ as a $C^\infty(M)$-multilinear function

$$\underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k \text{ copies}} \to C^\infty(M).$$

Suppose $X_i \in \mathfrak{X}(M)$ is φ-related to $Y_i \in \mathfrak{X}(N)$ for $i = 1, \dots, s$. Prove that

$$(\varphi^* A)(X_1, \dots, X_k) = A(Y_1, \dots, Y_k) \circ \varphi$$

as functions $M \to N$.

PROBLEM J.3. Let V be a vector space and suppose $\omega \in \bigwedge^h V^*$ and $\theta \in \bigwedge^k V^*$. Let $v_i \in V$ for $i = 1, \dots, h+k$. Prove that if we identify ω with an element of $\mathrm{Alt}_h(V)$, θ with an element of $\mathrm{Alt}_k(V)$, and $\omega \wedge \theta$ with an element of $\mathrm{Alt}_{h+k}(V)$, one has:

$$(\omega \wedge \theta)(v_1, \dots, v_{h+k}) = \frac{1}{h!k!} \sum_{\varrho \in \mathfrak{S}_{h+k}} \mathrm{sgn}(\varrho) \omega\big(v_{\varrho(1)}, \dots, v_{\varrho(h)}\big)\, \theta\big(v_{\varrho(h+1)}, \dots, v_{\varrho(h+k)}\big)$$

or equivalently

$$(\omega \wedge \theta)(v_1, \dots, v_{h+k}) = \sum_{\varrho \in \mathrm{Shuffle}(h,k)} \mathrm{sgn}(\varrho) \omega\big(v_{\varrho(1)}, \dots, v_{\varrho(h)}\big)\, \theta\big(v_{\varrho(h+1)}, \dots, v_{\varrho(h+k)}\big).$$

PROBLEM J.4. Let $\omega \in \bigwedge^h V^*$ and $\theta \in \bigwedge^k V^*$. Prove that

$$i_v(\omega \wedge \theta) = i_v \omega \wedge \theta + (-1)^h \omega \wedge i_v \theta.$$

PROBLEM J.5. Let $(M, \mathfrak{o})$ be an oriented smooth manifold with boundary. Let $\mu \in \Omega^m(M)$ be a volume form representing $\mathfrak{o}$. Let X be an outward pointing section.

(i) Prove that $i_X\mu$ restricts to define a volume form on ∂M.

(ii) Let $\partial\mathfrak{o}$ denote the orientation of ∂M determined by $i_X\mu$. Prove that (as the notation suggests) $\partial\mathfrak{o}$ only depends on $\mathfrak{o}$, and not on the particular choice of μ and X.

Problem J.6.

(i) Prove that S^m is orientable.

(ii) Prove that any Lie group is orientable.

(iii) Prove that $\mathbb{R}P^m$ is orientable if and only if n is odd. *Hint:* Consider the antipodal map $x \mapsto -x$ on S^m.

Problem J.7. Let

$$\mathbb{R}^m_- := \mathbb{R}^m_{u^1 \leq 0}, \qquad \mathbb{H}^m := \mathbb{R}^m_{u^m \geq 0}.$$

We can identify both $\partial\mathbb{R}^m_-$ and $\partial\mathbb{H}^m$ with $\mathbb{R}^{m-1}$. Endow both $\mathbb{R}^m_-$ and $\mathbb{H}^m$ with their standard orientation they inherit from $\mathbb{R}^m$. Show that the induced orientation on $\partial\mathbb{R}^m_-$ is equal to standard orientation on $\mathbb{R}^{m-1}$ for all m, but that the induced orientation on $\partial\mathbb{H}^m$ agrees with the standard orientation of $\mathbb{R}^{m-1}$ only when m is even.

Remark: This is the main reason we take our "standard" half-space to be $\mathbb{R}^m_-$, not $\mathbb{H}^m$, cf. Remark 24.20.

Problem J.8.

(i) Let V be a vector space of dimension n. A **symplectic form** on V is an element $\omega \in \mathrm{Alt}_2(V) \cong \bigwedge^2 V^*$ which is **non-degenerate** in the sense that $i_v(\omega) \equiv 0$ if and only if $v = 0$. Prove that if a symplectic form exists then $n = 2k$ is necessarily an even number.

(ii) A **symplectic manifold** is a smooth manifold M equipped with a *closed* differential 2-form ω such that ω_p is a symplectic form on T_pM for every $p \in M$. Prove that any symplectic manifold is orientable.

(iii) Let M be a smooth manifold. Define a 1-form $\Theta \in \Omega^1(T^*M)$ on the cotangent bundle via the formula:

$$\Theta_{p,\lambda}(\zeta) = \lambda\big(D\pi(p,\lambda)\zeta\big),$$

for $p \in M$, $\lambda \in T^*_pM$, and $\zeta \in T_{(p,\lambda)}T^*M$. Prove that $\omega := d\Theta$ is a symplectic form on T^*M. Thus every cotangent bundle is a symplectic manifold.

Problem J.9. Let M and N be smooth manifolds. Prove that if M has boundary and N does not, then $M \times N$ is a smooth manifold with boundary. Prove that if both M and N have non-empty boundary, then $M \times N$ is not a smooth manifold with boundary,

This problem is non-examinable.

Bonus Problem(s) for Sheet J

These problem(s) are hard, and are included for enthusiasts only. Solutions will not be provided.

Problem J.10. After making appropriate modifications, reprove all results in the course for manifolds with boundary.

Problem Sheet K

PROBLEM K.1. A singular k-cube $c\colon C^k \to M$ is said to be **degenerate** if there exists $1 \le i \le k$ such that c does not depend on u^i. Prove that if $c\colon C^k \to M$ is a degenerate singular k-cube then $\int_c \omega = 0$ for any $\omega \in \Omega^k(M)$.

PROBLEM K.2. Let $c\colon C^k \to M$ be a smooth singular k-cube in M and let $\varphi\colon C^k \to C^k$ be an orientation preserving diffeomorphism. Let $\tilde{c} := c \circ \varphi$. Prove that for any $\omega \in \Omega^k(M)$, one has

$$\int_c \omega = \int_{\tilde{c}} \omega.$$

As usual, think of this as meaning that φ is the restriction to C^k of an orientation preserving diffeomorphism of some neighbourhood.

PROBLEM K.3. Prove that there does not exist a compact symplectic manifold (M, ω) without boundary with the property that ω is exact.

See Problem J.8 if you forgot the definition of a symplectic manifold.

PROBLEM K.4. Find a closed $(m-1)$-form on $\mathbb{R}^m \setminus \{0\}$ that is not exact.

PROBLEM K.5. Let M be a smooth manifold, let $X \in \mathfrak{X}(M)$, and let A be a tensor field. Let Φ_t denote the flow of X. Prove that

$$\left.\frac{d}{dt}\right|_{t=t_0} \Phi_t^* A = \Phi_{t_0}^*(\mathcal{L}_X A).$$

PROBLEM K.6. Let $\varphi\colon M \to N$ be a diffeomorphism of connected oriented manifolds and let $\omega \in \Omega_c^m(N)$. Prove that

$$\int_M \varphi^*(\omega) = \pm \int_N \omega,$$

where the + signs occurs if and only if φ is orientation preserving.

PROBLEM K.7. Let G be a compact connected Lie group.

(i) Prove there exists a unique **normalised left-invariant** volume form μ on G, i.e. a volume form μ such that $\int_G \mu = 1$ and $l_g^* \mu = \mu$ for all $g \in G$.

Recall G is orientable by part (ii) of Problem J.6.

(ii) This allows us to define the integral of a *function* on G via:

$$\int_G f := \int_G f\mu, \qquad f \in C^\infty(G).$$

Prove that for all $f \in C^\infty(G)$ and $g \in G$, one has

$$\int_G f = \int_G (f \circ l_g) = \int_G (f \circ r_g), \qquad \forall\, f \in C^\infty(G),\ g \in G.$$

PROBLEM K.8. For this problem you may assume that for any compact connected orientable smooth manifold M^m, one has $H^m_{\mathrm{dR}}(M) \cong \mathbb{R}$, and that an explicit isomorphism is given by

This problem is non-examinable.

This is a special case of the de Rham Theorem 27.24.

$$\int : H^m_{\mathrm{dR}}(M) \to \mathbb{R}, \qquad [\omega] \mapsto \int_M \omega.$$

Let $\varphi\colon M \to N$ be a smooth map between compact connected orientable smooth manifolds of dimension m. Then $\varphi^*\colon H^m_{\mathrm{dR}}(N) \to H^m_{\mathrm{dR}}(M)$ is a linear map between one-dimensional vector spaces, and hence is multiplication by a number. We call this number the **degree** of φ. Explicitly,

$$\int_M \varphi^*\omega = \deg(\varphi) \int_N \omega, \qquad \omega \in \Omega^m(N).$$

(i) Let $q \in N$ denote a regular value of φ. Given $p \in \varphi^{-1}(q)$, let

$$\mathrm{sgn}_p(\varphi) := \begin{cases} +1, & \text{if } D\varphi(p) \text{ is orientation preserving,} \\ -1, & \text{if } D\varphi(p) \text{ is not orientation preserving.} \end{cases}$$

Prove that

$$\deg(\varphi) = \sum_{p \in \varphi^{-1}(q)} \mathrm{sgn}_p(\varphi).$$

Thus $\deg(\varphi)$ is an integer.

(ii) Prove the **Hairy Ball Theorem**: if m is even then any vector field on S^m has at least one zero.

Bonus Problem(s) for Sheet K

These problem(s) are hard, and are included for enthusiasts only. Solutions will not be provided.

PROBLEM K.9. Enjoy your holidays.

Problem Sheet L

Problem L.1. Let $\pi\colon E \to M$ be a vector bundle, let Δ denote a connection on E, and let $o\colon M \to E$ denote the zero section. Prove that

$$\Delta_{0_p} = Do(p)(T_pM), \qquad \forall p \in M,$$

where 0_p is the zero element of the vector space E_p.

Problem L.2. Let $\pi\colon E \to M$ be a vector bundle. Prove that a preconnection Δ on E is a vector subbundle of TE such that

$$(\pi_{TE}, D\pi)|_{\Delta}\colon \Delta \to E \oplus TM$$

is a vector bundle homomorphism from the composite bundle $\Delta \xrightarrow{\pi_{TE}} E \xrightarrow{\pi} M$ to the bundle $E \oplus TM$.

$$\begin{array}{ccc} \Delta & \xrightarrow{(\pi_{TE}, D\pi)|_{\Delta}} & E \oplus TM \\ {\scriptstyle \pi \circ \pi_{TE}} \big\downarrow & & \big\downarrow {\scriptstyle (\pi, \pi_{TM})} \\ M & \xrightarrow[\mathrm{id}]{} & M \end{array}$$

Problem L.3. Recall from Problem C.5 that if we let $\iota\colon S^m \hookrightarrow \mathbb{R}^{m+1}$ denote the inclusion then

$$D\iota(p)(T_pS^m) = \mathcal{J}_p(p^{\perp}),$$

where

$$p^{\perp} := \left\{ q \in \mathbb{R}^{m+1} \mid \langle p, q \rangle = 0 \right\},$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean dot product. Use this to prove that one can identify

$$T_{(p,\xi)}TS^m = \left\{ (u, v) \in \mathbb{R}^{2m+2} \mid \langle p, u \rangle = 0 = \langle p, v \rangle + \langle \xi, u \rangle \right\}.$$

Prove that

$$\Delta_{(p,\xi)} := \left\{ \big(v, -\langle \xi, v \rangle p\big) \mid v \in \mathbb{R}^{m+1},\ \langle p, v \rangle = 0 \right\} \subset T_{(p,\xi)}TS^m$$

defines a connection on TS^m.

Problem L.4. Take $m = 2$ and use the connection on TS^m from Problem L.3. Let $p_N = (0, 0, 1)$ denote the North pole.

(i) Let γ be a great circle. Compute $\mathbb{P}_\gamma\colon T_{\gamma(0)}S^2 \to T_{\gamma(0)}S^2$.

(ii) Given $s \in (-\pi.\pi)$, let

$$\gamma_s(t) := \big(\cos t \sin s, \sin t \sin s, \cos s \big)$$

Compute $\mathbb{P}_{\gamma_s}\colon T_{\gamma_s(0)}S^2 \to T_{\gamma_s(0)}S^2$.

Problem L.5. Let σ be a smooth effective left action of a Lie group G on a smooth manifold L, and suppose $L \to E \xrightarrow{\pi} M$ is a (G, σ)-fibre bundle. Let $\gamma\colon (a, b) \to M$ be a smooth curve. Prove that $\gamma^\star E \to (a, b)$ is a trivial bundle.

PROBLEM L.6. Let $\pi\colon E \to M$ be a vector bundle, and let Δ be a connection on E. Let $\gamma\colon [a,b] \to M$ be a smooth curve and let $t_0 \in [a,b]$. Prove that for any $v \in E_{\gamma(t_0)}$, there exists a unique horizontal section ρ of E along γ such that $\rho(t_0) = v$.

Problem Sheet M

Problem M.1. Let V and W be vector spaces, and suppose $f\colon V \to W$ is a continuous map which is differentiable at $0 \in V$ and homogeneous in the sense that $f(cv) = cf(v)$ for all $v \in V$ and $c \neq 0$. Prove that f is a linear map.

Problem M.2. Let $\pi\colon E \to M$ be a vector bundle of rank n with connection Δ. Fix $p \in M$ and let $\{v_1, \dots, v_n\}$ be a basis of E_p.

(i) Let $\psi_p\colon U_p \to \mathcal{O}_p$ be a ray parametrisation at p. For $\xi \in T_pM$ write $\gamma_{p,\xi}(t) := \psi_p(t\xi)$, as in (29.5). Prove there exists a local frame $\{e_1, \dots, e_n\}$ on U_p such that $e_i(p) = v_i$ and such that for all $\xi \in T_pM$, $e_i \circ \gamma_{p,\xi}$ is parallel along $\gamma_{p,\xi}$.

(ii) Let $\gamma\colon (-\varepsilon, \varepsilon) \to M$ be a smooth curve with $\gamma(0) = p$ and $\dot\gamma(t) \neq 0$ for all $t \in (-\varepsilon, \varepsilon)$. Deduce that there exists a local frame $\{e_1, \dots, e_n\}$ of E over an open set U containing p such that $e_i(p) = v_i$ and such that $e_i \circ \gamma$ is parallel along γ for each $i = 1, \dots, n$.

Problem M.3. Let Δ be a connection in a vector bundle $\pi\colon E \to M$ with associated parallel transport system $\mathbb{P}$ and covariant derivative $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$.

(i) Define the **dual parallel transport system** in the dual bundle E^* by declaring that a section $\nu \in \Gamma_\gamma(E^*)$ is parallel if and only if $\nu(\rho)$ is constant for every parallel section $\rho \in \Gamma_\gamma(E)$. Prove *directly* that this defines a parallel transport system.

You may skip the verification of Axiom (iv)' of Definition 29.11.

(ii) Define the **dual covariant derivative operator** $\nabla^*\colon \mathfrak{X}(M) \times \Gamma(E^*) \to \Gamma(E^*)$ defined by

$$(\nabla^*_X \sigma)(s) = X(\sigma(s)) - \sigma(\nabla_X s).$$

Prove *directly* that this is a covariant derivative operator in E^*.

(iii) The **dual connection** on E^* is the connection Δ^* whose associated parallel transport system is the dual parallel transport system from part (i) and whose associated covariant derivative operator is the dual covariant derivative operator from part (ii). How does one define Δ^* explicitly?

Problem M.4. Let E, F be two vector bundles over M with connections ∇^E and ∇^F.

(i) Prove that there is a unique connection on $E \otimes F$ which on decomposable sections $r \otimes s$ takes the form

$$\nabla^{\otimes}_X (r \otimes s) := \nabla^E_X r \otimes s + r \otimes \nabla^F_X s.$$

(ii) Prove that

$$(\nabla^{\mathrm{Hom}}_X \Phi)(s) := \nabla^F_X(\Phi(s)) - \Phi(\nabla^E_X s)$$

is a connection on $\mathrm{Hom}(E, F)$

The connections in part (i) and part (ii) are consistent with the connection on the dual bundle from Problem (ii) under the isomorphism $\mathrm{Hom}(E, F) \cong E^* \otimes F$ from Corollary 19.14.

Problem M.5. Let ∇ denote the connection on TS^m from Problem L.3.

(i) Find an explicit formula for the connection map $K\colon T(TS^m) \to TS^m$ and for the covariant derivative operator $\nabla\colon \mathfrak{X}(S^m)\times\mathfrak{X}(S^m) \to \mathfrak{X}(S^m)$.

(ii) Let p, q be two points in S^m such that $p \perp q$. Let $\gamma\colon [0, 2\pi] \to S^m$ denote the great circle $\gamma(t) = (\cos t)p+(\sin t)q$. Prove that $\nabla^{\gamma}_{T}\dot{\gamma} = 0$, where T is the vector field $\frac{\partial}{\partial t}$ on $[0, 2\pi]$.

(iii) Prove that $\mathrm{Hol}^{\nabla} = \mathrm{SO}(m)$ (in the sense of Corollary 32.16).

(iv) Compute the curvature tensor R^{∇}.

Problem M.6. Let $\pi\colon E \to M$ be a vector bundle with connection ∇. Let $F \subset E$ be a vector subbundle such that ∇ is reducible to F. Prove that ∇ restricts to define a connection on F.

Problem M.7. Suppose ∇ is a connection on the tangent bundle $\pi\colon TM \to M$ of a manifold M. Show that for each $X \in \mathfrak{X}(M)$ the operator $\nabla_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ extends uniquely to define a tensor derivation $\nabla_X : \mathscr{T}(M) \to \mathscr{T}(M)$.

Problem M.8. Let $\pi\colon E \to M$ be a vector bundle over a connected manifold M, and let Δ denote a connection on E. Let $\psi\colon \widetilde{M} \to M$ denote the universal covering of M. Prove that ∇ is flat if and only if $\psi^*E \to \widetilde{M}$ is the trivial bundle over $\widetilde{M}$ and the pullback connection $\psi^*\Delta$ is the trivial connection.

Problem M.9. Let G be a Lie group with Lie algebra $\mathfrak{g}$.

(i) Suppose $\beta\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ is a bilinear map. Prove there exists a unique connection ∇^{β} on $TG \to G$ which satisfies the following condition: if $\xi, \zeta \in \mathfrak{g}$ and X_ξ, X_ζ denote the corresponding left-invariant vector fields then

$$\nabla^{\beta}_{X_\xi}(X_\zeta) = X_{\beta(\xi,\zeta)}.$$

(ii) Prove that this connection is left-invariant in the sense that

$$(l_g)_*(\nabla^{\beta}_X Y) = \nabla^{\beta}_{(l_g)_*X}((l_g)_*(Y)), \qquad \forall\, X, Y \in \mathfrak{X}(G),\ \forall\, g \in G.$$

Deduce that the parallel transport determined by this connection is left-invariant in the sense that if ρ is a parallel section along a curve γ then $Dl_g(\gamma)\circ\rho$ is a parallel section along $l_g\circ\gamma$.

(iii) Prove moreover that any such left-invariant connection ∇ determines such a bilinear map β via

$$\beta(\xi,\zeta) := \nabla_{X_\xi}(X_\zeta)(e),$$

and hence that there is a bijective correspondence between bilinear maps $\mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ and left-invariant connections on TG.

Problem Sheet N

Problem N.1. Let $\pi\colon E \to M$ denote a vector bundle, and let ∇_1 and ∇_2 denote two connections on E.

(i) Prove that $\nabla_1 - \nabla_2$ defines an element $\Theta \in \Omega^1(M, \mathrm{End}(E))$.

(ii) If Δ_1 and Δ_2 are the distributions on E corresponding to ∇_1 and ∇_2 respectively, prove that for all $v \in E$ one has

$$\Delta_{2|v} = \left\{\zeta + \mathcal{J}_v\big(\Theta_{\pi(v)}(D\pi(v)\zeta)(v)\big) \mid \zeta \in \Delta_{1|v}\right\}$$

where Θ is in the previous part.

Θ is a 1-form with values in $\mathrm{End}(E)$. Thus for $p \in M$, $\xi \in T_pM$ and $v \in E_p$, the endomorphism $\Theta_p(\xi)$ can eat v to produce another element $\Theta_p(\xi)(v)$ in the fibre E_p.

(iii) Prove that

$$R^{\nabla_2} = R^{\nabla_1} - d^{\nabla_1}\Theta + [\Theta, \Theta],$$

where $[\Theta, \Theta] \in \Omega^2(M, \mathrm{End}(E))$ is defined by

$$[\Theta, \Theta](X, Y) = \Theta(X)\Theta(Y) - \Theta(Y)\Theta(X), \qquad X, Y \in \mathfrak{X}(M).$$

(iv) Conversely, prove that if ∇ is a connection on E and $\Theta \in \Omega^1(M, \mathrm{End}(E))$ then $\nabla_1 := \nabla + \Theta$ is another connection. Deduce that the space of connections on E is (non-canonically) isomorphic to $\Omega^1(M, \mathrm{End}(E))$.

(v) Use part (iii) to give another proof of Proposition 37.4.

Problem N.2. Let $\pi\colon E \to M$ denote a vector bundle with connection ∇. Let ∇^{End} denote the induced connection on $\mathrm{End}(E)$, and let d^{∇} and $d^{\nabla^{\mathrm{End}}}$ denote the corresponding exterior covariant differentials. Prove that for $\Theta \in \Omega^k(M, \mathrm{End}(E))$ and $\alpha \in \Omega(M, E)$ we have

$$d^{\nabla}(\Theta \wedge \alpha) = d^{\nabla^{\mathrm{End}}}\Theta \wedge \alpha + (-1)^k \Theta \wedge d^{\nabla}\alpha.$$

Problem N.3. Let $\pi\colon E \to M$ be a vector bundle of rank n over a connected manifold M. Fix a Lie subgroup $G \subset \mathrm{GL}(n)$.

(i) Let us say that a connection ∇ on G is a **G-connection** if $\mathrm{Hol}^{\nabla}(p) \subset G$, up to conjugation (cf. Corollary 32.16). Prove that this is well-defined (i.e. independent of the choice of p).

(ii) Fix a G-connection ∇_1, and let ∇_2 denote any other connection. Suppose that the difference $\nabla_1 - \nabla_2$ actually lies in $\Omega^1(M, \mathfrak{hol}^{\nabla_1}) \subset \Omega^1(M, \mathrm{End}(E))$. Prove that ∇_2 is also a G-connection.

Recall that $\mathfrak{hol}^{\nabla_1}$ is in particular a submanifold of $\mathrm{End}(E)$, so this assumption makes sense.

Problem N.4. Let (E, g) be a Riemannian vector bundle over M, and let ∇ be a metric connection. Fix $p \in M$. Prove that the holonomy group $\mathrm{Hol}^{\nabla}(p)$ is a subgroup of the orthogonal group

$$\mathrm{O}(E_p, g_p) := \{A \in \mathrm{GL}(E_p) \mid g_p(A(u), A(v)) = g_p(u, v),\ \forall\, u, v \in E_p\}.$$

Problem N.5. Let (E, g) be a Riemannian vector bundle. Given $u \in E_p$ define $u^\flat \in E_p^*$ by

$$u^\flat(v) := g_p(u, v)$$

(i) Prove that $\flat\colon E \to E^*$ is a vector bundle isomorphism.

(ii) Let $\sharp\colon E^* \to E$ denote the inverse of E (written $\lambda \mapsto \lambda^\sharp$). Prove that

$$g^*(\lambda, \eta) := g_p(\lambda^\sharp, \eta^\sharp), \qquad \lambda, \eta \in E_p^*$$

defines a Riemannian metric on E^*.

(iii) Prove that (E, g) and (E^*, g^*) are isometric vector bundles in the sense of Definition 37.7.

The vector bundle isomorphisms $\flat$ and $\sharp$ are usually called the **musical isomorphisms**.

Problem N.6. Let $\mathsf{q} \in \mathscr{P}_{\mathrm{inv}}(n)$ be an invariant homogeneous polynomial of odd degree $2k+1$. Prove that $\mathsf{CW}_E(\mathsf{q}) = 0$ for any vector bundle of rank n.

Problem N.7. Let $\pi\colon E \to M$ be a vector bundle of rank n. Prove that the Chern-Weil map

$$\mathsf{CW}_E\colon \mathscr{P}_{\mathrm{inv}}(n) \to H_{\mathrm{dR}}(M)$$

is an algebra homomorphism (where the algebra structure on the left-hand side is just the pointwise product of functions, and on the right-hand side it is the wedge product, cf. Definition 38.12).

Problem N.8. Suppose that E and F are two vector bundles over a smooth manifold M. Prove the **Whitney product formula** for the Pontryagin classes

$$p_k(E \oplus F) = \sum_{i=0}^{k} p_i(E) \wedge p_{k-i}(F).$$

For those of you who are familiar with Algebraic Topology: the statement would be more complicated if one worked with (singular) cohomology with coefficients in $\mathbb{Z}$, since then one would need to worry about 2-torsion elements.

Problem N.9. Prove directly that $p_k(TS^m) = 0$ for all $k > 0$. *Remark:* This shows that Pontryagin classes alone cannot determine a vector bundle up to isomorphism (since $TS^m \to S^m$ is not a trivial bundle).

i.e. don't just quote Proposition 38.19!

Problem Sheet O

Problem O.1. let V_1, V_2 and W be vector spaces. Let $\omega \in \Omega^h(M, V_1)$ and let $\theta \in \Omega^k(M, V_2)$, and let $\beta \colon V_1 \times V_2 \to W$ be a bilinear map. Prove that

$$d(\omega \wedge_\beta \theta) = d\omega \wedge_\beta \theta + (-1)^h \omega \wedge_\beta d\theta.$$

$\wedge_\beta$ was defined just before Proposition 36.5. This problem was meant to be on Problem Sheet N but I forgot to include it.

Problem O.2. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and suppose τ is a right action of G on a manifold P. Prove that the map $\xi \mapsto Z_\xi$ is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(P)$.

Problem O.3. Let P be a manifold and $\mathfrak{g}$ a Lie algebra. Let $\omega \in \Omega^1(P, \mathfrak{g})$. Prove that the 3-form $[[\omega, \omega], \omega] \in \Omega^3(P, \mathfrak{g})$ (defined as in Example 36.6) is identically zero.

Problem O.4. Let $\pi \colon P \to M$ denote a principal G-bundle, and let ϖ denote a connection on P with curvature form Ω. Fix $X, Y \in \mathfrak{X}(M)$, and let $\overline{X}$ and $\overline{Y}$ denote their horizontal lifts. Prove that for any $u \in P$ one has

$$\overline{[X, Y]}(u) - \left[\overline{X}, \overline{Y}\right](u) = D\tau^u(e)\Big(\Omega_p(\overline{X}(u), \overline{Y}(u))\Big).$$

Problem O.5. Let $\pi \colon P \to M$ denote a principal G-bundle, and let $\sigma \colon G \to \mathrm{GL}(V)$ denote a smooth effective representation of G. Let $\mu := D\sigma(e)$, and suppose $f \colon P \to V$ is an equivariant smooth function. Prove that for any $\xi \in \mathfrak{g}$, one has

$$Z_\xi(f) + \mu_\xi(f) = 0.$$

Problem O.6. Let $\pi \colon P \to M$ be a principal G-bundle. Let σ be a representation of G on a vector space V, and let $E = P \times_G V$ denote the associated vector bundle. Let ϖ denote a connection on P and let ∇ denote the associated connection on E. Fix $p \in M$. Then we can regard $\mathrm{Hol}^\varpi(p)$ and $\mathrm{Hol}^\nabla(p)$ as subgroups of G and $\mathrm{GL}(V)$ respectively, which are defined up to conjugation. Prove that (also up to conjugation)

$$\sigma\big(\mathrm{Hol}^\varpi(p)\big) = \mathrm{Hol}^\nabla(p).$$

Problem O.7. Let $\pi \colon E \to M$ be a vector bundle of rank n, and let $\mathrm{Fr}(E) \to M$ denote the principal $\mathrm{GL}(k)$-bundle. Then by Proposition 39.10 there is a bijective correspondence between connections ∇ on E and connections ϖ on $\mathrm{Fr}(E)$. Fix a Lie subgroup $G \subset \mathrm{GL}(kn$. Prove that a connection ∇ on E is a G-connection in the sense of Problem N.3 if and only if the corresponding connection ϖ on $\mathrm{Fr}(E)$ is reducible to G in the sense of Definition 42.5.

Problem O.8. Use the principal bundle version of the Bianchi Identity (i.e. (41.2)) to prove the vector bundle version (Theorem 36.21).

Problem O.9. Use the principal bundle version of the Ambrose–Singer Holonomy Theorem (Theorem 42.7) to prove the vector bundle version (Theorem 35.6).

Bonus Problem(s) for Sheet O

These problem(s) are hard, and are included for enthusiasts only.
Solutions will not be provided.

PROBLEM O.10. Develop the theory of characteristic classes for principal bundles.

Problem Sheet P

PROBLEM P.1. Let ∇ be a connection on M. Let (U, x) and (V, y) denote two charts on g such that $U \cap V \neq \emptyset$. Let Γ_{ij}^k denote the Christoffel symbols of x and $\tilde{\Gamma}_{ij}^k$ denote the Christoffel symbols of y, so that

$$\nabla_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right) = \Gamma_{ij}^k \frac{\partial}{\partial x^k}, \qquad \nabla_{\frac{\partial}{\partial y^i}}\left(\frac{\partial}{\partial y^j}\right) = \tilde{\Gamma}_{ij}^k \frac{\partial}{\partial y^k}.$$

Investigate the relationship between

$$\Gamma_{ij}^k|_{U\cap V} \qquad \text{and} \qquad \tilde{\Gamma}_{ij}^k|_{U\cap V}.$$

PROBLEM P.2. Let ∇ denote a connection on M, and let d^∇ denote the associated exterior covariant differential. Prove that

$$T^\nabla = d^\nabla(\mathrm{id}).$$

PROBLEM P.3. Let ∇ be a torsion-free connection on g with curvature tensor R^∇. Prove that for all $X, Y, Z \in \mathfrak{X}(M)$, one has

$$\big(\nabla_X R^\nabla\big)(Y, Z) + \big(\nabla_Y R^\nabla\big)(Z, X) + \big(\nabla_Z R^\nabla\big)(X, Y) = 0.$$

PROBLEM P.4. Consider S^m equipped with the metric g_{round} from part 46.13 of Examples 46.13. Prove that the Levi-Civita connection of g_{round} is the connection introduced in Problem L.3.

PROBLEM P.5. Let g be a Riemannian metric on M, and let ∇ denote the Levi-Civita connection of M.

(i) Prove that for all $X, Y, Z \in \mathfrak{X}(M)$,

$$\mathcal{L}_X g(Y, Z) = \mathcal{L}_X g(Y, Z) = \langle \nabla_Y(X), Z\rangle + \langle Y, \nabla_Z(X)\rangle.$$

(ii) We say that a vector field X is a **Killing field** if $\mathcal{L}_X g = 0$. Prove that a vector field is a killing field if and only if its maximal flow consists of local isometries.

PROBLEM P.6. Let $\varphi\colon M \to N$ be an isometric map between Riemannian manifolds. Prove that for $p \in M$ the restriction of $(\cdot)^\top$ to $T_{\varphi(p)}N$ is the orthogonal projection onto $D\varphi(p)(T_pM)$.

PROBLEM P.7. Let $\varphi\colon M \to N$ be a smooth normal covering map and g is a Riemannian metric on M which is invariant under all deck transformations. Prove there is a unique Riemannian metric on N such that φ is a Riemannian covering.

PROBLEM P.8. Let M be a smooth manifold and suppose σ is a smooth transitive left action of a Lie group G on M. Fix $p \in M$ and let H denote the isotropy group at p, so that $M \cong G/H$ is a homogeneous space. Let $\tau\colon H \to \mathrm{GL}(T_pM)$ denote the representation of H on T_pM given by

cf. Theorem 13.12.

cf. Proposition 13.11.

$$\tau_h(\xi) = D\mu_h(e)\xi, \qquad h \in H,\ \xi \in T_pM.$$

We say that a Riemannian metric g on M is **invariant** if $\mu_h\colon M \to M$ is an isometry for every $h \in G$. Prove that there is a bijective correspondence between invariant Riemannian metrics on M and inner products on T_pM that are invariant under τ_h for each $h \in H$.

PROBLEM P.9. Let M be a connected manifold and suppose ∇ is a torsion-free connection on M. Prove that ∇ is the Levi-Civita connection of some Riemannian metric g on M if and only if Hol^{∇} is conjugate in $\mathrm{GL}(m)$ to a subgroup of $\mathrm{O}(m)$.

PROBLEM P.10. Let M be a manifold of dimension two or three.

(i) Prove that the curvature tensor $\mathcal{R}_g^{\nabla}$ is completely determined by the Ricci tensor Ric_g.

(ii) Prove that a Riemannian metric g on M is Einstein if and only if g has constant curvature.

PROBLEM P.11. Let G denote a Lie group, and let $\mathfrak{g}$ denote the Lie algebra of G.

(i) Suppose $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is an inner product on $\mathfrak{g}$. Prove that $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ induces a left-invariant Riemannian metric ρ on G by

$$\rho_g(X_\xi(g), X_\zeta(g)) \coloneqq \langle \xi, \zeta\rangle_{\mathfrak{g}}, \qquad \forall\, \xi, \zeta \in \mathfrak{g},\ g \in G,$$

where X_ξ is the left-invariant vector field on G with $X_\xi(e) = \xi$. Prove moreover that every left-invariant Riemannian metric on G is of this form.

Recall (cf. Definition 44.6) that for for Lie groups we use the letter ρ as our default notation for a Riemannian metric.

(ii) Prove that the Riemannian metric ρ associated to $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is right-invariant (and hence bi-invariant) if and only if

$$\langle \mathrm{Ad}_g(\xi), \mathrm{Ad}_g(\zeta)\rangle_{\mathfrak{g}} = \langle \xi, \zeta\rangle_{\mathfrak{g}}, \qquad \forall\, \xi, \zeta \in \mathfrak{g},\ g \in G.$$

(iii) Assume now that G is connected. Prove that the Riemannian metric ρ associated to $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is bi-invariant if and only if ad_ξ is skew-symmetric with respect to $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ for all $\xi \in \mathfrak{g}$.

PROBLEM P.12. Let G denote a Lie group, and let $\mathfrak{g}$ denote the Lie algebra of G. Let ∇^c denote the connection on G defined by

$$\nabla^c_{X_\xi}(X_\zeta) = c\,[X_\xi, X_\zeta], \qquad \forall\, \xi, \zeta \in \mathfrak{g}.$$

This is the connection on G given by taking $\beta = c\,[\cdot,\cdot]$ in Problem M.9.

Let ρ denote a bi-invariant Riemannian metric on G.

(i) Prove that ∇^c is complete for any $c \in \mathbb{R}$.

(ii) Prove that ∇^c is metric with respect to ρ for all $c \in \mathbb{R}$.

(iii) Prove that $\nabla^{\frac{1}{2}}$ is torsion-free (and hence is equal to the Levi-Civita connection of (G, ρ)).

(iv) Prove that $\nabla^{\frac{1}{2}}$ is right-invariant in the sense that

$$(r_g)_*(\nabla^{\frac{1}{2}}_X Y) = \nabla^{\frac{1}{2}}_{(r_g)_* X}((r_g)_* Y), \qquad \forall\, X, Y \in \mathfrak{X}(G),\ \forall\, g \in G.$$

We already know from Problem M.9 that $\nabla^{\frac{1}{2}}$ is left-invariant.

(v) Compute the curvature tensor $R^{\nabla^{\frac{1}{2}}}$ of $\nabla^{\frac{1}{2}}$.